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ON SOME EXTREMAL PROBLEM IN CLASSES OF BOUNDED UNIVALENT FUNCTIONS

1. Introduction

Denote by S the class of functions F holomorphic and univalent in the disc $E = \{z: |z| < 1\}$, with the Taylor expansion

$$F(z) = z + \sum_{n=2} A_{nF} z^n, \quad z \in E.$$

Let $S(M)$, $M > 1$, be the subclass of the family S, consisting of functions satisfying in the disc E the condition $|F(z)| \leqslant M$, whereas $S_R(M)$ - the subclass of $S(M)$, consisting of functions with real coefficients.

In [17], [18] and [4] (cf. also [3]), for $N = 6$, $N = 8$ and any even N, successively, there has been determined a problem of estimating the N-th coefficient of functions of the classes $S_R(M)$ when M is sufficiently large. In the method applied in these papers use is made of, among others, the differential-functional equation of extremal functions in the classes $S_R(M)$ [2], yet with the avoidance of complicated integration of this equation. It seems natural to apply this method to the examination of coefficients of functions of the classes $S(M)$. One knows in these classes the differential-functional equation of extremal functions [1]. Besides, it turned out more than once that if, in any extremal problem in the class $S_R(M)$, the extremal function was a function $F(z)$, $z \in E$, then, in an analogous problem in $S(M)$, such was a function $e^{-i\theta}F(e^{i\theta}z)$, $0 \leqslant \theta \leqslant 2\pi$.

In the present paper the functional $\mathrm{Re}\, A_{NF}$ defined in the class $S(M)$ is considered, where N is any even positive integer, $N \geqslant 6$. By applying the method in question, it has been shown that there exists a constant M_N^* such that, for all $M > M_N^*$ and any function $W = F_0(z)$, $z \in E$, realizing the maximum of this functional (for $N \geqslant 8$, some additional assumption is needed), the domain $F_0(E)$ is the disc $|W| < M$ from which exactly one analytic arc issuing from some point W_{0F} on the circle $|W| = M$ has been removed.

An analogous problem, consisting in characterizing domains obtained by extremal mappings, was considered in the class S by, among others, A.C.Schaeffer and D.C.Spencer. These authors proved, in particular, that each function, for which any linear functional defined in the class S attains its maximum, maps the disc E onto a plane from which exactly one analytic arc issuing from the point ∞ has been removed ([9], p.149, lemma XXX). The analogous theorem in the class of bounded functions would, however, be false, which can be shown by, at least, a function realizing the maximum of the functional $\mathrm{Re}\, A_{3F}$ defined in the class $S(M)$ (cf. e.g., [10]).

Let us still recall that coefficients of functions of the classes $S(M)$ were exactly estimated in the case when M was sufficiently close to 1 ([13], [14], [15]; [12]). For M sufficiently large, such estimations are known only for $N = 2$, 3, 4 ([7]; [10], [16], [5]; [11]).

The theorems included in the present paper may become starting-point for further investigations of coefficients with even indices.

Z.J.Jakubowski has raised a hypothesis that, similarly as in the classes $S_R(M)$ (cf. [4]), the only functions in the classes $S(M)$, for which $\mathrm{Re}\, A_{NF}$ ($N = 6, 8, \ldots$), attains maximum, are functions of the form

$$e^{-i\theta}P_M(e^{i\theta}z), \quad z \in E, \quad \theta = \frac{2\pi j}{N-1}, \quad j=0,1,\dots,N-2,$$

where $W = P_M(z)$ is a Pick function given by the equation

(1) $$\frac{W}{\left(1 - \frac{W}{M}\right)^2} = \frac{z}{(1 - z)^2}, \quad z \in E,$$

and satisfying the condition $P_M(0) = 0$.

2. Remarks and auxiliary theorems

Let N be any fixed even positive integer, $N \geqslant 6$, M - any real number, $M > 1$. The functional

(2) $$J_N(F) = \operatorname{Re} A_{NF}, \quad F \in S(M),$$

is continuous, whereas the class $S(M)$ - compact; consequently, the family of functions for which this functional attains its maximum is non-empty. This family will be denoted by $\mathcal{F}_N(M)$.

