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M.S. Khan

REMARKS ON SOME FIXED POINT THEOREMS

The well-known fixed point theorem of Lj.B.Čirič has been used to prove some results concerning common fixed points of a pair of mappings defined on a complete metric space. Our work refines some known results.

1. Introduction

Let (X,d) be a metric space. Then clearly a fixed point of a mapping $S:X \to X$ is a common fixed point of S and the identity mapping I_X on X. Motivated by this, Jungck [5] obtained the following generalization of the Banach Contraction Principle by replacing I_X by a continuous mapping $T:X \to X$.

T h e o r e m A. A continuous self-mapping T of a complete metric space (X,d) has a fixed point iff there exists a real number $\alpha \in (0,1)$ and a mapping $S:X \to X$ which commutes with T and satisfies

(i) $S(X) \subset T(X)$,

(ii) $d(Sx,Sy) \leqslant \alpha d(Tx,Ty)$ for all $x,y \in X$.

Indeed, S and T have a unique common fixed point.

Following is yet another generalization of Banach Contraction Principle which is due to Čirič [2].

T h e o r e m B. Let S be a self-mapping of a metric space (X,d). If the inequality

$$(*) \quad d(Sx,Sy) \leqslant \alpha \max\{d(x,y),\ d(x,Sx),\ d(y,Sy),\ d(x,Sy),\ d(y,Sx)\}$$

holds for all $x,y \in X$, $\alpha \in (0,1)$, and X is S-orbitally Complete, then S has a unique fixed point.

Mappings satisfying (*) are called quasi-contractions.

More recently, Das and Naik [3] proved the following result which unifies Theorem A and Theorem B.

Theorem C. Let (X,d) be a complete metric space. Let T be a continuous self-mapping on X and S be any self-mapping on X that commutes with T. Further, let S and T satisfy

(i) $S(X) \subset T(X)$,

(ii) $d(Sx,Sy) \leq \alpha \max\{d(Tx,Ty),d(Tx,Sx),d(Ty,Sy),d(Tx,Sy),d(Ty,Sx)\}$

holds for all $x,y \in X$ and $\alpha \in (0,1)$.

Then S and T have a unique common fixed point.

It may be mentioned that Theorem C has also been obtained earlier by Ranganathan [7]. Also note that in the proof of Theorem A, use is made of the Continuity of S, which is a consequence of (i), but no continuity argument of S has been used in the proof of Theorem C. It is known that every complete metric space is orbitally complete but the converse is not true (see Čirič [1]). Therefore Theorem B, as claimed by Das-Naik [3] and Ranganathan [7], does not follow from Theorem C by taking $T = I_X$.

In this note we show among other things that Theorem C actually follows from Theorem B. To do this we first prove a coincidence theorem which is subsequently utilized to establish the main theorem.

2. Main results

Theorem 2.1. Let S and T be two mappings of a non-empty set X into a metric space (Y,d) satisfying the following conditions for some non-negative real number $\alpha < 1$:

(i) $S(X) \subset T(X)$

(ii) for all $x,y \in X$ the inequality

$$d(Sx,Sy) \leq \alpha \max\{d(Sx,Tx),d(Sy,Ty),d(Sx,Ty),d(Sy,Tx),d(Tx,Ty)\}.$$

If $S(X)$ or $T(X)$ is complete under the metric d; then there exists a point $a \in X$ such that $S(a) = T(a)$.

P r o o f . Let U be a choice function for the family $\{T^{-1}(y): y \in S(X)\}$. Then $S \circ U$ is a mapping of $S(X)$ into itself and satisfies

$$d(S\circ Ux, S\circ Uy) \leqslant \alpha \max\{d(S\circ Ux,x), d(S\circ Ux,x), d(S\circ Uy,y), d(S\circ Ux,y), d(S\circ Uy,x), d(x,y)\}$$

for all $x,y \in S(X)$. So if $S(X)$ is complete, then by Theorem B we have $S\circ U(b) = b$ for some $b \in S(X)$. Put $a = Ub$. Then $a \in T^{-1}(b)$ and we have $b = T(a)$. Hence $S(a) = S\circ U(b) = b = T(a)$, as required. If $T(X)$ is complete then we consider a choice function for the family $\{S^{-1}(y): y \in T(X)\}$.

