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APPLICATIONS OF INEQUALITIES OF GARABEDIAN-SCHIFFER TYPE FOR PAIRS AND BOUNDED FUNCTIONS NOT ASSUMING CERTAIN VALUES

Introduction

Two functions, F and G, are called a pair (F,G) if they are univalent in $\Delta = \{z : |z| < 1\}$,

$$F(z) = a(z + a_2z^2 + \dots), \quad G(z) = b(z + b_2z^2 + \dots),$$

and such that

$$F(z)G(\zeta) \neq 1$$

for any $(z,\zeta) \in \Delta \times \Delta$. The class of all pairs (F,G) as defined above is denoted by A. This class was intensely investigated in the work of J.A.Hummel and M.Schiffer [2], and class close to those belonging to the class A were examined by A.Seiler [3].

In [6] we proved inequalities of Garabedian-Schiffer ([1]) type for pairs of vector functions [4] and presented a throught discussion of these inequalities. The present paper is a continuation of studies presented in [6] and deals with the problem of maximization of certain functionals in the class A of pairs (F,G) not assuming two values. In particular we give corollaries concerning bounded functions, Bieberbach--Eilenberg ones and Grunsky-Shah functions, not assuming certain values.

1. The area inequality

For any fixed $u,v\in\mathbb{C}$ such that $u\neq v$, $uv\neq 0$, let $A_{u,v}$ be a subclass of A of those pairs (F,G) for which $u,v\in\mathbb{C}\setminus[F(\Delta)\cup 1/G(\Delta)]$. Let $(F,G)\in A_{u,v}$, and let

$$(1)\quad\begin{cases}\hat{F}(z)=\left[\dfrac{u-F(z)}{v-F(z)}\right]^{1/2}=\sqrt{\dfrac{u}{v}}+\dfrac{a(u-v)}{2v\sqrt{uv}}\,z+\ldots=\hat{a}+\hat{a}_1z+\hat{a}_2z^2+\ldots,\\[2ex]\hat{G}(z)=\left[\dfrac{1-vG(z)}{1-uG(z)}\right]^{1/2}=1+\dfrac{b(u-v)}{2}\,z+\ldots=\hat{b}+\hat{b}_1z+\hat{b}_2z^2+\ldots\,.\end{cases}$$

Hence

$$(2)\qquad F(z)=\frac{u-v\,\hat{F}^2(z)}{1-\hat{F}^2(z)}\,,\quad G(z)=\frac{1-\hat{G}^2(z)}{v-u\,\hat{G}^2(z)}\,.$$

We define the coefficients $a_{qp}=a_{qp}(u,v)$, $b_{qp}=b_{qp}(u,v)$, $c_{qp}=c_{qp}(u,v)$, $d_{qp}=d_{qp}(u,v)$, $q,p=0,1,2,\ldots,$ generated in the bicylinder $\Delta\times\Delta$ by the functions:

$$\log\frac{(z-\zeta)[\hat{F}(z)+\hat{F}(\zeta)]}{\hat{F}(z)-\hat{F}(\zeta)}=\sum_{q,p=0}^{\infty}a_{qp}z^q\zeta^p,$$

$$\log\frac{(z-\zeta)[\hat{G}(z)+\hat{G}(\zeta)]}{\hat{G}(z)-\hat{G}(\zeta)}=\sum_{q,p=0}^{\infty}b_{qp}z^q\zeta^p,$$

$$\log\frac{1+\hat{F}(z)\,\hat{G}(\zeta)}{1-\hat{F}(z)\,\hat{G}(\zeta)}=\sum_{q,p=0}^{\infty}c_{qp}z^q\zeta^p,$$

$$\log\frac{1+\hat{G}(z)\,\hat{F}(\zeta)}{1-\hat{G}(z)\,\hat{F}(\zeta)}=\sum_{q,p=0}^{\infty}d_{qp}z^q\zeta^p.$$

Hence, in particular, we have

$$(3)\quad a_{00}=\log\frac{4uv}{a(u-v)},\ c_{00}=\log\frac{\sqrt{u}+\sqrt{v}}{\sqrt{u}-\sqrt{v}},\ b_{00}=\log\frac{4}{b(u-v)},\ d_{00}=\log\frac{\sqrt{v}+\sqrt{u}}{\sqrt{v}-\sqrt{u}},$$

$$(4)\quad \begin{aligned} a_{11} &= a_2{}^2 - a_3 - \frac{1}{8}a^2\left(\frac{1}{u}-\frac{1}{v}\right)^2, \quad b_{11}=b_2{}^2-b_3-\frac{1}{8}b^2(u-v)^2, \\ c_{11} &= d_{11} = -\frac{1}{2}ab\frac{u+v}{\sqrt{uv}}. \end{aligned}$$

Assume that

$$(5)\quad H_0(w)=P_0(w)-Q_0(w),\quad H_1(w)=-\sqrt{(u-w)(v-w)}\left[\frac{P_1(w)}{\hat{a}_1\hat{b}}+\frac{Q_1(w)}{\hat{b}_1\hat{a}}\right],$$

where

$$(6)\quad P_0(w)=\log\frac{\hat{w}(w)+\sqrt{u/v}}{\hat{w}(w)-\sqrt{u/v}},\qquad Q_0(w)=\log\frac{1+\hat{w}(w)}{1-\hat{w}(w)},$$

$$\hat{w}(w)=\left[\frac{u-w}{v-w}\right]^{1/2},\quad P_1(w)=\frac{a}{\sqrt{uvw}},\quad Q_1(w)=b$$

and let

$$(7)\quad H_0\circ F(z)=\sum_{q=0}^{\infty}A_qz^q-\log z,\ H_0\circ 1/G(z)=\sum_{q=0}^{\infty}B_qz^q+\log z,$$

$$(|z|=r,\ 0<r<1)$$

$$(8)\qquad H_1\circ F(z)=\sum_{q=-\infty}^{\infty}C_qz^q,\quad H_1\circ 1/G(z)=\sum_{q=-\infty}^{\infty}D_qz^q.$$

A simple consequence of Theorem 1 [6] is

Theorem 1. If $(F,G) \in A_{u,v}$, then

$$(9)\qquad \sum_{q=1} q\left(|A_q|^2 + |B_q|^2\right) \leqslant 2\,\mathrm{Re}\{A_0 - B_0\},$$

and for $|\varepsilon| = |\tau| = 1$,

$$(10)\qquad -\mathrm{Re}\left\{\bar{\varepsilon}^2 C_1 C_{-1} + \bar{\tau}^2 D_1 D_{-1}\right\} \leqslant |C_{-1}|^2 + |D_{-1}|^2,$$

with that equality takes place if and only if

$$(11)\qquad C_1 = -\varepsilon^2 \bar{C}_{-1},\quad D_1 = -\tau^2 \bar{D}_{-1},\quad C_q = D_q = 0,\quad q = 2,3,\ldots .$$

2. Maximization of the functional $|ab|$

We prove the following

Theorem 2. If $(F,G) \in A_{u,v}$, then

$$(12)\qquad |ab| \leqslant 16\,\frac{uv}{\left|\sqrt{u} + \sqrt{v}\right|^4}.$$

Equality in the case when $u/v > 0$ takes place only for the pairs (F,G) (Figure 2) defined by formulae (2), where the functions $\hat{F}$ and $\hat{G}$ (Figure 1) satisfy the equations

$$(13)\qquad \frac{\sqrt[4]{u/v}}{\hat{F}(z)} - \frac{\hat{F}(z)}{\sqrt[4]{u/v}} = \left[\sqrt[4]{\frac{v}{u}} - \sqrt[4]{\frac{u}{v}}\right]\frac{1+\varepsilon z}{1-\varepsilon z},\quad \frac{\sqrt[4]{v/u}}{\hat{G}(z)} - \frac{\hat{G}(z)}{\sqrt{v/u}} = \left[\sqrt[4]{\frac{v}{u}} - \sqrt[4]{\frac{u}{v}}\right]\frac{1+\tau z}{1-\tau z}$$

and $|\varepsilon| = |\tau| = 1$.