It is self-evident that, if $F \in S(M)$, then, for each $\theta \in \langle 0, 2\pi)$, the function

$$F_\theta(z) = e^{-i\theta}F(e^{i\theta}z), \quad z \in E,$$

also belongs to the class $S(M)$. Hence it immediately appears that, if $F \in \mathcal{F}_N(M)$, then, for $\theta = \frac{2\pi j}{N-1}$, $j=0,1,2,\dots,N-2$, the function F_θ also belongs to $\mathcal{F}_N(M)$.

Moreover, note that, for each function $F \in \mathcal{F}_N(M)$, we have: $\operatorname{Re} A_{NF} = A_{NF}$.

L e m m a 1. Let $n \geqslant 2$ be any fixed positive integer. If the only functions in the class S, for which $\operatorname{Re} A_{nF} = n$, are the Koebe functions

$$K_\varepsilon(z) = \frac{z}{(1 - \varepsilon z)^2}, \quad z \in E, \quad |\varepsilon| = 1,$$

then, for each function $F \in S$,

$$\operatorname{Re} A_{nF} \leqslant n.$$

P r o o f . First of all, let us observe that, if F is a function of the class S, then, for each number α, $0 < \alpha < 1$, the function

$$F_{\alpha}(z) = \frac{1}{\alpha} F(\alpha z) = z + \alpha A_{2F} z^2 + \dots + \alpha^{n-1} A_{nF} z^n + \dots$$

is holomorphic and univalent in the disc $|z| < \frac{1}{\alpha}$; so it also belongs to the class S.

Suppose, despite of the proposition, that there exists a function $F_0 \in S$ such that

$$\operatorname{Re} A_{nF_0} > n.$$

It follows from the above remark that, for $\alpha_0 = \sqrt[n-1]{\frac{n}{\operatorname{Re} A_{nF_0}}}$ $(0 < \alpha_0 < 1)$, the function F_{α_0} belongs to S. But $\operatorname{Re} A_{nF_{\alpha_0}} = n$, so, in accordance with the assumption,

$$F_{\alpha_0}(z) = \frac{z}{(1 - \varepsilon z)^2}, \quad z \in E, \quad |\varepsilon| = 1.$$

This contradicts the fact that the function F_{α_0} is holomorphic in the disc $|z| < \frac{1}{\alpha_0}$, and thus, in particular, at the point $z = \overline{\varepsilon}$.

L e m m a 2. Let N be any fixed even positive integer, $N \geqslant 6$. For $N \geqslant 8$, assume in addition that the only functions in the class S, for which $\operatorname{Re} A_{NF} = N$, are the Koebe functions

$$(3) \qquad K_j(z) = \frac{z}{(1-\varepsilon_j z)^2}, \quad z \in E, \quad \varepsilon_j = e^{\frac{2\pi j}{N-1}i},$$
$$j=0,1,\dots,N-2.$$

Then there exists a constant $\hat{M}_N$ ($\hat{M}_N > 1$) such that, for all $M > \hat{M}_N$ and each function $F \in \mathcal{F}_N(M)$,

$$A_{2F} \neq 0.$$

The proof will be carried out for $N = 6$. In doing this, essential use will be made of the following Pederson result [6]: for each function $F \in S$, the estimation

$$(4) \qquad \operatorname{Re} A_{6F} \leqslant 6$$

holds, with that equality in (4) takes place only for functions (3) ($N = 6$, $j = 0, 1, 2, 3, 4$).

