DEMONSTRATIO MATHEMATICA
Vol. XV No 2 1982

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INVARIANT SUBMANIFOLDS OF f_λ-MANIFOLD

Invariant subspaces in almost complex X^{2n} and invariant submanifolds of an almost contact manifold have been studied by Yano and Schouton [3], Yano and Ishihara [4], respectively. Upadhyay and Gupta [2] defined and studied the f_λ-manifold. The purpose of the present paper is to study the invariant submanifolds of f_λ-manifold.

1. Preliminaries

Let $\tilde{M}$ be an m-dimensional C^∞ Riemannian manifold imbedded in an n-dimensional C^∞ Riemannian manifold M, where $m < n$ and the imbedding is denoted by $\Phi : \tilde{M} \to M$. Let B be the mapping induced by Φ, that is, $B = d\Phi : T(\tilde{M}) \to T(M)$, where $T(\tilde{M})$ and $T(M)$ are tangent bundles of $\tilde{M}$ and M respectively. If $T(\tilde{M},M)$ is the set of all vectors tangent to $\Phi(\tilde{M})$, then $B : T(\tilde{M}) \to T(\tilde{M},M)$ is an isomorphism [1].

The set of all vectors normal to $\Phi(\tilde{M})$ forms a vector bundle $N(\tilde{M},M)$ over $\Phi(\tilde{M})$ and is called the normal bundle of $\tilde{M}$. The vector bundle induced from $N(\tilde{M},M)$ by Φ is denoted by $N(\tilde{M})$. Let us denote by $\psi : N(\tilde{M}) \to N(\tilde{M},M)$ the natural isomorphism.

Throughout this paper, we use the following notations and conventions:

(i) $\mathcal{J}^r_s(\tilde{M})$ denotes the space of all C^∞ tensor fields of the type (r,s) associated with $T(\tilde{M})$.

(ii) $\mathcal{U}^r_s(\tilde{M})$ denotes the space of all C^∞ tensor fields of the type (r,s) normal to $\tilde{M}$.

An element of $\mathfrak{J}^1_0(M)$ is a vector field on $\tilde{M}$ and an element of $\mathfrak{U}^1_0(\tilde{M})$ is a vector field normal to $\tilde{M}$.

Let X and Y be any vector fields defined along $\Phi(\tilde{M})$ and tangential to $\Phi(\tilde{M})$. Let $\bar{X}$ and $\bar{Y}$ be the local extensions of X and Y. Then $[\bar{X},\bar{Y}]$ is a vector field tangential to M and its restriction $[\bar{X},\bar{Y}]|\Phi(\tilde{M})$ to $\Phi(\tilde{M})$ can be determined independently from the choice of local extensions $\bar{X}$ and $\bar{Y}$. Thus we can define $[X,Y]$ by

$$[X,Y] = [\bar{X},\bar{Y}]|\Phi(\tilde{M}). \tag{1.1}$$

Since B is an isomorphism, therefore for all $\tilde{X},\tilde{Y} \in \mathfrak{J}^1_0(\tilde{M})$ we have

$$[B\tilde{X},B\tilde{Y}] = B[\tilde{X},\tilde{Y}]. \tag{1.2}$$

D e f i n i t i o n 1.1. If in an n-dimensional C^∞ manifold M, a C^∞ tensor field f of the type (1,1) satisfies

$$f^2 - \lambda^2 I = 0, \tag{1.3}$$

where λ is a complex number not equal to zero and I is the unit tensor field; then f is said to possess a π-structure on M and M is called a π-manifold [2].

Let us assume that M is a C^∞ f_λ-manifold endowed with a $C^\infty(1,1)$ tensor field f satisfying [2]

$$f^3 - \lambda^2 f = 0, \tag{1.4}$$

where λ is a complex number not equal to zero, that is, f is a f_λ-structure on M.