P r o o f . From inequality (9) it follows that

(14) $$\operatorname{Re}\{A_0 - B_0\} \geqslant 0;$$

equality holds if and only if

(15) $$A_q = B_q = 0, \quad q = 1,2,\ldots, \quad \operatorname{Re}\{A_0\} = \operatorname{Re}\{B_0\}.$$

Since, in virtue of (5) - (7),

$$A_0 = a_{00} - c_{00}, \quad B_0 = d_{00} - b_{00},$$

therefore, in view of (3), inequality (14) will take the form (12).

Suppose that in (12) equality holds, and let $u/v > 0$. Then, from (15) it follows that, for the extremal pair (F,G), the equations

$$H_0 \circ F(z) = A_0 - \log z, \quad H_0 \circ 1/G(z) = B_0 + \log z, \quad \operatorname{Re}\{A_0 - B_0\} = 0$$

are satisfied. Consequently, taking $A_0 = -\log \varepsilon$, $B_0 = \log \tau$, we have

(16) $$-\operatorname{Re}\{H_0 \circ F(z)\} = \log|\varepsilon z|, \quad \operatorname{Re}\{H_0 \circ 1/G(z)\} = \log|\tau z|, \quad |\varepsilon\tau| = 1,$$

and hence it appears that the equations (13) are satisfied and $|\varepsilon\tau| = 1$.

Note that $0, \pm\sqrt[4]{u/v}\, i \in D(\hat{F},\hat{G}) = \mathbb{C} \setminus [\hat{F}(\Delta) \cup -\hat{F}(\Delta) \cup 1/\hat{G}(\Delta) \cup -1/\hat{G}(\Delta)]$. Really, since $H_0 \not\equiv 0$, equality in (9) takes place if and only if the measure of the set $D(F,G) = \mathbb{C} \setminus [F(\Delta) \cup 1/G(\Delta)]$ is equal to zero. Consequently, the images of Δ through the mappings F and $1/G$ must fill up, together with the boundaries, the plane $\mathbb{C}$, and since

$$\hat{F}(z) = \left[\frac{u - F(z)}{v - F(z)}\right]^{1/2}, \quad 1/\hat{G}(z) = \left[\frac{u - 1/G(z)}{v - 1/G(z)}\right]^{1/2},$$

also the images of Δ through the mappings $\hat{F}$, $-\hat{F}$, $1/\hat{G}$, $-1/\hat{G}$ must fill up, together with the boundaries, the plane $\mathbf{C}$.

Since the functions f_1 and g_1,

$$f_1(z) = \frac{1 + \varepsilon z}{1 - \varepsilon z}, \quad g_1(z) = \frac{1 + \tau z}{1 - \tau z},$$

are univalent in Δ, the functions $\hat{f}_1$ and $\hat{g}_1$,

$$\hat{f}_1(w) = \frac{\sqrt[4]{u/v}}{w} - \frac{w}{\sqrt[4]{u/v}}, \quad \hat{g}_1(w) = \frac{\sqrt[4]{v/u}}{w} - \frac{w}{\sqrt[4]{v/u}},$$

are univalent in, respectively, $\hat{F}(\Delta)$ and $\hat{G}(\Delta)$. In consequence, there do not exist in $\hat{F}(\Delta)$ distinct points w_1, w_2 such that $w_1 w_2 = -\sqrt{u/v}$, and there do not exist in $\hat{G}(\Delta)$ distinct points w_1, w_2 such that $w_1 w_2 = -\sqrt{v/u}$.

Suppose that $\sqrt[4]{u/v}\, i \in \hat{F}(\Delta)$. Then, from the openness of the set $\hat{F}(\Delta)$, it follows that, for n sufficiently large,

$$w_{1n} = \sqrt[4]{u/v}\, \exp[i(\pi/2+1/n)], \quad w_{2n} = -\bar{w}_{1n} \in \hat{F}(\Delta).$$

However, this is impossible since $w_{1n} w_{2n} = -\sqrt{u/v}$. Now, suppose that $-\sqrt[4]{u/v}\, i \in \hat{F}(\Delta)$, and put $w'_{1n} = \bar{w}_{1n}$, $w'_{2n} = \bar{w}'_{2n}$. Then w'_{1n}, $w'_{2n} \in \hat{F}(\Delta)$ and $w'_{1n} w'_{2n} = -\sqrt{u/v}$, which is also impossible. In an analogous way, we check that $\pm\sqrt[4]{u/v}\, i \notin -\hat{F}(\Delta) \cup 1/\hat{G}(\Delta) \cup -1/\hat{G}(\Delta)$.

If $\sqrt[4]{u/v}\, i \in \partial\, [\hat{F}(\Delta) \cup -\hat{F}(\Delta)]$, then there exists some $z' \in \partial\Delta$ such that

$$\hat{f}_1(\sqrt[4]{u/v}\ i) = \left(\sqrt[4]{v/u} - \sqrt[4]{u/v}\right)\frac{1+\varepsilon z'}{1-\varepsilon z'},$$

whence, by equating the real parts, we get that $|\varepsilon| = 1$ and, in consequence, also $|\tau| = 1$, and thus equations (16) take the form (13). We come to a similar conclusion if $-\sqrt[4]{u/v}\ i \in [\hat{F}(\Delta) \cup -\hat{F}(\Delta)]$ or $\sqrt[4]{u/v}\ i \in \partial[1/\hat{G}(\Delta) \cup -1/\hat{G}(\Delta)]$ or $-\sqrt[4]{u/v}\ i \in \partial[1/\hat{G}(\Delta) \cup -1/\hat{G}(\Delta)]$. So, $|\varepsilon| = |\tau| = 1$ and, in consequence, $\pm\sqrt[4]{u/v} \in D(\hat{F},\hat{G})$.

The functions f_1 and g_1 transform Δ conformally onto the set $\{\eta : \mathrm{Re}\,\eta > 0\}$, whereas $\mathrm{Re}\ \hat{f}_1(w) = 0$ if and only if $\mathrm{Re}\ w = 0$ or $|w| = \sqrt[4]{u/v}$, and $\mathrm{Re}\ \hat{g}_1(w) = 0$ if and only if $\mathrm{Re}\ w = 0$ or $|w| = \sqrt[4]{v/u}$.

It remains to notice that, if $0 < u/v < 1$, the functions $\hat{f}_1$, respectively $\hat{g}_1$, transform the sets $\{w: |w| < \sqrt[4]{u/v},\ \mathrm{Re}\ w > 0\}$, respectively $\{w: |w| < \sqrt[4]{v/u},\ \mathrm{Re}\ w > 0\}$, conformally onto the set $\{\eta : \mathrm{Re}\,\eta > 0\}$, and if $1 < u/v$, then they transform the sets $\{w: |w| > \sqrt[4]{u/v},\ \mathrm{Re}\ w > 0\}$, respectively $\{w: |w| > \sqrt[4]{v/u},\ \mathrm{Re}\ w > 0\}$, conformally onto the set $\{\eta : \mathrm{Re}\,\eta < 0\}$, whereas the functions f_1, $|\varepsilon| = 1$, respectively g_1, $|\tau| = 1$, transform Δ conformally onto the set $\{\eta : \mathrm{Re}\ \eta > 0\}$. The extremal domains are illustrated by Figures 1 and 2.