So suppose that, for $N = 6$, the proposition of the lemma does not hold. Then there exists an increasing sequence $(M_h)_{h=1,2,\dots}$ of real numbers and its correspondent sequence $(F_h)_{h=1,2,\dots}$ of functions belonging respectively to the families $\mathcal{F}_6(M_h)$, $h = 1,2,\dots$, and such that

$$(5) \qquad A_{2F_h} = 0, \quad h = 1,2,\dots .$$

In virtue of the compactness of the class S, from the sequence $(F_h)_{h=1,2,\dots}$ one may extract a subsequence $(F_{h_m})_{m=1,2,\dots}$ converging almost uniformly to some function $\hat{F} \in S$. From (5) and the Weierstrass theorem we then have

$$(6) \qquad \begin{cases} \lim\limits_{m\to\infty} A_{2F_{h_m}} = A_{2\hat{F}} = 0, \\ \lim\limits_{m\to\infty} A_{6F_{h_m}} = A_{6\hat{F}}. \end{cases}$$

It is known that, for each $M > 1$, the Pick function $W = P_M(z)$, $z \in E$, defined by equation (1) and satisfying the condition $P_M(0) = 0$, belongs to the class $S(M)$. Moreover, denoting

$$P_M(z) = z + \sum_{n=2}^{\infty} P_{n,M}z^n, \quad z \in E,$$

we have (cf. e.g., [4]):

$$\lim_{M \to \infty} P_{n,M} = n, \quad n = 2,3,\ldots .$$

From this and the Pederson result

$$P_{6M_{h_m}} \leqslant A_{6F_{h_m}} < 6$$

and, in consequence, $A_{6\hat{F}} = 6$. So $\hat{F}$ is one of Koebe functions (3), which contradicts (6).

For $N \geqslant 8$, in view of an additional assumption and Lemma 1, our reasoning runs in the same way as that for $N = 6$.

Let N still be any fixed even positive integer, $N \geqslant 6$. In the sequel, we shall assume (similarly as in Lemma 2) that, for $N \geqslant 8$, the only functions in the class S, for which $\operatorname{Re} A_{NF} = N$, are functions (3).

Denote

$$\mathcal{F}_N = \bigcup_{M \in (\hat{M}_N, \infty)} \mathcal{F}_N(M), \tag{7}$$

where $\hat{M}_N$ is the constant defined in Lemma 2. Let us divide family (7) into the following N-1 subclasses

$$\mathcal{F}_N^{(j)} = \left\{F \in \mathcal{F}_N : \frac{\pi}{N-1}(2j-1) \leqslant \operatorname{Arg} A_{2F} < \frac{\pi}{N-1}(2j+1)\right\}, \tag{8}$$

$$j=0,1,\ldots,N-2,$$

which is relevant for our further considerations.

Families (8) are, of course, non-empty and disjoint. What is more, for each $M > \hat{M}_N$ and each $j = 0,1,\dots,N-2$:

$$\mathcal{F}_N^{(j)} \cap \mathcal{F}_N(M) \neq \emptyset \quad \text{and} \quad \bigcup_{j=0}^{N-2} \mathcal{F}_N^{(j)} = \mathcal{F}_N.$$

L e m m a 3. Let j be any fixed number of the set $\{0, 1, \dots, N-2\}$, whereas $(M_h)_{h=1,2,\dots}$ - any sequence of real numbers, such that $\lim_{h\to\infty} M_h = \infty$, $M_h > \hat{M}_N$, for $h = 1,2,\dots$. From each of the families $\mathcal{F}_N^{(j)} \cap \mathcal{F}_N(M_h)$, $h = 1,2,\dots$, let us choose arbitrarily one function $F_h^{(j)}$. Then the sequence $(F_h^{(j)})_{h=1,2,\dots}$ is almost uniformly convergent in the disc E to the function K_j given by formula (3).

We carry out the proof by contradiction, fixing (with no loss of generality), for instance, $j = 0$. In this case it suffices to make use of the compactness of the class S and the Pederson result (for $N = 6$), or Lemma 1 (for $N \geqslant 8$).

3. Fundamental theorems

Let us proceed to fundamental theorems on the images of the disc E, obtained by extremal mappings belonging to the families $\mathcal{F}_N(M)$, where M is sufficiently large.