If we put

$$s = \left(\frac{f}{\lambda}\right)^2, \quad t = I - \left(\frac{f}{\lambda}\right)^2, \tag{1.5}$$

I being the unit tensor field; then we have

$$s + t = I, \quad st = ts = 0; \tag{1.6}$$

$$s^2 = s, \quad t^2 = t. \tag{1.7}$$

Thus there exist two complementary distributions S and T corresponding to the projection operators s and t respectively. These projection operators satisfy the following relations [2]

$$fs = sf = f, \quad ft = tf = 0; \tag{1.8}$$

$$f^2s = \lambda^2 s, \quad f^2t = tf^2 = 0; \tag{1.9}$$

that is, f acts on S as a π-structure operator and on T as a null operator.

Such a manifold M always admits a Riemannian metric tensor $\bar{G}$ such that

$$\bar{G}(\bar{X},\bar{Y}) = \bar{G}(f\bar{X},f\bar{Y}) + \bar{G}(t\bar{X},\bar{Y}), \tag{1.10}$$

for all $\bar{X}, \bar{Y} \in \mathcal{J}_0^1(M)$. Then in view of (1.8) and (1.10), we have

$$\bar{G}(\bar{X},f\bar{Y}) = \bar{G}(f\bar{X},f^2\bar{Y}) + \bar{G}(t\bar{X},f\bar{Y}), \tag{1.11}$$

$$\bar{G}(f\bar{X},\bar{Y}) = \bar{G}(f^2\bar{X},f\bar{Y}). \tag{1.12}$$

Let us define $\tilde{g}$ and g^* on $\tilde{M}$ and $N(\tilde{M})$ respectively as follows

$$\tilde{g}(\tilde{X},\tilde{Y}) = \bar{G}(B\tilde{X},B\tilde{Y}) \circ \Phi \tag{1.13}$$

and

$$g^*(N,N') = \bar{G}(\psi N, \psi N'), \tag{1.14}$$

for all $N, N' \in \mathcal{U}_0^1(\tilde{M})$.

It can be easily shown that $\tilde{g}$ is a Riemannian metric tensor in $\tilde{M}$ which is called the induced metric tensor of $\tilde{M}$ and g^* is a tensor field which defines an inner product in $N(\tilde{M})$. The tensor g^* is called the induced metric tensor of $N(\tilde{M})$.

Let $\bar{\nabla}$ be the Riemannian connection determined by $\bar{G}$ in M, then $\bar{\nabla}$ induces a connection ∇ in $\Phi(\tilde{M})$ defined by [4]

$$\nabla_X Y = \bar{\nabla}_{\bar{X}} \bar{Y} \Big| \Phi(\tilde{M}), \tag{1.15}$$

where X,Y are C^∞ vector fields defined along $\Phi(\tilde{M})$ and tangential to $\Phi(\tilde{M})$.
This in view of (1.1) and (1.15), we have

$$\nabla_X Y - \nabla_Y X = [X,Y]. \tag{1.16}$$

2. Invariant submanifolds of f_λ-manifold

Let $\tilde{M}$ be a C^∞ m-dimensional manifold imbedded in a C^∞ f_λ-manifold M endowed with (1,1) tensor field f satisfying (1.4).

D e f i n i t i o n 2.1. $\tilde{M}$ is said to be an invariant submanifold of M if the tangent space $T_p(\Phi(\tilde{M}))$ of $\Phi(\tilde{M})$ is invariant by the linear mapping f at each point p in $\Phi(\tilde{M})$ [4].

In this paper, we shall assume that $\tilde{M}$ is an invariant submanifold of M. Therefore for $\tilde{X} \in \mathfrak{J}^1_0(\tilde{M})$, we have

$$fB\tilde{X} = BX^o, \tag{2.1}$$

where X^o is some vector field in $\tilde{M}$. Thus we define a (1,1) tensor field $\tilde{f}$ in $\tilde{M}$, that is, a mapping

$$\tilde{f} : \mathfrak{X}(\tilde{M}) \longrightarrow \mathfrak{X}(\tilde{M}) \quad \text{by} \quad \tilde{f}\tilde{X} = X^o.$$