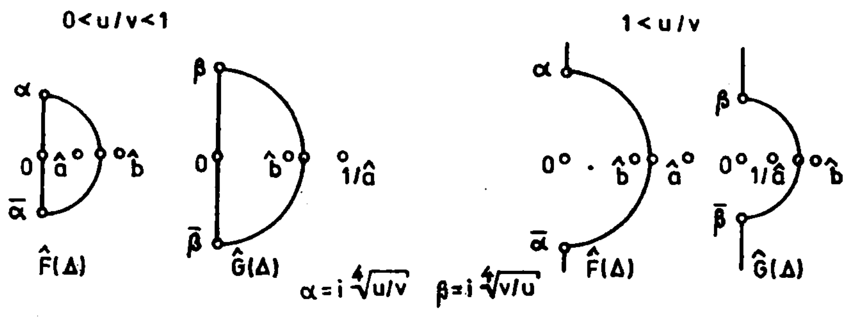


Figure 1

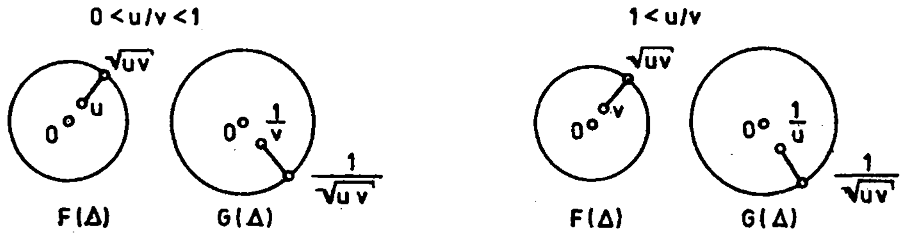


Figure 2

3. Maximization of derivatives for pairs not assuming two values

If $(F,G) \in A_{u,v}$, and $z_1, z_2 \in \Delta$, let

$$p(z) = \frac{z+z_1}{1+\bar{z}_1 z},\ q(z) = \frac{z+z_2}{1+\bar{z}_2 z},\ \tilde{p}(z) = \frac{z-z_1}{1-\bar{z}_1 z},\ \tilde{q}(z) = \frac{z-z_2}{1-\bar{z}_2 z},$$

while

$$(17) \qquad \tilde{u} = \frac{u - F(z_1)}{1 - u\, G(z_2)}, \qquad \tilde{v} = \frac{v - F(z_1)}{1 - v\, G(z_2)}.$$

There holds

Theorem 3. If $(F,G) \in A_{u,v}$, then $(\tilde{F},\tilde{G}) \in A_{\tilde{u},\tilde{v}}$, where

$$\tilde{F}(z) = \frac{F\circ p(z) - F(z_1)}{1 - F\circ p(z)G(z_2)}, \qquad \tilde{G}(z) = \frac{G\circ q(z) - G(z_2)}{1 - G\circ q(z)F(z_1)}$$

and

$$(18) \quad F(z) = \frac{\tilde{F}\circ\tilde{p}(z) - \tilde{F}(-z_1)}{1 - \tilde{F}\circ\tilde{p}(z)\tilde{G}(-z_2)}, \qquad G(z) = \frac{\tilde{G}\circ\tilde{q}(z) - \tilde{G}(-z_2)}{1 - \tilde{G}\circ\tilde{q}(z)\tilde{F}(-z_1)}.$$

As an immediate corollary of Theorems 2, 3 we get

T h e o r e m 4. If $(F,G) \in A_{u,v}$, $z_1, z_2 \in \Delta$, then

$$|F'(z_1)G'(z_2)| \leqslant$$

$$\leqslant \frac{16\,|1-F(z_1)G(z_2)|^2\,\left|[u-F(z_1)][1-vG(z_2)][v-F(z_1)][1-uG(z_2)]\right|}{(1-|z_1|^2)(1-|z_2|^2)\left|\sqrt{[u-F(z_1)][1-vG(z_2)]}+\sqrt{[v-F(z_1)][1-uG(z_2)]}\right|^4}\,.$$

In the case when $\tilde{u}/\tilde{v} > 0$, where $\tilde{u}$ and $\tilde{v}$ are defined by equalities (17), equality holds only for the pairs $(F,G) \in$ $\in A_{u,v}$ defined by equations (18), where

$$\tilde{F}(z) = \frac{\tilde{u} - \tilde{v}\hat{F}^2(z)}{1 - \hat{F}^2(z)}, \qquad \tilde{G}(z) = \frac{1 - \hat{G}^2(z)}{\tilde{v} - \tilde{u}\hat{G}^2(z)},$$

with that

$$\frac{\sqrt[4]{\tilde{u}/\tilde{v}}}{\hat{F}(z)} - \frac{F(z)}{\sqrt[4]{\tilde{u}/\tilde{v}}} = \left[\sqrt[4]{\frac{\tilde{v}}{\tilde{u}}} - \sqrt[4]{\frac{\tilde{u}}{\tilde{v}}}\right]\frac{1+\varepsilon z}{1-\varepsilon z},$$

$$\frac{\sqrt[4]{\tilde{v}/\tilde{u}}}{\hat{G}(z)} - \frac{\hat{G}(z)}{\sqrt[4]{\tilde{v}/\tilde{u}}} = \left[\sqrt[4]{\frac{\tilde{v}}{\tilde{u}}} - \sqrt[4]{\frac{\tilde{u}}{\tilde{v}}}\right]\frac{1+\tau z}{1-\tau z}, \quad |\varepsilon| = |\tau| = 1.$$

4. Maximization of derivatives for bounded functions

Let S_1 stand for the class of all functions F, univalent in Δ, of the form

$$F(z) = b(z + a_2 z^2 + \ldots), \quad (0 < |b| \leqslant 1),$$

such that $|F(z)| < 1$ for $z \in \Delta$. For any fixed $u \in \Delta \setminus \{0\}$, let S_{1u} denote a subclass of S_1 of those functions F for which $u \notin F(\Delta)$.

From Theorem 4, if $z_0 = z_1 = \bar{z}_2$, $v = 1/\bar{u}$, and $G = \bar{F}$, where $\bar{F}(z) = \overline{F(\bar{z})}$, there follows

T h e o r e m 5. If $F \in S_{1u}$, $z_0 \in \Delta$, then

$$|F'(z_0)| \leqslant 4\,\frac{(1-|F(z_0)|^2)\,\big|[u-F(z_0)]\,[1-\bar{u}\,F(z_0)]\big|}{(1-|z_0|^2)\,\big[|u-F(z_0)|+|1-\bar{u}\,F(z_0)|\big]^2}.$$

Equality takes place only for the function $F \in S_{1u}$ satisfying the equation

$$F(z) = \frac{\tilde{F}\circ\tilde{p}(z) - \tilde{F}(-z_0)}{1-\tilde{F}\circ\tilde{p}(z)\overline{\tilde{F}(-z_0)}}, \quad \tilde{p}(z) = \frac{z-z_0}{1-\bar{z}_0 z},$$

where

$$\tilde{F}(z) = \frac{|\tilde{u}|^2 - \hat{F}^2(z)}{\bar{\tilde{u}}\left[1-\hat{F}^2(z)\right]}, \quad \left(\tilde{u} = \frac{u-F(z_0)}{1-u\,\overline{F(z_0)}}\right)$$

while the function $\hat{F}$ satisfies the equation

$$\frac{\sqrt{|\tilde{u}|}}{\hat{F}(z)} - \frac{\hat{F}(z)}{\sqrt{|\tilde{u}|}} = \left(1/\sqrt{|\tilde{u}|} - \sqrt{|\tilde{u}|}\right)\frac{1+\varepsilon z}{1-\varepsilon z}, \quad |\varepsilon| = 1.$$

The domains $\hat{F}(\Delta)$ and $\tilde{F}(\Delta)$ are illustrated by Figure 3.