T h e o r e m 1. There exists a constant M_6^* ($M_6^* > 1$) such that, for each $M > M_6^*$, any function $W = F(z)$, $z \in E$, of the class $S(M)$, for which the functional

$$J_6(F) = \operatorname{Re} A_{6F}, \qquad F \in S(M),$$

attains its maximum, maps the disc E onto the disc $|W| < M$ from which exactly one analytic arc issuing from some point W_{0F} on the circle $|W| = M$ has been removed.

P r o o f . 1) We shall first factorize, for M sufficiently large, both sides of a suitable equation of extremal functions.

Let $F \in \mathcal{F}_6(M)$, $M > 1$. By the theorem of Z.Charzyński ([1], pp.5-6), the function $w = f(z) = \frac{1}{M} F(z)$, $z \in E$, satisfies the following differential-functional equation

$$(9) \qquad \left(\frac{zw'}{w}\right)^2 \mathfrak{M}_F(w) = \mathfrak{N}_F(z), \quad 0 < |z| < 1,$$

where

$$(10) \qquad \mathfrak{M}_F(w) = \sum_{m=2}^{6} \left[\frac{A_{6F}^{(m)}}{M^{m-1} w^{m-1}} + \frac{\overline{A_{6F}^{(m)}}}{M^{m-1}} w^{m-1}\right] - \mathfrak{P}_F,$$

$$(11) \qquad \mathfrak{N}_F(z) = \sum_{m=2}^{6} \left[\frac{(7-m) A_{7-m,F}}{z^{m-1}} + (7-m)\bar{A}_{7-m,F} z^{m-1}\right] + 5A_{6F} - \mathfrak{P}_F,$$

$$\mathfrak{P}_F = \min_{0 \leq x \leq 2\pi} \operatorname{Re}\left[\sum_{m=2}^{6} \frac{A_{6F}^{(m)}}{M^{m-1}} e^{ix(m-1)}\right],$$

$$F^m(z) = \sum_{n=m}^{\infty} A_{nF}^{(m)} z^n, \quad z \in E, \quad m = 1, 2, \ldots,$$

$$A_{nF}^{(1)} = A_{nF}, \quad n = 2, 3, \ldots; \quad A_{1F} = 1.$$

Functions (10) and (11) are non-negative on the circles $|w| = 1$, $|z| = 1$, respectively, and each of them has on the respective circle at least one double zero. Besides, from the forms of these functions it follows that, if $\mathfrak{M}_F(w_0) = 0$, then $\mathfrak{M}_F\left(\frac{1}{\bar{w}_0}\right) = 0$, and if $\mathfrak{N}_F(z_0) = 0$, then $\mathfrak{N}_F\left(\frac{1}{\bar{z}_0}\right) = 0$.

Let us first examine the location of zeros of the function $\mathcal{N}_F$ for M sufficiently large.

First of all, note that each of functions of the family $\mathcal{F}_6 = \bigcup_{M\in(\hat{M}_6,\infty)} \mathcal{F}_6(M)$ ($\hat{M}_6$ is the constant defined in Lemma 2) belongs to one and only one family $\mathcal{F}_6^{(j)}$, $j = 0,1,2,3,4$, defined by formula (8). It follows from Lemma 3, that, for each number $\varepsilon > 0$ and any compact subset Δ of the complex plane, there exists a constant $\tilde{M}_6$ ($\tilde{M}_6 > \hat{M}_6$) such that, for all $M > \tilde{M}_6$, if $F \in \mathcal{F}_6^{(j)}$, then, for each $z \in \Delta$:

$$(12) \qquad |z^5(\mathcal{N}_F(z) - \mathcal{N}_j(z))| < \varepsilon,$$

where

$$\mathcal{N}_j(z) = \sum_{m=2}^{6}\left[\frac{(7-m)^2\varepsilon_j^{6-m}}{z^{m-1}} + (7-m)^2\,\overline{\varepsilon_j^{6-m}}\,z^{m-1}\right] + 30, \quad \varepsilon_j = e^{\frac{2\pi i}{5}j}$$

