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ON THE CONTINUOUS SOLUTIONS OF A FUNCTIONAL EQUATION CONTAINING ITERATIONS OF THE UNKNOWN FUNCTION

Let there be given the functional equation

$$(1) \qquad \varphi(x) = h(x,\varphi(x),\varphi(\hat{f}_1^1[x,\varphi]),\dots,\varphi(\hat{f}_n^1[x,\varphi])),$$

where the expressions $\hat{f}_j^1[x,\varphi]$ $(j=1,\dots,n)$ are defined by the recurrent formula

$$(2) \begin{cases} \hat{f}_j^{q_j}[x,\varphi] := f_j^{q_j}(x,\varphi(x)) \\ \hat{f}_j^m[x,\varphi] := f_j^m(x,\varphi(\hat{f}_j^{m+1}[x,\varphi])), \quad m=1,\dots,q_j-1, \ j=1,\dots,n, \end{cases}$$

and q_j $(j=1,\dots,n)$ are fixed natural numbers. The functions h, f_j^m $(m=1,\dots,q_j,\ j=1,\dots,n)$ involved in equation (1) are given functions, whereas φ is the unknown function.

The problem of the existence of continuous (or lipschitzian) solutions of a functional equation containing iterations of the unknown function was studied by many authors (see e.g. [1] - [6]).

The existence of a unique solution of equation (1) in the class of functions lipschitzian in arbitrary metric spaces was investigated by the author of this paper in [3]. Here we are going to prove by means of Schauder's principle a theorem on the existence of solutions of equation (1) in a class which is narrower than C^0, but contains the class of lipschitzian functions. This theorem yields a generalization of Theorem 3 established by H. Adamczyk in paper [1].

In order to prove the existence of a solution φ of equation (1) we admit that the given functions h, f_j^m satisfy the following assumptions:

1° $$h : X\times Y^{n+1} \longrightarrow Y$$

$$f_j^m : X\times Y \longrightarrow X \quad m = 1,\dots,q_j; \quad j = 1,\dots,n,$$

where (X,ϱ) is a compact and connected metric space and $(Y, \|\cdot\|)$ is a finite-dimensional Banach space.

2° There exist $\xi \in X$, $\eta \in Y$ such that

$$f_j^m(\xi,\eta) = \xi, \qquad h(\xi, \eta,\dots, \eta) = \eta, \tag{3}$$

$$m = 1,\dots,q_j, \quad j = 1,\dots,n.$$

3° The function h is continuous in its domain of definition and satisfies the following condition:
for arbitrary points $x,\bar{x} \in X$, $y_i,\bar{y}_i \in Y$ $(i = 0,\dots,n)$ such that

$$\|y_i - \bar{y}_i\| \leqslant 2\omega(\varrho(x,\bar{x})), \quad i = 0,1,\dots,n$$

holds

$$\|h(x,y_0,y_1,\dots,y_n) - h(\bar{x},\bar{y}_0,\dots,\bar{y}_n)\| \leqslant \omega(\varrho(x,\bar{x})), \tag{4}$$

where ω is a real function defined, continuous and strictly increasing in the interval $\langle 0,d\rangle$, with $d = \operatorname{diam} X$, and satisfying the equality $\omega(0) = 0$.

4° The functions f_j^m $(m = 1,\dots,q_j;\ j = 1,\dots,n)$ are continuous and satisfy the conditions:

(a) for arbitrary points $x,\bar{x} \in X$, $u,\bar{u} \in Y$ such that

$$\|u - \bar{u}\| \leqslant 2\,\omega\,(\varrho(x,\bar{x}))$$

there holds the inequality

$$(5)\quad \varrho(f_j^m(x,u),\ f_j^m(\bar{x},\bar{u})) \leqslant \varrho(x,\bar{x}), \quad m=1,\dots,q_j,\ j=1,\dots,n$$

(b) to any positive number ζ there exists a positive number k_ζ such that for all $x,\bar{x} \in X$, $u,\bar{u} \in Y$ such that $\varrho(x,\bar{x}) \geqslant \zeta$ and $\|u - \bar{u}\| \leqslant 2\omega(\varrho(x,\bar{x}))$ the following inequality holds

$$(6)\qquad \varrho(f_j^1(x,u),\ f_j^1(\bar{x},\bar{u})) \geqslant k_\zeta, \quad j=1,\dots,n.$$