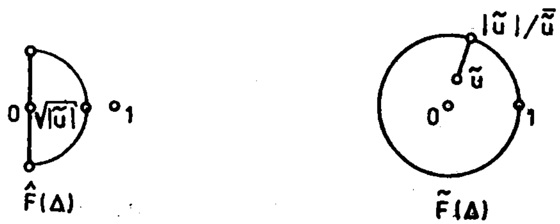


Figure 3

5. Maximization of the third order functional for pairs and characterization of all extremal pairs

We shall now prove the following

Theorem 6. If $(F,G) \in A_{u,v}$, then

$$\left|C\,\frac{buv}{a}\right| + \left|D\,\frac{a}{buv}\right| \leqslant \left|\frac{buv}{a}\right| + \left|\frac{a}{buv}\right|, \tag{19}$$

where

$$C = a_3 - a_2^2 + \frac{1}{8}a^2\left[\left(\frac{1}{u}-\frac{1}{v}\right)^2 + \frac{4}{\sqrt{uv}}\left(\frac{1}{u}+\frac{1}{v}\right)\right],$$

$$D = b_3 - b_2^2 + \frac{1}{8}b^2\left[(u-v)^2 + 4\sqrt{uv}(u+v)\right].$$

In the case when $u/v > 0$, equality takes place only for the pairs $(F,G) \in A_{u,v}$, where the functions F and G (Figure 5) are defined by the formulae (2), while the functions $\hat{F}$ and $\hat{G}$ (Figure 4),

$$\hat{F}(z) = \sqrt{u/v} + \hat{a}_1 z + \hat{a}_2 z^2 + \ldots, \quad \hat{G}(z) = 1 + \hat{b}_1 z + \hat{b}_2 z^2 + \ldots,$$

satisfy the equations:

$$\frac{\sqrt[4]{v/u}-\sqrt[4]{u/v}}{\dfrac{\hat{F}(z)}{\sqrt[4]{u/v}}-\dfrac{\sqrt[4]{u/v}}{\hat{F}(z)}} - \frac{\dfrac{\hat{F}(z)}{\sqrt[4]{u/v}}-\dfrac{\sqrt[4]{u/v}}{\hat{F}(z)}}{\sqrt[4]{v/u}-\sqrt[4]{u/v}} = 2\sqrt{\frac{v}{u}}\,|\hat{a}_1|\,\frac{\sqrt{v}+\sqrt{u}}{\sqrt{v}-\sqrt{u}}\,\frac{z\varepsilon}{(z\tau-\zeta_1)(z\tau+\bar{\zeta}_1)},$$

$$\frac{\sqrt[4]{v/u}-\sqrt[4]{u/v}}{\dfrac{\hat{G}(z)}{\sqrt[4]{v/u}}-\dfrac{\sqrt[4]{v/u}}{\hat{G}(z)}} - \frac{\dfrac{\hat{G}(z)}{\sqrt[4]{v/u}}-\dfrac{\sqrt[4]{v/u}}{\hat{G}(z)}}{\sqrt[4]{v/u}-\sqrt[4]{u/v}} = 2|\hat{b}_1|\,\frac{\sqrt{v}+\sqrt{u}}{\sqrt{v}-\sqrt{u}}\,\frac{z\tau}{(z\tau-\zeta_2)(z\tau+\bar{\zeta}_2)}, \tag{20}$$

with that $\hat{a}_1 = \varepsilon|\hat{a}_1|$, $\hat{b}_1 = \tau|\hat{b}_1|$, $|\zeta_1| = |\zeta_2| = 1$,

$$-2\mathrm{Im}\{\zeta_1\}i = \bar{\varepsilon}\,\frac{\hat{a}_2}{\hat{a}_1} - \sqrt{\frac{v}{u}}\,\frac{u-v+4\sqrt{uv}}{2(u-v)}\,|\hat{a}_1|,\quad \left|\mathrm{Im}\{\zeta_1\}\right| \leqslant 1 - \frac{1}{2}\sqrt{\frac{v}{u}}\left|\hat{a}_1\,\frac{\sqrt{v}+\sqrt{u}}{\sqrt{v}-\sqrt{u}}\right|,$$

$$-2\mathrm{Im}\{\zeta_2\}i = \bar{\tau}\,\frac{\hat{b}_2}{\hat{b}_1} - \frac{u-v+4\sqrt{uv}}{2(u-v)}\,|\hat{b}_1|,\qquad \left|\mathrm{Im}\{\zeta_2\}\right| \leqslant 1 - \frac{1}{2}\left|\hat{b}_1\,\frac{\sqrt{v}+\sqrt{u}}{\sqrt{v}-\sqrt{u}}\right|.$$

And so, for every $\hat{a}_1$, $\hat{b}_1$, the functions $\hat{F}$, $\hat{G}$ belong to the one-parameter families, where the parameters are, respectively, $\hat{a}_2$, $\hat{b}_2$ or ζ_1, ζ_2.

P r o o f . Since, in view of (4)-(6) and (8),

$$C_{-1} = -1/(\hat{a}_1\hat{b}),\quad D_{-1} = 1/(\hat{b}_1\hat{a}),\quad C_1 = -C_{-1}C,\quad D_1 = -D_{-1}D,$$

ineauality (10) will take the form

$$\mathrm{Re}\left\{\bar{\varepsilon}^2\, C_{-1}{}^2C + \bar{\tau}^2\, D_{-1}{}^2D\right\} \leqslant \left|C_{-1}\right|^2 + \left|D_{-1}\right|^2,\ |\varepsilon| = |\tau| = 1, \tag{21}$$

whence, in view of the arbitrariness of ε and τ, taking account of (1), we obtain at once inequality (19).

Note that, if, for the pair (F,G), equality in (21) does hold, then this equality holds also for the pairs (F_σ, G_δ), where $F_\sigma(z) = F(\sigma z)$, $G_\delta(z) = G(\delta z)$, $|\sigma| = |\delta| = 1$. So we may assume, after some eventual rotation, that (F,G) is such a pair for which $\hat{a}_1 = \varepsilon|\hat{a}_1|$, $\hat{b}_1 = \tau|\hat{b}_1|$.

Now, note that the function H_1 can be represented in the form

$$H_1(w) = -\frac{1}{\hat{w}(w)-\hat{a}} - \frac{1}{\hat{w}(w)+\hat{a}} + \frac{1}{\hat{a}}\left[\frac{\hat{w}(w)}{1-\hat{w}(w)} + \frac{\hat{w}(w)}{1+\hat{w}(w)}\right] =$$

$$= \frac{2(\sqrt{\hat{a}} + 1/\sqrt{\hat{a}})\left[\frac{\hat{w}(w)}{\sqrt{\hat{a}}} - \frac{\sqrt{\hat{a}}}{\hat{w}(w)}\right]}{1 + \hat{a}^2 - \hat{a}\left[\frac{\hat{w}^2(w)}{\hat{a}} + \frac{\hat{a}}{\hat{w}^2(w)}\right]},$$

and $C_0 = -C_{-1}\hat{C}$, $D_0 = -D_{-1}\hat{D}$,

$$\hat{C} = \frac{\hat{a}_2}{\hat{a}_1} - \frac{u-v+4\sqrt{uv}}{2(u-v)}\sqrt{\frac{v}{u}}\,\hat{a}_1, \qquad \hat{D} = \frac{\hat{b}_2}{\hat{b}_1} - \frac{u-v+4\sqrt{uv}}{2(u-v)}\,\hat{b}_1.$$

So, taking account by (11),

$$H_1 \circ F(z) = \frac{1}{\hat{a}_1\hat{b}}\left[-\frac{1}{z} + \hat{C} + z\varepsilon^2\right], \quad H_1 \circ 1/G(z) = -\frac{1}{\hat{b}_1\hat{a}}\left[-\frac{1}{z} + \hat{D} + z\tau^2\right]$$

and, in consequence,

$$(22)\quad \frac{1 + \frac{u}{v} - \sqrt{\frac{u}{v}}\left[\frac{\hat{F}^2(z)}{\sqrt{u/v}} + \frac{\sqrt{u/v}}{\hat{F}^2(z)}\right]}{\frac{\hat{F}(z)}{\sqrt[4]{u/v}} - \frac{\sqrt[4]{u/v}}{F(z)}} = 2|\hat{a}_1|\left(\sqrt[4]{v/u} + \sqrt[4]{u/v}\right)\frac{z\varepsilon}{z^2\varepsilon^2 + \hat{C}z - 1},$$

$$(23)\quad \frac{1 + \frac{u}{v} - \sqrt{\frac{u}{v}}\left[\frac{\hat{G}^2(z)}{\sqrt{v/u}} + \frac{\sqrt{v/u}}{\hat{G}^2(z)}\right]}{\frac{\hat{G}(z)}{\sqrt[4]{v/u}} - \frac{\sqrt[4]{v/u}}{\hat{G}(z)}} = 2|\hat{b}_1|\sqrt{\frac{u}{v}}\left(\sqrt[4]{v/u} + \sqrt[4]{u/v}\right)\frac{z\tau}{z^2\tau^2 + \hat{D}z - 1}.$$

Since $H_1 \not\equiv 0$, equality in (21) takes place if and only if the area $D(F,G)$ is equal to zero. Hence it appears that the set $D(F,G)$ possesses no interior points and also the set

$D(\hat{F},\hat{G})$ possesses no interior points; the functions $\hat{F}$ and $-\hat{F}$ are defined by equation (22) whereas the functions $\hat{G}$ and $-\hat{G}$ are defined by (23).