$$j = 0,1,2,3,4.$$

It is easy to verify that

$$\mathcal{N}_j(z) = \frac{1}{z^5}(z + \bar{\varepsilon}_j)^2\mathcal{L}_j(z),$$

where

$$\mathcal{L}_j(z) = z^8 + 2\bar{\varepsilon}_j z^7 + 4\bar{\varepsilon}_j^2 z^6 + 6\bar{\varepsilon}_j^3 z^5 + 9\bar{\varepsilon}_j^4 z^4 +$$

$$+ 6z^3 + 4\varepsilon_j^4 z^2 + 2\varepsilon_j^3 z + \varepsilon_j^2, \qquad j = 0,1,2,3,4.$$

We shall prove that the point $z = -\bar{\varepsilon}_j$ is the only zero of the function $\mathcal{N}_j$ on the circle $|z| = 1$. To this end, let us observe that the function

$$\hat{\mathcal{N}}_j(z) = \frac{1}{z^5}(z-\bar{\varepsilon}_j)^2 \mathcal{L}_j(z) = \sum_{m=1}^{3}\left[(\varepsilon_j z)^{2m-1} + \frac{1}{(\varepsilon_j z)^{2m-1}}\right] - 6$$

has, with the exception of the points $z = \bar{\varepsilon}_j$ and $z = -\bar{\varepsilon}_j$, the same zeros as the function $\mathcal{N}_j$. The equality $\hat{\mathcal{N}}_j(e^{ix}) = 0$ $(0 \leqslant x < 2\pi)$ is liable to hold only if

$$\cos 5\left(\frac{2\pi j}{5} + x\right) + \cos 3\left(\frac{2\pi j}{5} + x\right) + \cos\left(\frac{2\pi j}{5} + x\right) = 3,$$

i.e., if $x = -\frac{2\pi j}{5}$ or $x = 2\pi - \frac{2\pi j}{5}$. Consequently, the only zero of the function $\hat{\mathcal{N}}_j$ on the circle $|z| = 1$ is the point $z = \bar{\varepsilon}_j$ not being, as can be seen, a zero of the polynomial $\mathcal{L}_j$. So, indeed, the only zero of the function $\mathcal{N}_j$ on the circle $|z| = 1$ is the point $z = -\bar{\varepsilon}_j$.

Taking account of the symmetry of functions $\mathcal{N}_j$, we may therefore represent each of them in the form

$$(13) \qquad \mathcal{N}_j(z) = \frac{(z+\bar{\varepsilon}_j)^2}{z^5}\prod_{m=1}^{4}(z - z_m^{(j)})\left(z - \frac{1}{z_m^{(j)}}\right),$$

$$j = 0,1,2,3,4;$$

where $|z_m^{(j)}| < 1$, $m = 1, 2, 3, 4$ $(j = 0, 1, 2, 3, 4)$.

Let us now fix arbitrarily j $(j = 0,1,2,3,4)$ and surround all zeros of the function $\mathcal{N}_j$ with sufficiently small disjoint discs. By (12) and the Hurwitz theorem, there exists a constant $M^{(j)}$ $(M^{(j)} > \hat{M}_6)$ such that, for all $M > M^{(j)}$, zeros of function (11) corresponding to any function $F \in \mathcal{F}_6^{(j)} \cap \mathcal{F}_6(M)$ lie, respectively, in chosen neighbourhoods of zeros of the function $\mathcal{N}_j$, with that in each of these neighbourhoods the number of zeros of both the functions is the same (multiplicities being taken into consideration). Consequently, for $M > M^{(j)}$ and any function $F \in \mathcal{F}_6^{(j)} \cap \mathcal{F}_6(M)$, function (11) has, according to (13), four zeros: z_{mF} (m=1,2,3,4) in the disc E and four: $\frac{1}{\bar{z}_{mF}}$ (m=1,2,3,4) outside this disc. Besides, from the theorem of Z.Charzyński [1] we know that this function has at least one double zero in the circle $|z|=1$. Consequently, for $M > M^{(j)}$ and $F \in \mathcal{F}_6^{(j)} \cap \mathcal{F}_6(M)$, we have