Assume, moreover, the notation $\hat{f}_0^1[x,u] := x,\ \ x \in X.$

Theorem. Under the assumptions 1° - 4° the equation (1) has at least one solution $\varphi : X \to Y$ bounded and satisfying the conditions

$$(7)\qquad \varphi(\xi) = \eta,\ \|\varphi(x) - \varphi(\bar{x})\| \leqslant \omega(\varrho(x,\bar{x})), \quad x,\bar{x} \in X.$$

Proof. To prove the theorem we shall make use of Schauder's fixed point theorem as well as of the Arzelá-Ascoli theorem (see [7], p.164).

Consider the space $\mathcal{F}$ whose points are the functions $\varphi:X \to Y$ continuous and bounded in X. Let the norm of the point φ of the space $\mathcal{F}$ be defined by the equality

$$(8)\qquad |||\varphi||| := \sup_{x \in X} \|\varphi(x)\|.$$

The space $\mathcal{F}$ under the norm (8) is a Banach space. In this space we shall consider the set A of all its points φ which satisfy the conditions (7).

We shall prove that the set A is convex. Indeed, for arbitrary $\varphi_1, \varphi_2 \in A$ and $t \in \langle 0,1 \rangle$ we have, by the first of conditions (7), the equality

$$(t\varphi_1 + (1-t)\varphi_2)(\xi) = t\varphi_1(\xi) + (1-t)\varphi_2(\xi) = \eta.$$

Using the property of the norm and the second of conditions (7) we get the inequality

$$\|(t\varphi_1+(1-t)\varphi_2)(x) - (t\varphi_1+(1-t)\varphi_2)(\bar{x})\| \leqslant \omega(\varrho(x,\bar{x}), \qquad x,\bar{x} \in X.$$

Hence $t\varphi_1 + (1-t)\varphi_2$ satisfies conditions (7) and A is a convex set.

We shall now prove that A is a set of equicontinuous functions. Let ε be an arbitrary real positive number. There exists a $\delta = \omega^{-1}(\varepsilon)$ such that the condition $\varrho(x,\bar{x}) < \delta$, $(x,\bar{x} \in X)$, implies the inequality $\|\varphi(x)-\varphi(\bar{x})\| < \varepsilon$ for any $\varphi \in A$. Indeed, for $\varphi \in A$ we have (by inequality (7))

$$\|\varphi(x) - \varphi(\bar{x})\| \leqslant \omega(\varrho(x,\bar{x})) < \omega(\omega^{-1}(\varepsilon) = \varepsilon,$$

which means that the functions φ of the set A are equicontinuous.

Taking into account the form of equation (1) we submit the set A to the operation $\psi = T[\varphi]$ defined by the equality

$$(9) \qquad \psi(x) = h(x,\varphi(x),\varphi(\hat{f}_1^1[x,\varphi]),\ldots,\varphi(\hat{f}_n^1[x,\varphi])), \qquad x \in X.$$

We shall prove that the operation T maps the set A into itself. Let $\varphi \in A$. The first of conditions (7) is satisfied for $\psi = T[\varphi]$, since by assumption 2^0 we have

$$\varphi(\xi) = h(\xi,\varphi(\xi), \varphi(\hat{f}_1^1[\xi,\varphi]),\ldots,\varphi(\hat{f}_n^1[\xi,\varphi])) =$$

$$= h(\xi,\eta,\ldots,\eta) = \eta.$$

Let us now show that the second of conditions (7) is also satisfied for $\psi = T[\varphi]$. To this aim we shall first prove that for all $x,\bar{x} \in X$ the following inequality holds

$$\varrho(\hat{f}_i^m[x,\varphi],\ \hat{f}_i^m[\bar{x},\varphi]) \leqslant \varrho(x,\bar{x}), \qquad m = 1,\dots,q_i; \quad i = 1,\dots,n.$$

Now, for all $x,\bar{x} \in X$ we have, by virtue of definition (2), of the second of conditions (7) and of assumption 4^0(a), the inequality

$$\varrho(\hat{f}_i^{q_i}[x,\varphi],\hat{f}_i^{q_i}[\bar{x},\varphi]) = \varrho(f_i^{q_i}(x,\varphi(x)),\ f_i^{q_i}(\bar{x},\varphi(\bar{x}))) \leqslant \varrho(x,\bar{x}),$$

$(i = 1,\dots,n)$.