From the univalence of the functions f_2 and g_2 in Δ,

$$f_2(z) = \frac{z\varepsilon}{z^2\varepsilon^2 + \bar{\varepsilon}\,\hat{C}\,z\varepsilon - 1}, \quad g_2(z) = \frac{z\tau}{z^2\tau^2 + \bar{\tau}\,D\,z\tau - 1},$$

the univalence of the functions $\hat{f}_2$ and $\hat{g}_2$ in, respectively, $\hat{F}(\Delta)$ and $\hat{G}(\Delta)$,

$$\hat{f}_2(w) = \frac{1 + \frac{u}{v} - \sqrt{\frac{u}{v}}\left[\frac{w^2}{\sqrt{u/v}} + \frac{\sqrt{u/v}}{w^2}\right]}{\frac{w}{\sqrt[4]{u/v}} - \frac{\sqrt[4]{u/v}}{w}},$$

$$\hat{g}_2(w) = \frac{1 + \frac{u}{v} - \sqrt{\frac{u}{v}}\left[\frac{w^2}{\sqrt{v/u}} + \frac{\sqrt{v/u}}{w^2}\right]}{\frac{w}{\sqrt[4]{v/u}} - \frac{\sqrt[4]{v/u}}{w}},$$

follows in virtue of (22) and (23). Consequently, there do not exist in $\hat{F}(\Delta)$ distinct points w_1, w_2 such that $w_1w_2 = -\sqrt{u/v}$ or

$$\left(\frac{w_1}{\sqrt[4]{u/v}} - \frac{\sqrt[4]{u/v}}{w_1}\right)\left(\frac{w_2}{\sqrt[4]{u/v}} - \frac{\sqrt[4]{u/v}}{w_2}\right) = -\left(\sqrt[4]{v/u} - \sqrt[4]{u/v}\right)^2,$$

and there do not exist in $\hat{G}(\Delta)$ distinct points w_1,w_2 such that $w_1w_2 = -\sqrt{v/u}$ or

$$\left(\frac{w_1}{\sqrt[4]{v/u}} - \frac{\sqrt[4]{v/u}}{w_1}\right)\left(\frac{w_2}{\sqrt[4]{v/u}} - \frac{\sqrt[4]{v/u}}{w_2}\right) = -\left(\sqrt[4]{v/u} - \sqrt[4]{u/v}\right)^2.$$

Note that $\pm\sqrt[4]{u/v}\, i \in D(\hat{F},\hat{G})$. For, if $\sqrt[4]{u/v}\, i \in \hat{F}(\Delta)$, then, for n sufficiently large, we would have that $w_{1n}, w_{2n} \in \hat{F}(\Delta)$, which is impossible. And, if $-\sqrt[4]{u/v}\, i \in \hat{F}(\Delta)$, then $w'_{1n}, w'_{2n} \in \hat{F}(\Delta)$, which is also impossible. Analogously, we check that $\pm\sqrt[4]{u/v}\, i \notin -\hat{F}(\Delta) \cup 1/\hat{G}(\Delta) \cup -1/\hat{G}(\Delta)$.

Of course, $0 \in D(\hat{F},\hat{G})$.

So, suppose that $\sqrt[4]{u/v}\, i \in \partial\left[\hat{F}(\Delta) \cup -\hat{F}(\Delta)\right]$. Then there exists some $z' \in \partial\Delta$ such that

$$\hat{f}_2\left(\sqrt[4]{u/v}\; i\right) = 2|\hat{a}_1|\left(\sqrt[4]{v/u} + \sqrt[4]{u/v}\right) f_2(z'),$$

whence it follows that $\operatorname{Re}\{\bar{\varepsilon}\hat{C}\} = 0$ and thus, that

$$f_2(z) = \frac{z\varepsilon}{(z\varepsilon - \zeta_1)(z\varepsilon + \zeta_1)}, \tag{24}$$

where $|\zeta_1| = 1$, $\bar{\varepsilon}\hat{C} = -(\zeta_1 - \bar{\zeta}_1) = -2\operatorname{Im}\zeta_1\, i$. In consequence, from (22) it follows that $\pm\sqrt[4]{u/v} \in \partial\left[\hat{F}(\Delta) \cup -\hat{F}(\Delta)\right] \subset D(\hat{F},\hat{G})$, whence, in turn, by (23), it follows that

$$g_2(z) = \frac{z\tau}{(z\tau - \zeta_2)(z\tau + \bar{\zeta}_2)}, \tag{25}$$

where $|\zeta_2| = 1$, $\bar{\tau}\hat{D} = -(\zeta_2 - \bar{\zeta}_2) = -2\operatorname{Im}\{\zeta_2\} i$. Similarly, we verify that the functions f_2 and g_2 are, respectively, of the form (24) and (25) also in the remaining cases, i.e., if $-\sqrt[4]{u/v}\, i \in \partial\left[\hat{F}(\Delta) \cup -\hat{F}(\Delta)\right]$ or $\sqrt[4]{u/v}\, i \in \partial\left[1/\hat{G}(\Delta) \cup -1/\hat{G}(\Delta)\right]$ or $-\sqrt[4]{u/v}\, i \in \partial\left[1/\hat{G}(\Delta) \cup -1/\hat{G}(\Delta)\right]$.

Consequently, we have demonstrated that 0, $\pm\sqrt[4]{u/v}\, i$, $\pm\sqrt[4]{u/v} \in D(\hat{F},\hat{G})$, and, since

$$1/H_1(w) = \frac{1}{2}\sqrt{\frac{u}{v}}\,\frac{\sqrt{v}-\sqrt{u}}{\sqrt{v}+\sqrt{u}}\left[\frac{\sqrt[4]{v/u}-\sqrt[4]{u/v}}{\dfrac{\hat{w}(w)}{\sqrt[4]{u/v}}-\dfrac{\sqrt[4]{u/v}}{\hat{w}(w)}}-\frac{\dfrac{\hat{w}(w)}{\sqrt[4]{u/v}}-\dfrac{\sqrt[4]{u/v}}{\hat{w}(w)}}{\sqrt[4]{v/u}-\sqrt[4]{u/v}}\right],$$

that equations (22) and (23) take the form (20).

Functions (24) and (25) transform Δ conformally onto, respectively, the sets

$$\mathbf{C}\setminus\left[\left\{\eta: -\infty<\operatorname{Im}\eta\leqslant-\frac{1}{|\zeta_k-1|^2}, \operatorname{Re}\eta=0\right\}\cup\left\{\eta: \frac{1}{|\zeta_k+1|^2}\leqslant\operatorname{Im}\eta<\infty, \operatorname{Re}\eta=0\right\}\right],$$

$$k=1,2,$$

whereas $\operatorname{Re}\hat{f}_2(w)=0$ if and only if $\operatorname{Re} w=0$ or $|w|=\sqrt[4]{u/v}$ or

$$\left|\frac{w}{\sqrt[4]{u/v}}-\frac{\sqrt[4]{u/v}}{w}\right|^2=\left(\sqrt[4]{v/u}-\sqrt[4]{u/v}\right)^2, \tag{26}$$

while $\operatorname{Re}\hat{g}_2(w)=0$ if and only if $\operatorname{Re} w=0$ or $|w|=\sqrt[4]{v/u}$ or

$$\left|\frac{w}{\sqrt[4]{v/u}}-\frac{\sqrt[4]{v/u}}{w}\right|^2=\left(\sqrt[4]{v/u}-\sqrt[4]{u/v}\right)^2. \tag{27}$$

Let us still notice that $\hat{f}_2$ $(\hat{g}_2)$ transforms the set $\{w: |w|<\sqrt[4]{u/v}, \operatorname{Re} w>0\}$ (the set $\{w: |w|<\sqrt[4]{v/u}, \operatorname{Re} w>0\}$), when