From (2.1) we have

$$f(B\tilde{X}) = B(\tilde{f}\tilde{X}). \tag{2.2}$$

T h e o r e m 2.1. Let N and $\tilde{N}$ be the Nijenhuis tensors of M and $\tilde{M}$ determined by the (1,1) tensor fields f and $\tilde{f}$, respectively. Then N and $\tilde{N}$ are related as follows

$$N(B\tilde{X},B\tilde{Y}) = B\tilde{N}(\tilde{X},\tilde{Y}). \tag{2.3}$$

P r o o f . By virtue of (1.2) and (2.2), we have

$$N(B\tilde{X},B\tilde{Y}) = [fB\tilde{X},fB\tilde{Y}] - f[B\tilde{X},fB\tilde{Y}] - f[fB\tilde{X},B\tilde{Y}] + f^2[B\tilde{X},B\tilde{Y}] =$$
$$= B([\tilde{f}\tilde{X},\tilde{f}\tilde{Y}] - \tilde{f}[\tilde{X},\tilde{f}\tilde{Y}] - \tilde{f}[\tilde{f}\tilde{X},\tilde{Y}] + \tilde{f}^2[\tilde{X},\tilde{Y}]) =$$
$$= B\tilde{N}(\tilde{X},\tilde{Y}).$$

Hence the result.

For the invariant submanifold $\tilde{M}$ of f_λ-manifold M, we shall consider the following two cases:

Case I. The distribution T is never tangential to $\Phi(\tilde{M})$, that is, to any vector field of the type $t\bar{X}$, where $\bar{X}$ is a vector field tangential to $\Phi(\tilde{M})$.

Case II. The distribution T is always tangential to $\Phi(M)$.

Let us consider the Case I, that is, the distribution T is never tangential to the invariant submanifold $\Phi(\tilde{M})$. In this case, any vector field of the type $t\bar{X}$ is independent of any vector field of the same frame $B\tilde{X}$ for $\tilde{X} \in J_0^1(\tilde{M})$. Applying f to (2.2), we obtain

$$f^2B\tilde{X} = B\tilde{f}^2\tilde{X}.$$

Since any vector field tangential to $\Phi(\tilde{M})$ is not contained in the distribution T, therefore, the vector fields of the type $B\tilde{X}$ are in the distribution S. Thus in consequence of (1.9), we have

$$B\tilde{f}^2\tilde{X} = \lambda^2 B\tilde{X}$$

from which it follows that

$$\tilde{f}^2\tilde{X} = \lambda^2\tilde{X}. \tag{2.4}$$

Consequently, the (1,1) tensor field $\tilde{f}$ in $\tilde{M}$ is a π-structure on the invariant submanifold $\tilde{M}$.

Let us define a tensor field $\bar{S}$ of the type (1,2) in M as follows

$$\bar{S}(\bar{X},\bar{Y}) = N(\bar{X},\bar{Y}) + \bar{\nabla}_{\bar{X}}(t\bar{Y}) - \bar{\nabla}_{\bar{Y}}(t\bar{X}) - t[\bar{X},\bar{Y}] \tag{2.5}$$

for any vector fields $\bar{X}, \bar{Y} \in \mathfrak{J}^1_0(M)$.

T h e o r e m 2.2. Let the distribution T be never tangential to $\Phi(\tilde{M})$. Then the (1,2) tensor field $\bar{S}$ defined in M is given by

$$\bar{S}(B\tilde{X},B\tilde{Y}) = N(B\tilde{X},B\tilde{Y}) = B\tilde{N}(\tilde{X},\tilde{Y}) \tag{2.6}$$

for $\tilde{X}, \tilde{Y} \in \mathfrak{J}^1_0(M)$.

P r o o f . Since any vector field tangential to $\Phi(\tilde{M})$ is not contained in the distribution T, therefore in consequence of (1.6), we have

$$t(B\tilde{X}) = 0,$$

for any $\tilde{X} \in \mathfrak{J}^1_0(\tilde{M})$. Hence in view of (2.3) and (2.5), the result follows.