Hence, by the second of conditions (7), we get

$$\|\varphi(\hat{f}_i^{q_i}[x,\varphi]) - \varphi(\hat{f}_i^{q_i}[\bar{x},\varphi])\| \leqslant \omega(\varrho(x,\bar{x}), \qquad (i=1,\dots,n),$$

From this inequality and from assumption 4^0(a) it follows that

$$\varrho(\hat{f}_i^{q_i-1}[x,\varphi],\ \hat{f}_i^{q_i-1}[\bar{x},\varphi]) = \varrho(f_i^{q_i-1}(x,\varphi(\hat{f}_i^{q_i}[x,\varphi])),$$

$$f_i^{q_i-1}(\bar{x},\varphi(\hat{f}_i^{q_i}[\bar{x},\varphi]))) \leqslant \varrho(x,\bar{x}) \qquad \text{for all} \quad x,\bar{x} \in X,\ i=1,\dots,n.$$

An analogous argument leads to the following estimate

$$\varrho(\hat{f}_i^1[x,\varphi],\ \hat{f}_i^1[\bar{x},\varphi]) \leqslant \varrho(x,\bar{x}) \qquad \text{for} \quad x,\bar{x} \in X,\ i=1,\dots,n.$$

Hence, using the second of conditions (7) and the notation $\hat{f}_0^1[x,\varphi] = x$ we obtain for $i = 0,1,\dots,n$ the inequality

$$(10) \qquad \|\varphi(\hat{f}_i^1[x,\varphi]) - \varphi(\hat{f}_i^1[\bar{x},\varphi])\| \leqslant \omega(\varrho(x,\bar{x})), \quad x,\bar{x} \in X.$$

From inequality (10), assumption 3^0 and definition (9) it follows that

$$\|\psi(x) - \psi(\bar{x})\| \leqslant \omega(\varrho(x,\bar{x})), \qquad x,\bar{x} \in X,$$

whence we conclude that $T(A) \subset A$.

We shall prove that the operation T defined by equation (9) is continuous; that is to say, we have to show that if $\varphi_p, \varphi \in A$ satisfy the relation

$$\lim_{p\to\infty} |||\varphi_p - \varphi||| = 0, \tag{11}$$

then for the points $\psi_p = T[\varphi_p]$, $\psi = T[\varphi]$ of the set $T(A)$ the following equality holds

$$\lim_{p\to\infty} |||\psi_p - \psi||| = 0.$$

Let ε be an arbitrary real positive number. To the number ε we choose an $\bar{\varepsilon} = \bar{\varepsilon}(\varepsilon)$ such that

$$0 < \bar{\varepsilon} < \varepsilon \tag{12}$$

and that, for any fixed $x \in X$, the set X_ε of points $\bar{x} \in X$ defined as follows

$$X_\varepsilon = \left\{\bar{x} \in X : \omega^{-1}\left(\frac{\bar{\varepsilon}}{k}\right) \leqslant \varrho(x,\bar{x}) < \omega^{-1}\left(\frac{\bar{\varepsilon}}{2}\right)\right\}, \quad k > 2 \tag{13}$$

is non-void.