$0 < u/v < 1$ (when $0 < v/u < 1$), and the set $\{w: |w| > \sqrt[4]{u/v},$ $\operatorname{Re} w > 0\}$ (the set $\{w: |w| > \sqrt[4]{v/u}, \operatorname{Re} w > 0\}$), when $1 < u/v$ (when $1 < v/u$), conformally onto the set

$$\mathbf{C} \setminus \{\eta : |\operatorname{Im}\eta| \geqslant 2\sqrt[4]{u/v}\,\left|1 - \sqrt[4]{u/v}\right|, \operatorname{Re}\eta = 0\}.$$

Consequently,

$$\text{(28)} \qquad \max\left\{\frac{1}{|\zeta_k - i|^2}, \frac{1}{|\zeta_k + i|^2}\right\} \leqslant \frac{\sqrt{u/v}}{|\hat{a}_1|}\left|\frac{\sqrt{v} - \sqrt{u}}{\sqrt{v} + \sqrt{u}}\right|, \quad k=1$$

$$\leqslant \frac{1}{|\hat{b}_1|}\left|\frac{\sqrt{v} - \sqrt{u}}{\sqrt{v} + \sqrt{u}}\right|, \quad k=2.$$

If $0 \leqslant \operatorname{Im}\{\zeta_1\} \leqslant 1$, then $2(1 - \operatorname{Im}\{\zeta_1\}) = |\zeta_1 - i|^2 \leqslant |\zeta_1 + i|^2$ and, in virtue of (28),

$$1 - \frac{1}{2}\frac{|\hat{a}_1|}{\sqrt{u/v}}\left|\frac{\sqrt{v} + \sqrt{u}}{\sqrt{v} - \sqrt{u}}\right| \geqslant \operatorname{Im}\{\zeta_1\} \geqslant 0.$$

Whereas, if $-1 \leqslant \operatorname{Im}\{\zeta_1\} \leqslant 0$, then, proceeding analogously, we shall get that

$$-\left(1 - \frac{1}{2}\sqrt{v/u}\,\left|\hat{a}_1\,\frac{\sqrt{v} + \sqrt{u}}{\sqrt{v} - \sqrt{u}}\right|\right) \leqslant \operatorname{Im}\{\zeta_1\} \leqslant 0.$$

And so,

$$\left|\operatorname{Im}\{\zeta_1\}\right| \leqslant 1 - \frac{1}{2}\sqrt{v/u}\,\left|\hat{a}_1\,\frac{\sqrt{v} + \sqrt{u}}{\sqrt{v} - \sqrt{u}}\right|.$$

Similarly, we prove that

$$\left|\mathrm{Im}\{\zeta_2\}\right| \leqslant 1 - \frac{1}{2}\left|\hat{b}_1 \frac{\sqrt{v}+\sqrt{u}}{\sqrt{v}-\sqrt{u}}\right| .$$

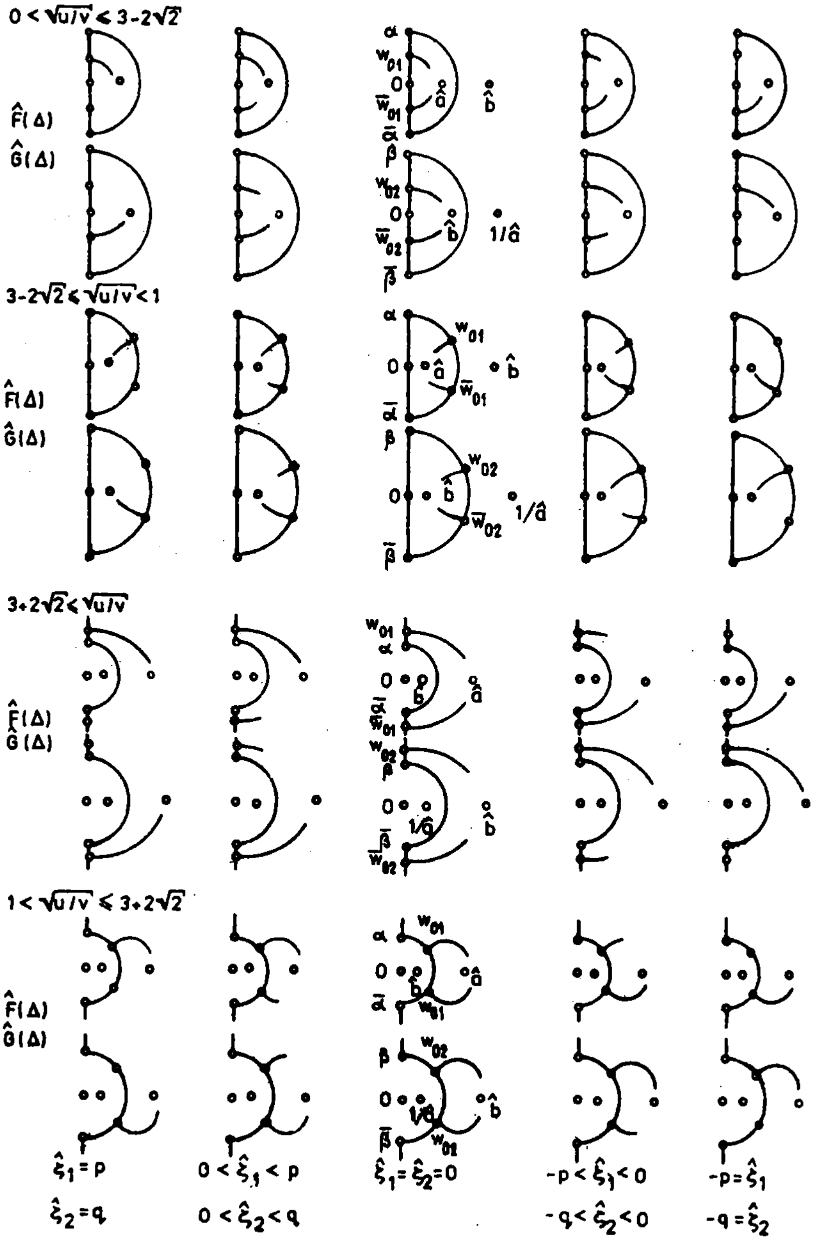
The minimum of the modulus of the function $\hat{f}_2$ $(\hat{g}_2)$ on the boundary of the domain $\{w: |w| < \sqrt[4]{u/v},\ \mathrm{Re}\ w > 0\}$, when $0 < u/v < 1$, $(\{w: |w| < \sqrt[4]{v/u},\ \mathrm{Re}\ w > 0\}$, when $0 < v/u < 1)$, or on the boundary of the domain $\{w: |w| > \sqrt[4]{u/v},\ \mathrm{Re}\ w > 0\}$, when $1 < u/v < \infty$, $(\{w: |w| > \sqrt[4]{v/u},\ \mathrm{Re} w > 0\}$, when $1 < v/u < \infty)$, is attained at the points w_{01} and $\overline{w}_{01}$ $(w_{02}$ and $\overline{w}_{02})$ with that $w_{01} = \sqrt[4]{u/v}\, w_0$ $(w_{02} = \sqrt[4]{v/u}\, w_0)$, while

$$w_0 = \sqrt{\frac{u/v - 4\sqrt{u/v} + 1 - \left|1-\sqrt{u/v}\right|\sqrt{u/v - 6\sqrt{u/v}+1}}{2\sqrt{u/v}}}\, i, \quad \text{when } \sqrt{u/v} \in (0; 3-2\sqrt{2}],$$

$$w_0 = \sqrt{\frac{u/v - 4\sqrt{u/v} + 1 + \left|1-\sqrt{u/v}\right|\sqrt{u/v - 6\sqrt{u/v}+1}}{2\sqrt{u/v}}}\, i, \quad \text{when } \sqrt{u/v} \in [3+2\sqrt{2}; \infty),$$

$$|w_0| = 1,\ \mathrm{Im}\{w_0\} = \frac{1}{2}\,\frac{\left|1-\sqrt{u/v}\right|}{\sqrt[4]{u/v}}, \quad \text{when } \sqrt{u/v} \in [3-2\sqrt{2}; 1) \cup (1; 3+2\sqrt{2}].$$