$$(14) \qquad \mathfrak{N}_F(z) = \frac{(z-z_{0F})^2}{z^5} \prod_{m=1}^{4} (z - z_{mF}) \left(z - \frac{1}{\bar{z}_{mF}}\right),$$

where $|z_{mF}| < 1$, $m = 1,2,3,4$; $|z_{0F}| = 1$.

From equation (9) we further infer that the points $w_{mF} = f(z_{mF})$, $m = 1,2,3,4$, are zeros of function (10) since $f'(z_{mF}) \neq 0$ $(m = 1,2,3,4)$. So, again by the theorem of Z.Charzyński, we have

$$(15) \qquad \mathfrak{M}_F(w) = \frac{(w-w_{0F})^2}{w^5} \prod_{m=1}^{4} (w - w_{mF}) \left(w - \frac{1}{\bar{w}_{mF}}\right)$$

where $|w_{mF}| < 1$, $m = 1,2,3,4$; $|w_{0F}| = 1$.

Let $M_6^* = \max_{0 \leq j \leq 4} M^{(j)}$.

We have demonstrated that, if $M > M_6^*$, then, for any function $F \in \mathcal{F}_6(M)$, the function $w = f(z) = \frac{1}{M} F(z)$, $z \in E$, satisfies equation (9) in which the functions $\mathfrak{M}_F$ and $\mathfrak{N}_F$ are of forms (15) and (14), respectively.

2) We shall now make use of the well-known Royden theorem ([8], p.660) by which each function $w = f(z) = \frac{1}{M} F(z)$, $z \in E$, where $F \in \mathcal{F}_6(M)$, $M > 1$, maps the disc E onto the disc $|w| < 1$ lacking a finite number of analytic arcs $l_1, l_2, \ldots, l_k$ $(k = k(f),\ k \geq 1)$ with the following properties ([9], parts III and IV):

1° The arcs $l_1, \ldots, l_k$ run in the disc $|w| < 1$, with the exception of, at most, their ends.

2° They are disjoint, with the exception of, at most, their ends.

3° The union of the arcs $l_1,\ldots,l_k$ and of the circle $|w| = 1$ constitues a continuum.

4° Along each of the arcs

(16) $$\mathrm{Re}\int\left[\mathfrak{M}_F(w)\right]^{1/2}\frac{dw}{w} = \text{const},$$

where $\mathfrak{M}_F$ is function (10).

5° Each common point of an arc and the circle $|w| = 1$, or of two arcs, is a zero of function (10). The number of arcs defined by equation (16), meeting in such a zero, depends on the multiplicity of the zero; in particular, in a double zero four arcs (16) meet, forming one with another, respectively, angles of measure $\frac{\pi}{2}$ ([9], part III).

6° At least one of ends of each arc $l_1,\ldots,l_k$ is a zero of function (10).

Let us take any function $F \in \mathcal{F}_6(M)$, where $M > M_6^*$, and its correspondent function $w = f(z) = \frac{1}{M}F(z)$, $z \in E$. We shall show that the number k of arcs as described is equal to 1.

Note that, according to property 3°, at least one of the arcs $l_1,\ldots,l_k$ must have a point in common with the circle $|w| = 1$. Without loss of generality, let us assume that it is the arc l_1. In view of 5°, the common point of the arc l_1 and the circle $|w| = 1$ is a zero of function (15), and thereby, it is the point w_{0F}. Since on the circle $|w| = 1$ condition (16) is satisfied, two of the arcs described in property 5° are arcs of this circle, forming each with the other an angle of measure π. Consequently, only one arc, namely l_1, may enter the interior of the circle. Besides, it follows from 5° and (15) that no other arc $(l_2,\ldots,l_k)$ has a point in common with the circle $|w| = 1$. So, in accordance with 3°, the union of arcs $l_2,\ldots,l_k$ is the empty set, or any of these arcs has a point in common with the end of the arc l_1, lying in the disc $|w|<1$. The second clause of this alternative, however, cannot hold since such a common point would be a zero of function (15), and thus, one of the points w_{mF}, $m = 1,2,3,4$. This is impossible since each of these points is an interior point of the domain $f(E)$.