Let us now examine the distance between the points ψ_p, ψ of the set $T(A)$. For this purpose we shall estimate the distance $\|\psi_p(x) - \psi(x)\|$ which is given by the formula

$$\begin{aligned} \|\psi_p(x) - \psi(x)\| = \\ = \|h(x, \varphi_p(x), \varphi_p(\hat{f}_1^1[x,\varphi_p]),\dots,\varphi_p(\hat{f}_n^1[x,\varphi_p])) - \\ - h(x,\varphi(x),\varphi(\hat{f}_1^1[x,\varphi]),\dots,(\hat{f}_n^1[x,\varphi]))\| \end{aligned} \tag{14}$$

and satisfies the inequality

$$(15)\qquad \|\psi_p(x) - \psi(x)\| \leqslant$$

$$\leqslant \|h(x,\varphi_p(x),\ \varphi_p(\hat{f}_1^1[x,\varphi_p]),\dots,\varphi_p(\hat{f}_n^1[x,\varphi_p])) -$$

$$- h(\bar{x},\varphi(\bar{x}),\ \varphi(\hat{f}_1^1[\bar{x},\varphi]),\dots,\ \varphi(\hat{f}_n^1[\bar{x},\varphi]))\| +$$

$$+ \|h(\bar{x},\varphi(\bar{x}),\ \varphi(\hat{f}_1^1[\bar{x},\varphi]),\dots,\ \varphi(\hat{f}_n^1[\bar{x},\varphi])) -$$

$$- h(x,\varphi(x),\ \varphi(\hat{f}_1^1[x,\varphi]),\dots,\ \varphi(\hat{f}_n^1[x,\varphi]))\|$$

for any fixed $x \in X$ and for $\bar{x} \in X_\varepsilon$. Consider the second term of the sum (15). Now, by formulas (7) and (9) we get

$$\|h(\bar{x},\varphi(\bar{x}),\ \varphi(\hat{f}_1^1[\bar{x},\varphi]),\dots,\ \varphi(\hat{f}_n^1[\bar{x},\varphi])) -$$

$$- h(x,\varphi(x),\ \varphi(\hat{f}_1^1[x,\varphi]),\dots,\ \varphi(\hat{f}_n^1[x,\varphi]))\| \leqslant \omega(\varrho(x,\bar{x})).$$

Taking into account the definition (13) we obtain for the second term of the sum (15) the following estimate

$$(16)\qquad \|h(\bar{x},\varphi(\bar{x}),\ \varphi(\hat{f}_1^1[\bar{x},\varphi]),\dots,\ \varphi(\hat{f}_n^1[\bar{x},\varphi])) -$$

$$- h(x,\varphi(x),\ \varphi(\hat{f}_1^1[x,\varphi]),\dots,\ \varphi(\hat{f}_n^1[x,\varphi]))\| <$$

$$< \omega\left(\omega^{-1}\left(\frac{\varepsilon}{2}\right)\right) = \frac{\varepsilon}{2}, \qquad x \in X,\quad \bar{x} \in X_\varepsilon.$$

Turning now to the first term of the sum (15) let us consider the expression $\|\varphi_p(x) - \varphi(\bar{x})\|$ which we majorize as follows

$$(17)\qquad \|\varphi_p(x) - \varphi(\bar{x})\| \leqslant \|\varphi_p(x) - \varphi(x)\| + \|\varphi(x) - \varphi(\bar{x})\| .$$

From condition (11) it follows that to the previously fixed number $\bar{\varepsilon} > 0$ we can find an $N_{\bar{\varepsilon}}$ such that for $p > N_{\bar{\varepsilon}}$ the following inequality holds

$$\|\varphi_p(x) - \varphi(x)\| < \frac{\bar{\varepsilon}}{k}.$$

By definition (13) we have for any $\bar{x} \in X_\varepsilon$

$$\omega^{-1}\left(\frac{\bar{\varepsilon}}{k}\right) \leqslant \varrho(x,\bar{x}),$$

The function ω is strictly increasing, hence

$$\frac{\bar{\varepsilon}}{k} \leqslant \omega(\varrho(x,\bar{x})).$$

Therefore

(18) $$\|\varphi_p(x) - \varphi(x)\| \leqslant \omega(\varrho(x,\bar{x})) \quad \text{for} \quad x \in X,\ \bar{x} \in X_\varepsilon.$$

Making use of (7), (17) and (18) we get (for $p > N_{\bar{\varepsilon}}$) the relation

(19) $$\|\varphi_p(x) - \varphi(\bar{x})\| \leqslant 2\omega(\varrho(x,\bar{x})) \quad \text{for} \quad x \in X,\ \bar{x} \in X_\varepsilon.$$