The slits of the extremal domains $\hat{F}(\Delta)$ $(-\hat{F}(\Delta))$, issuing from the points w_{01} and $\overline{w}_{01}$ $(-w_{01}$ and $-\overline{w}_{01})$, lie on the curves defined by equation (26), whereas those of the extremal domains $\hat{G}(\Delta)$ $(-\hat{G}(\Delta))$, issuing from the points w_{02} and $\overline{w}_{02}$ $(-w_{02}$ and $-\overline{w}_{02})$, lie on the curves defined by (27).



$$\hat{\xi}_1 = \mathrm{Im}\zeta_1, \quad p = 1 - \frac{1}{2}\sqrt{v/u}\left|\hat{a}_1 \frac{\sqrt{v}+\sqrt{u}}{\sqrt{v}-\sqrt{u}}\right|, \quad q = 1 - \frac{1}{2}\left|\hat{b}_1 \frac{\sqrt{v}+\sqrt{u}}{\sqrt{v}-\sqrt{u}}\right|, \quad \hat{\xi}_2 = \mathrm{Im}\zeta_2$$

Figure 4

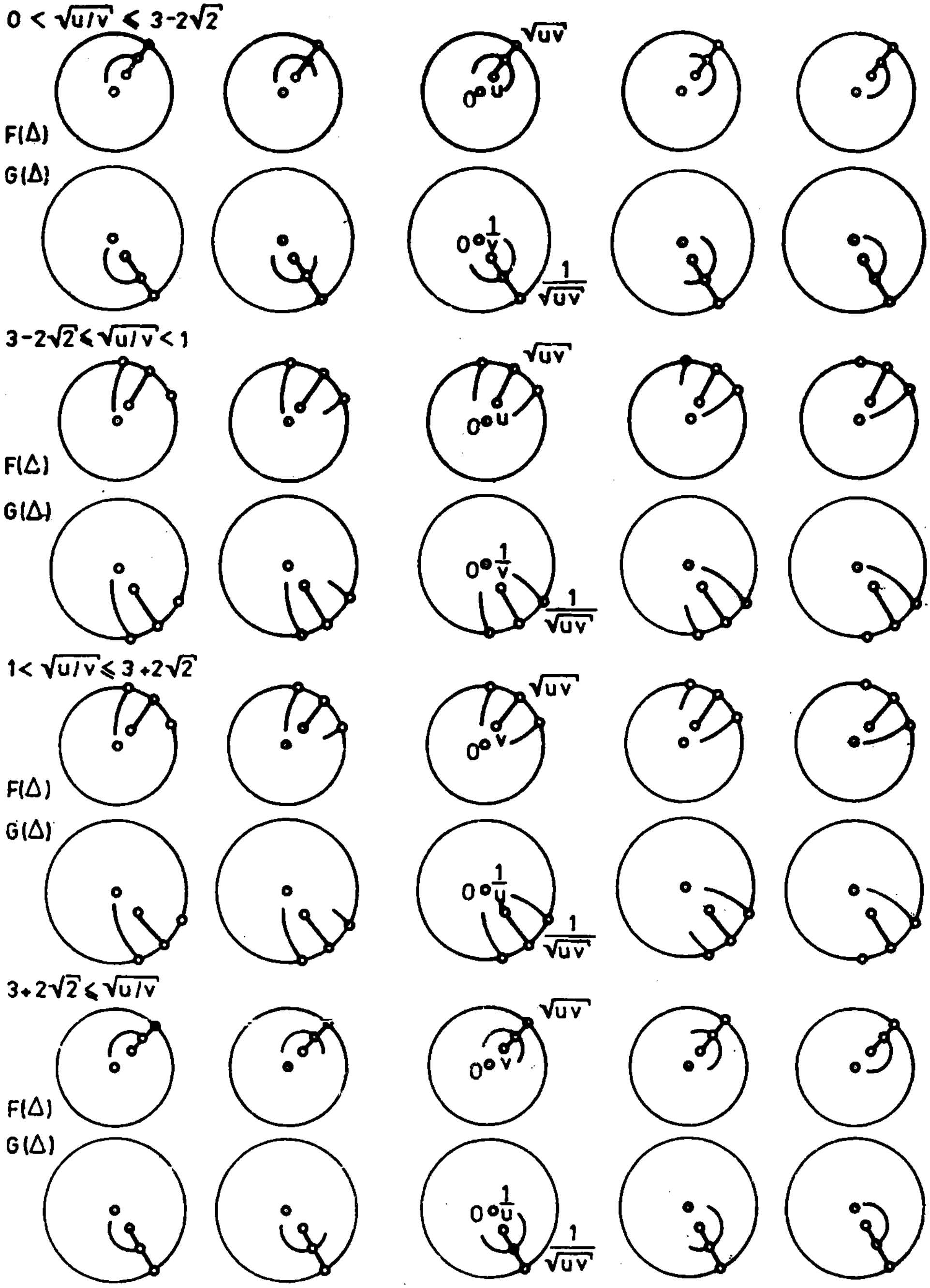

0 < √u/v ≤ 3-2√2
√uv
0 u
F(Δ)
G(Δ)
0 1/v
1/√uv
3-2√2 ≤ √u/v < 1
√uv
0 u
F(Δ)
G(Δ)
0 1/v
1/√uv
1< √u/v ≤ 3+2√2
√uv
0 v
F(Δ)
G(Δ)
0 1/u
1/√uv
3+2√2 ≤ √u/v
√uv
0 v
F(Δ)
G(Δ)
0 1/u
1/√uv

Figure 5

6. Estimation of the third order functional for bounded functions

From Theorem 6 one directly obtain the corresponding result in the class of S_{1u}.

Theorem 7. If $F \in S_{1u}$, then

$$\left|a_3-a_2^2\right| \leqslant 1 - \frac{1}{8}\,|b|^2\left[(|u| - 1/|u|)^2 + 4(|u| + 1/|u|)\right].$$

Equality takes place only for the functions $F \in S_{1u}$ (Figure 6) defined by the formula

$$F(z) = \frac{|u|^2 - \hat{F}^2(z)}{\bar{u}[1 - \hat{F}^2(z)]},$$

where the functions $\hat{F}$, $\hat{F}(z) = |u| + \hat{a}_1 z + \hat{a}_2 z^2 + \dots$, satisfy the equation

$$\frac{1/\sqrt{|u|}-\sqrt{|u|}}{\dfrac{\hat{F}(z)}{\sqrt{|u|}} - \dfrac{\sqrt{|u|}}{\hat{F}(z)}} - \frac{\dfrac{\hat{F}(z)}{\sqrt{|u|}} - \dfrac{\sqrt{|u|}}{\hat{F}(z)}}{1/\sqrt{|u|}-\sqrt{|u|}} = 2\left|\frac{\hat{a}_1}{u}\right| \frac{1+|u|}{1-|u|} \frac{\gamma z}{(\gamma z - \zeta_0)(\gamma z + \bar{\zeta}_0)},$$

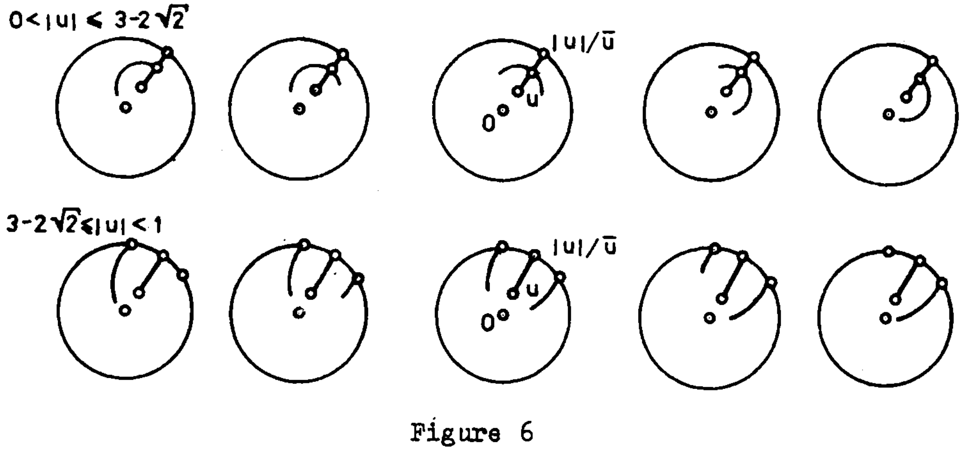


Figure 6

with that $\hat{a}_1 = \gamma|\hat{a}_1|$, $|\zeta_0| = 1$,

$$-2\mathrm{Im}\{\zeta_0\}i = \bar{\gamma}\ \frac{\hat{a}_2}{\hat{a}_1} + \frac{|u|^2+4|u|-1}{2(1-|u|^2)}\left|\frac{\hat{a}_1}{u}\right|,\quad |\mathrm{Im}\ \zeta_0| \leqslant 1-\frac{1}{2}\left|\frac{\hat{a}_1}{u}\right|\ \frac{1+|u|}{1-|u|}.$$

Consequently, for every $\hat{a}_1$, the functions $\hat{F}$ belong to the one-parameter family, where the parameter is $\hat{a}_2$ or ζ_0.