R e m a r k s :

(i) Machuca [6] obtained a coincidence theorem for mappings satisfying a contraction type condition which is a special case of a quasi-contraction. Hence Theorem 2.1 generalizes Machuca's result.

(ii) Theorem 2.1 offers criteria for the existence of solutions for equations of the form $Sx = Tx$, and $Sx = y$, $y \in X$ by specializing T to the constant mapping with value y.

Our main theorem reads as follows.

T h e o r e m 2.2. Let S and T be commuting mappings of a non-empty set X into a metric space (Y,d) and satisfying conditions (i) and (ii) of Theorem 2.1. If $S(X)$ or $T(X)$ is complete under the metric d, then S and T have a unique common fixed point.

P r o o f . By Theorem 2.1, there exists a point $a \in X$ such that $S(a) = T(a)$. Then commutativity of S and T gives $T(Ta) = T(Sa) = TS(a) = ST(a) = S(T(a)) = S(Sa)$. So using condition (ii) we have $d(Sa, S(Sa)) = 0$, that is, $S(S(a)) = Sa$. Hence $T(S(a)) = S(Sa) = Sa$. Thus $Sa(=Ta)$ is a common fixed point of S and T. The uniqueness of the common fixed point of S and T is routine and follows once again from condition (ii). This completes the proof.

Remark. In Theorem C the continuity of T was used both by Das-Naik [3] and Ranganathan [7]. Our Theorem 2.2 shows that even the continuity of T can be dispensed with.

Finally, we wish to present an application of Theorem 2.2

Theorem 2.3. Let (X,d) be a non-empty complete metric space and S, T be mappings of the product space $X \times X$ into X such that

$$S(X \times X) \subset T(X \times X) \quad \text{and} \quad S(T(x,y),y) = T(S(x,y),y)$$

for all $x,y \in X$. If there exists a constant α with $0 \leqslant \alpha < 1$ and

$$(**) \quad d(S(x,y),S(x',y')) \leqslant \alpha \max\{d(T(x,y),T(x',y')),d(T(x,y),S(x,y)), d(T(x',y'),S(x',y')),d(T(x,y),S(x',y')),d(T(x',y'),S(x,y))\}$$

for all $(x,y),(x',y') \in X \times X$, then there exists exactly one point $a \in X$ such that $S(a,y) = a = T(a,y)$ for all $y \in X$.

Proof. Let $y = y'$ and $x \neq x'$. Then by $(**)$ for every $y \in X$ we have

$$d(S(x,y),S(x',y)) \leqslant \alpha \max\{d(T(x,y),T(x',y)),d(T(x,y),S(x,y)), d(T(x',y),S(x',y)),d(T(x,y),S(x',y)),d(T(x',y),S(x,y))\}.$$

Therefore by Theorem 2.2, for each $y \in X$ there exists only one $x(y) \in X$ such that

$$S(x(y),y) = x(y) = T(x(y),y).$$

Now we claim that $x(y) = x(y')$ for $y \neq y'$. Suppose not, then by $(**)$ we would have

$$d(x(y),x(y')) \leqslant \alpha\, d(x(y),x(y')),$$

as $\alpha < 1$, we find that $x(\cdot)$ is some constant $a \in X$. So $S(a,y) = a = T(a,y)$ for all $y \in X$. Unicity of a is obvious.

R e m a r k . Theorem 2.3 extends a result of Iséki [4].

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DEPARTMENT OF PURE MATHEMATICS, LA TROBE UNIVERSITY,
BUNDOORA, VICTORIA. 3083, AUSTRALIA
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