D e f i n i t i o n 2.2. The f_λ-structure f is said to be a normal in M if $\bar{S} = 0$.

T h e o r e m 2.3. An invariant submanifold $\tilde{M}$ imbedded in a f_λ-manifold M such that the distribution T is never tangential to $\Phi(\widetilde{M})$ is a π-manifold with induced π-structure $\widetilde{f}$. If the f_λ-structure f is normal in M, then the π-structure $\widetilde{f}$ is integrable in $\widetilde{M}$.

P r o o f . The proof follows by virtue of the equations (2.4), (2.6) and Definition 2.2.

Next we shall consider the Case II, that is, the distribution T is always tangential to the invariant submanifold $\Phi(\widetilde{M})$. Therefore for $\widetilde{X} \in J_0^1(\widetilde{M})$, we have

$$tB\widetilde{X} = BX^0, \tag{2.7}$$

where X^0 is some vector field in $\widetilde{M}$.

Let us define a (1,1) tensor field $\widetilde{t}$ in $\widetilde{M}$ such that $\widetilde{t}\widetilde{X} = X^0$. Then the equation (2.7) can be expressed as

$$tB\widetilde{X} = B\widetilde{t}\widetilde{X}. \tag{2.8}$$

Also we can define a (1,1) tensor field $\widetilde{s}$ on $\widetilde{M}$ by

$$sB\widetilde{X} = B\widetilde{s}\widetilde{X}. \tag{2.9}$$

Since in M the relation $s + t = I$ (I being the unit tensor field) holds, therefore the (1,1) tensor field $\widetilde{s}$ in $\widetilde{M}$ is well defined.

T h e o r e m 2.4. The (1,1) tensor fields $\widetilde{t}$ and $\widetilde{s}$ in $\widetilde{M}$ defined by (2.8) and (2.9) respectively satisfy the following relations

$$\widetilde{s} + \widetilde{t} = \widetilde{I}, \qquad \widetilde{s}\widetilde{t} = \widetilde{t}\widetilde{s} = 0; \tag{2.10}$$

$$\widetilde{s}^2 = \widetilde{s}, \qquad \widetilde{t}^2 = \widetilde{t}. \tag{2.11}$$

P r o o f . We have

$$s + t = I.$$

Operating the above equation with $B\tilde{X}$, we obtain

$$sB\tilde{X} + tB\tilde{X} = IB\tilde{X},$$

which in view of (2.8) and (2.9) becomes

$$B\tilde{s}\tilde{X} + B\tilde{t}\tilde{X} = B\tilde{I}\tilde{X}.$$

This implies that

$$\tilde{s}\tilde{X} + \tilde{t}\tilde{X} = \tilde{X}.$$

That is $\tilde{s} + \tilde{t} = \tilde{I}$.

Next operating $st = ts = 0$ by $B\tilde{X}$ and making use of (2.8) and (2.9) we get

$$B\tilde{s}\tilde{t}\tilde{X} = 0.$$

This implies that $\tilde{s}\tilde{t} = 0$.

Similarly, making use of (2.8) and (2.9) in (1.7) we can prove that

$$\tilde{s}^2 = \tilde{s}, \quad \tilde{t}^2 = \tilde{t}.$$

This (2.10) and (2.11) hold in $\tilde{M}$. Hence $\tilde{s}$ and $\tilde{t}$ are complementary projection operators in $\tilde{M}$, given by

$$\tilde{s} = \left(\frac{\tilde{f}}{\lambda}\right)^2, \quad \tilde{t} = \tilde{I} - \left(\frac{\tilde{f}}{\lambda}\right)^2.$$

This proves the theorem.