7. Estimations of Schwarz' derivative and two derivative

The Schwarz derivative $\{h;z\}$ of the function h is defined as follows

$$\{h;z\} = 6\left[\frac{\partial^2}{\partial z\partial\zeta}\log\frac{h(z)-h(\zeta)}{z-\zeta}\right]_{\zeta=z} = \frac{h'''(z)}{h'(z)} - \frac{3}{2}\left(\frac{h''(z)}{h'(z)}\right)^2.$$

Theorem 7 can be generalized as follows:

Theorem 8. If $F \in S_{1u}$, $z_0 \in \Delta$, then

$$\left|\{F;z_0\}\right| \leqslant \frac{6}{(1-|z_0|^2)^2} - \frac{3}{2}\,\frac{|F'(z_0)|^2}{[1-|F(z_0)|^2]^2}\times$$

$$\times\left[\frac{(|u-F(z_0)|^2-|1-\bar{u}F(z_0)|^2)^2}{|[u-F(z_0)]\ [1-\bar{u}F(z_0)]|^2} + 4\,\frac{|u-F(z_0)|^2+|1-\bar{u}F(z_0)|^2}{|[u-F(z_0)]\ [1-\bar{u}F(z_0)]|}\right].$$

Equality holds only for the function $F \in S_{1u}$ satisfying the equation

$$F(z) = \frac{\tilde{F}\circ\tilde{p}(z)-\tilde{F}(-z_0)}{1-\tilde{F}\circ\tilde{p}(z)\overline{\tilde{F}(-z_0)}},\quad \tilde{F}(z) = \frac{|\tilde{u}|^2-\hat{F}^2(z)}{\bar{\tilde{u}}[1-\hat{F}^2(z)]},\quad \tilde{p}(z) = \frac{z-z_0}{1-\bar{z}_0 z},$$

where the function $\hat{F}$, $\hat{F}(z) = |\tilde{u}| + \hat{a}_1 z + \hat{a}_2 z^2 + \dots$, satisfies the equation

$$\frac{1/\sqrt{|\tilde{u}|}-\sqrt{|\tilde{u}|}}{\frac{\hat{F}(z)}{\sqrt{|\tilde{u}|}} - \frac{\sqrt{|\tilde{u}|}}{\hat{F}(z)}} - \frac{\frac{\hat{F}(z)}{\sqrt{|\tilde{u}|}} - \frac{\sqrt{|\tilde{u}|}}{\hat{F}(z)}}{1/\sqrt{|\tilde{u}|}-\sqrt{|\tilde{u}|}} = 2\left|\frac{\hat{a}_1}{\tilde{u}}\right| \frac{1+|\tilde{u}|}{1-|\tilde{u}|} \frac{\gamma z}{(\gamma z - \zeta_0)(\gamma z + \bar{\zeta}_0)}$$

with that $\hat{a}_1 = \gamma|\hat{a}_1|$, $|\zeta_0| = 1$,

$$-2\,\mathrm{Im}\{\zeta_0\}i = \bar{\gamma}\,\frac{\hat{a}_2}{\hat{a}_1} + \frac{|\tilde{u}|^2+4|\tilde{u}|-1}{2(1-|\tilde{u}|^2)}\left|\frac{\hat{a}_1}{\tilde{u}}\right|, \quad |\mathrm{Im}\,\zeta_0| \leqslant 1 - \frac{1}{2}\left|\frac{\hat{a}_1}{\tilde{u}}\right| \frac{1+|\tilde{u}|}{1-|\tilde{u}|}.$$

P r o o f . If

$$(29) \qquad \tilde{F}(z) = \frac{F\circ p(z) - F(z_0)}{1 - F\circ p(z)\overline{F(z_0)}}, \quad p(z) = \frac{z+z_0}{1+\bar{z}_0 z}, \quad \tilde{u} = \frac{u-F(z_0)}{1-u\overline{F(z_0)}}$$

then it follows from Theorem 3 that $\tilde{F} \in S_{1\tilde{u}}$. So, if $\tilde{F}(z) = \tilde{b}(z+\tilde{a}_2 z^2+\dots)$, then $\tilde{b} = F'(z_0)(1-|z_0|^2)(1-|F(z_0)|^2)^{-1}$, $\tilde{a}_3 - \tilde{a}_2^2 = \{F;z_0\}(1-|z_0|^2)^2/6$, and the proposition follows from Theorem 7 applied to the function $\tilde{F} \in S_{1\tilde{u}}$.

As a simple corollary from Theorem 7 we have

T h e o r e m 9 . If $F \in S_{1u}$, $z_0 \in \Delta$, then

$$\left|\frac{z_0 F''(z_0)}{F'(z_0)} + 2\,\frac{z_0 F'(z_0)\overline{F(z_0)}}{1-|F(z_0)|^2} + \frac{2|z_0|^2}{|z_0|^2-1}\right| \leqslant \frac{4|z_0|}{1-|z_0|^2} - \frac{|z_0 F'(z_0)|}{2\,[1-|F(z_0)|^2]} \times$$

$$\times\left\{\frac{(|u-F(z_0)| - |1-\bar{u}F(z_0)|)^2}{|[u-F(z_0)]\,[1-\bar{u}F(z_0)]|} + 4\,\frac{|u-F(z_0)| + |1-\bar{u}F(z_0)|}{|[u-F(z_0)]\,[1-\bar{u}F(z_0)]|^{1/2}}\right\},$$

for $z_0 \neq 0$,

and

$$|a_2| \leqslant 2 - \frac{1}{4}|b| \left[(\sqrt{|u|} - 1/\sqrt{|u|})^2 + 4(\sqrt{|u|} + 1/\sqrt{|u|}) \right].$$

P r o o f . Let $\tilde{F}_0$ be a function defined by the formula $\tilde{F}_0(z) = (\tilde{F}(z^2))^{1/2}$, where the mapping $\tilde{F}$ is defined by (29). Consequently, $\tilde{F}_0(z) = \tilde{b}^{1/2}(z+\tilde{a}_2 z^3/2 + \dots)$, with that

$$\tilde{a}_2 = \frac{1}{2}\frac{F''(z_0)}{F'(z_0)}(1 - |z_0|^2) - \bar{z}_0 + \frac{F'(z_0)\overline{F(z_0)}}{1 - |F(z_0)|^2}(1 - |z_0|^2)$$

and $\tilde{F}_0 \in S_{1\sqrt{\bar{u}}}$. The desired result follows immediately from Theorem 7.

R e m a r k s . 1^0 From Theorem 4, if $z_0 = z_1 = z_2$, $v = 1/u$, and $G = F$, respectively $z_0 = z_1 = \bar{z}_2$, $v = -1/\bar{u}$, and $G = -\bar{F}$, we obtain analogues of Theorem 5 in the class of Bieberbach-Eilenberg functions, respectively in the class of Grunsky-Shah functions, not assuming certain values.
2^0 The analogue to Theorem 6 in the class A of pairs was given in paper [5]. 3^0 From Theorems 6 and 3 one easily obtains analogues of Theorems 8 and 9 in the class of pairs, of Bieberbach-Eilenberg functions, and of Grunsky-Shah functions.

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