We have thus proved that, if $M > M_6^*$, then, for any function $F \in \mathcal{F}_6(M)$, the function $w = f(z) = \frac{1}{M} F(z)$, $z \in E$, maps the disc E onto the disc $|w|<1$ from which exactly one analytic arc issuing from some point w_{0F} on the circle $|w| = 1$ had been removed.

Hence we immediately obtain the proposition of the theorem.

T h e o r e m 2. Let N be any fixed even positive integer, $N \geqslant 8$. Assume that the only functions in the class S, for which $\mathrm{Re}\, A_{NF} = N$, are Koebe functions (3). Then there exists a constant M_N^* $(M_N^* > 1)$ such that, for each $M > M_N^*$, any function $W = F(z)$, $z \in E$, of the class $S(M)$, for which functional (2) attains its maximum, maps the disc E onto the disc $|W| < M$ from which exactly one analytic arc issuing from some point W_{0F} on the circle $|W| = M$ has been removed.

The proof runs analogously as that of Theorem 1. By the theorem of Z.Charzyński [1], each function $w = f(z) = \frac{1}{M} F(z)$, $z \in E$, where $F \in \mathcal{F}_N(M)$ $(M > 1)$, satisfies differential-functional equation (9), where

$$(17) \qquad \mathfrak{M}_F(w) = \sum_{m=2}^{N} \left(\frac{A_{NF}^{(m)}}{M^{m-1} w^{m-1}} + \frac{\overline{A_{NF}^{(m)}}}{M^{m-1}} w^{m-1} \right) - \varphi_F,$$

$$(18) \quad \mathfrak{N}_F(z) = \sum_{m=2}^{N} \left[\frac{N-m+1}{z^{m-1}} A_{N-m+1,F} + (N-m+1)\bar{A}_{N-m+1,F} z^{m-1} \right] +$$

$$+ (N-1) A_{NF} - \varphi_F,$$

$$\Psi_F = \min_{0 \le x \le 2\pi} \operatorname{Re} \left[\sum_{m=2}^{N} \frac{A_{NF}^{(m)}}{M^{m-1}} e^{ix(m-1)} \right].$$

Functions (17) and (18) have the same properties as functions (10) and (11), respectively.

In virtue of the assumption and Lemma 3, we obtain, similarly as in Theorem 1, that there exists a constant M_N^* ($M_N^* > 1$) such that, for all $M > M_N^*$ and any function $F \in \mathcal{F}_N(M)$, functions (18) and (17) are, respectively, of the forms

$$\mathfrak{N}_F(z) = \frac{(z - z_{OF})^2}{z^{N-1}} \prod_{m=1}^{N-2} (z - z_{mF}) \left(z - \frac{1}{\bar{z}_{mF}} \right),$$

$$|z_{OF}| = 1, \quad |z_{mF}| < 1, \quad m = 1,2,\dots,N-2.$$

$$\mathfrak{M}_F(w) = \frac{(w - w_{OF})^2}{w^{N-1}} \prod_{m=1}^{N-2} (w - w_{mF}) \left(w - \frac{1}{\bar{w}_{mF}} \right),$$

$$|w_{OF}| = 1, \quad w_{mF} = f(z_{mF}), \; m=1,2,\dots,N-2,$$

$$w = f(z) = \frac{1}{M} F(z), \; z \in E.$$

Hence, basing ourselves on the Royden theorem and properties of Γ-structures [9], we get the proposition of the theorem.

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