Now in consequence of (1.4) and (2.2), we have

$$B\tilde{f}^3\tilde{X} = f^3(B\tilde{X}) = \lambda^2 f(B\tilde{X}) = \lambda^2 B\tilde{f}\tilde{X}.$$

This implies that

(2.12) $$\tilde{f}^3 - \lambda^2\tilde{f} = 0.$$

Hence $\tilde{f}$ acts as a f_λ-structure in $\tilde{M}$ and is called the induced f_λ-structure on $\tilde{M}$. The Riemannian metric $\tilde{g}$ given by [3]

(2.13) $$\tilde{g}(\tilde{X},\tilde{Y}) = \tilde{g}(\tilde{f}\tilde{X},\tilde{f}\tilde{Y}) + \tilde{g}(\tilde{t}\tilde{X},\tilde{Y})$$

also holds for $\tilde{M}$.

Let $\tilde{\nabla}$ be an operator in $\tilde{M}$ defined by

(2.14) $$B(\tilde{\nabla}_{\tilde{X}}\tilde{Y}) = \bar{\nabla}_{B\tilde{X}}B\tilde{Y},$$

where $\bar{\nabla}$ is the Riemannian connection in M.
It can be easily shown that the operator $\tilde{\nabla}$ is a connection in $\tilde{M}$. Now by virtue of (1.1) and (2.14), we have

(2.15) $$\tilde{\nabla}_{\tilde{X}}\tilde{Y} - \tilde{\nabla}_{\tilde{Y}}\tilde{X} = [\tilde{X},\tilde{Y}].$$

Thus $\tilde{\nabla}$ is a Riemannian connection in $\tilde{M}$.

Let us define a (1,2) tensor field $\tilde{S}$ in $\tilde{M}$ as follows

(2.16) $$\tilde{S}(\tilde{X},\tilde{Y}) = \tilde{N}(\tilde{X},\tilde{Y}) + \tilde{\nabla}_{\tilde{X}}\tilde{t}\tilde{Y} - \tilde{\nabla}_{\tilde{Y}}(\tilde{t}\tilde{X}) - \tilde{t}[\tilde{X},\tilde{Y}]$$

for $\tilde{X},\tilde{Y} \in J_0^1(M)$.

Now in view of (1.2), (2.3), (2.5), (2.8), (2.14) and (2.16), we have

(2.17) $$\bar{S}(B\tilde{X},B\tilde{Y}) = B\tilde{S}(\tilde{X},\tilde{Y}).$$

Definition 2.3. The f_λ-structure $\tilde{f}$ is said to be normal in $\tilde{M}$ if $\tilde{S} = 0$.

T h e o r e m 2.5. An invariant submanifold $\tilde{M}$ imbedded in a f_λ-manifold M such that the distribution T is always tangential to $\Phi(\tilde{M})$, is a f_λ-manifold with induced f_λ-structure f. If the f_λ-structure f is normal in M, then the f_λ-structure $\tilde{f}$ is normal in $\tilde{M}$.

P r o o f . The proof follows by virtue of the equations (2.12), (2.17), Theorem 2.4 and Definitions 2.2 and 2.3.

Acknowledgements: We are thankful to Dr.M.D.Upadhyay for his valuable suggestions in the preparation of this paper. The first author is thankful to C.S.I.R. Complex, New Delhi for financial assistance as Pool Officer and the second author is also thankful to C.S.I.R., New Delhi for providing research fellowship.

REFERENCES

[1] S. I s h i h a r a , K. Y a n o : On integrability conditions of a structure f satisfying $f^3+f = 0$, Quart. J. Math. Oxford 15 (1964) 217-222.

[2] M.B. U p a d h y a y , V.C. G u p t a : Integrability conditions of a structure f_λ satisfying $f^3-\lambda^2 f = 0$, Publ. Math. 24 (1977) 249-255.

[3] K. Y a n o , J.A. S c h o u t o n : On invariant subspaces in almost complex X^{2n}, Indiana Math. J. 17 (1968) 261-267.

[4] K. Y a n o , S. I s h i h a r a : Invariant submanifolds of an almost contact manifold, Kodai Math. Sem. Rep. 21 (1969) 350-367.

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Received May 31, 1980.