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SOME REMARKS ON NULL GEODESIC COLLINEATIONS IN 2-RECURRENT RIEMANNIAN MANIFOLDS

1. Introduction

A non-flat n-dimensional $(n > 2)$ Riemannian manifold is said be of recurrent curvature [9] (briefly, a recurrent manifold) if its curvature tensor satisfies the condition

$$R_{hijk,l} = c_l R_{hijk} \tag{1}$$

for some non-zero vector field c_j, where the comma indicates covariant differentiation with respect to the metric.

As a generalization of the concept of a recurrent manifold, Lichnerowicz [4] initiated investigations of n-dimensional $(n > 2)$ Riemannian manifolds whose curvature tensors satisfy the relation of the form

$$R_{hijk,lm} = e_{lm} R_{hijk}. \tag{2}$$

Non-flat manifolds of such a type, i.e. satisfying (2) for some tensor e_{ij}, are called second-order recurrent or, briefly, 2-recurrent manifolds.

According to Katzin and Levine [3] a Riemannian manifold is said to admit a symmetry called a null geodesic collineation if there exists a vector field v such that

$$L\Gamma^h_{ij} = g^{hr} g_{ij} Q_{,r}, \tag{3}$$

where Q is a certain function, and $L\Gamma_{ij}^{h}$ denotes the Lie derivative with respect to v.

If $Q = const$, the null geodesic collineation is an affine one.

Roter proved [6] that a null geodesic collineation in a locally symmetric as well as in recurrent manifold is necessarily an affine one.

The purpose of the present paper is to obtain some generalizations of his results.

Throughout this note we assume that all considered manifolds are connected, of class C^{∞} and have indefinite metric forms.

2. Preliminary results

In the sequel we need the following lemmas:

Lemma 1 ([1], Theorem 1). If B_{hijk} is a generalized curvature tensor ([5], [8]) on a Riemannian manifold M satisfying the condition

$$B_{hijk,lm} - B_{hijk,ml} = 0 \tag{4}$$

and a_{ij}, b_{ij} are symmetric tensor fields such that

$$a_{ij,lm} - a_{ij,ml} = b_{im}g_{jl} + b_{jm}g_{il} - b_{il}g_{jm} - b_{jl}g_{im}, \tag{5}$$

then

$$\left(b_{lm} - \frac{b}{n} g_{lm}\right)\left(B_{hijk} - \frac{S}{n(n-1)} (g_{hk}g_{ij} - g_{hj}g_{ik})\right) = 0, \tag{6}$$

where $b = g^{rs}b_{rs}$, $S = g^{rs}g^{ij}B_{rijs}$.

Lemma 2 ([2], Theorem 1). If a vector field P_i satisfies the equation

$$v_r R^{r}{}_{ijk} = P_k g_{ij} - P_j g_{ik} \tag{7}$$

for some vector field v_i, where g_{ij} and $R^h{}_{ijk}$ are the metric and curvature tensors of the manifold M respectively, then

$$(8) \qquad P_h \left(B_{lijk} - \frac{S}{n(n-1)} (g_{lk}g_{ij} - g_{lj}g_{ik}) \right) = 0.$$

L e m m a 3 ([6]). If a Riemannian manifold admits a null geodesic collineation, then the following relations

$$(9) \qquad a_{hi,j} = A_h g_{ij} + A_i g_{hj},$$

$$(10) \qquad a_{hi,jk} - a_{hi,kj} = A_{h,k}g_{ij} + A_{i,k}g_{hj} - A_{h,j}g_{ik} - A_{i,j}g_{hk}$$

hold, where

$$(11) \qquad A^h = g^{hr}Q_{,r} \quad \text{and} \quad a_{ij} = Lg_{ij}.$$

3. Main results

Now we shall prove the main results of this paper.

T h e o r e m . If a 2-recurrent manifold M admits a null geodesic collineation, then this collineation is necessarily an affine one.

P r o o f . The condition (2) implies

$$R_{hijk,lm} - R_{hijk,ml} = (e_{lm} - e_{ml}) R_{hijk},$$

whence, by Lemma 2 of [7], we get

$$(12) \qquad R_{hijk,lm} - R_{hijk,ml} = 0$$

everywhere on M.

If we set $B_{hijk} = R_{hijk}$, $a_{ij} = Lg_{ij}$, $b_{ij} = A_{i,j} = Q_{,i,j}$, $P_i = -b_{,i}$ and $v_i = A_i$, then in view of (12) and (10), we see that the equations (4) and (5) are satisfied. Hence, by virtue of Lemma 1, the condition (6) holds.

We may assume that

$$A_{i,j} = \frac{b}{n} g_{ij} \tag{13}$$

in some neighbourhood U. Otherwise M would be of constant curvature and our assertion would follow from Theorem 2 of [6].

Therefore, differentiating (13) covariantly and making use of Ricci identity, we obtain (7).

Hence, in view of (12) and Lemma 2, $P_j = -b_{,j} = 0$ in U. But the last result, together with (7), yields

$$A_r R^r{}_{ijk} = 0,$$

which, by covariant differentiation, implies

$$\frac{b}{n} R_{lijk} + A_r R^r{}_{ijk,l} = 0. \tag{14}$$

On the other hand, as an immediate consequence of (10) and (13), we get

$$a_{kr} R^r{}_{ijm} + a_{ir} R^r{}_{kjm} = 0, \tag{15}$$

and therefore,

$$a_{kr,pl} R^r{}_{ijm} + a_{kr,p} R^r{}_{ijm,l} + a_{kr,l} R^r{}_{ijm,p} + a_{kr} R^r{}_{ijm,pl} +$$

$$+ a_{ir,pl} R^r{}_{kjm} + a_{ir,p} R^r{}_{kjm,l} + a_{ir,l} R^r{}_{kjm,p} + a_{ir} R^r{}_{kjm,pl} = 0.$$

The last relation, because of (9), (13), (14), (2) and (15), gives

$$A_k(R_{pijm,l} + R_{lijm,p}) + A_i(R_{pkjm,l} + R_{lkjm,p}) = 0$$

whence

$$A_i(R_{jmkp,l} + R_{jmkl,p}) = 0 \quad \text{in} \quad U.$$

If $R_{hijk,l} + R_{hijl,k} = 0$, Bianchi's identity yields $R_{hijk,l} = 0$, and our assertion follows immediately from Theorem 2 of [6].

The last remark completes the proof.

Since a non-flat Riemannian manifold $(n > 2)$ whose curvature tensor satisfies (1) as well as $R_{hijk,lm} = 0$ is 2-recurrent, we have

C o r o l l a r y 1. Let M be a non-flat Riemannian manifold $(n > 2)$ whose curvature tensor satisfies $R_{hijk,lm}=0$. If M admits a null geodesic collineation, then this collineation is an affine one.

C o r o l l a r y 2. ([6], Theorem 4). If a recurrent manifold admits a null geodesic collineation, then this collineation is an affine one.

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