By a similar argument we can majorize the expression

$$\|\varphi_p(\hat{f}_i^1[x,\varphi_p]) - \varphi(\hat{f}_i^1[\bar{x},\varphi])\|, \quad \text{for } i=1,2,\dots,n.$$

Now, applying the triangular inequality to the norm we get

$$\|\varphi_p(\hat{f}_i^1[x,\varphi_p]) - \varphi(\hat{f}_i^1[\bar{x},\varphi])\| \leqslant$$
$$\|\varphi_p(\hat{f}_i^1[x,\varphi_p]) - \varphi(\hat{f}_i^1[x,\varphi_p])\| + \|\varphi(\hat{f}_i^1[x,\varphi_p]) - \varphi(\hat{f}_i^1[\bar{x},\varphi])\|,$$

whence, by the second of conditions (7) we obtain

$$(20)\qquad \|\varphi_p(\hat{f}_i^1[x,\varphi_p]) - \varphi(\hat{f}_i^1[\bar{x},\varphi])\| \leqslant \|\varphi_p(\hat{f}_i^1[x,\varphi_p]) - $$

$$-\varphi(\hat{f}_i^1[x,\varphi_p])\| + \omega(\varrho(\hat{f}_i^1[x,\varphi_p],\ \hat{f}_i^1[\bar{x},\varphi])).$$

From assumption (11) it follows that to the previously fixed number $\bar{\varepsilon} > 0$ we can choose an $N'_{\bar{\varepsilon}}$ such that for $p > N'_{\bar{\varepsilon}}$ the following inequality holds

$$(21)\qquad \|\varphi_p(\hat{f}_i^1[x,\varphi_p]) - \varphi(\hat{f}_i^1[x,\varphi_p])\| < \frac{\bar{\varepsilon}}{k_1}\,, \qquad k_1 > 0.$$

Let us note, moreover, that by a procedure similar to the proof of inequality (10) we deduce from conditions (11), (19) and assumption 4^0(a), for all $x \in X$, $\bar{x} \in X_{\varepsilon}$, the inequality

$$(22)\qquad \|\varphi_p(\hat{f}_i^2[x,\varphi_p]) - \varphi(\hat{f}_i^2[\bar{x},\varphi])\| \leqslant 2\omega(\varrho(x,\bar{x})),$$

$$i = 1,2,\dots,n, \quad p > N''_{\bar{\varepsilon}}.$$

To the number $\zeta = \omega^{-1}\left(\frac{\bar{\varepsilon}}{k}\right)$ we can choose, by assumption 4^0(b), a number $k_{\zeta} = \omega^{-1}\left(\frac{\bar{\varepsilon}}{k_1}\right)$ such that for all the $x,\bar{x} \in X$ for which $\varrho(x,\bar{x}) \geqslant \omega^{-1}\left(\frac{\bar{\varepsilon}}{k}\right)$ and (22) are satisfied the following inequality holds

$$\varrho(\hat{f}_i^1[x,\varphi_p],\ \hat{f}_i^1[\bar{x},\varphi]) \geqslant \omega^{-1}\left(\frac{\bar{\varepsilon}}{k_1}\right).$$

From this inequality and from the fact that ω is a monotone increasing function it follows that

$$\frac{\bar{\varepsilon}}{k_1} \leqslant \omega(\varrho(\hat{f}_i^1[x,\varphi_p],\ \hat{f}_i^1[\bar{x},\varphi])).$$

Combining this with inequality (21) we get

$$(23)\qquad \|\varphi_p(\hat{f}_i^1[x,\varphi_p]) - \varphi(\hat{f}_i^1[x,\varphi_p])\| <$$

$$< \omega(\varrho(\hat{f}_i^1[x,\varphi_p],\ \hat{f}_i^1[\bar{x},\varphi])) \quad \text{for} \quad i=1,\dots,n,\ x\in X,\ \bar{x}\in X_\varepsilon.$$

Using (20) and (23) we obtain for $p > \max(N'_{\bar{\varepsilon}},\ N''_{\bar{\varepsilon}})$ the relation

$$(24)\quad \|\varphi_p(\hat{f}_i^1[x,\varphi_p]) - \varphi(\hat{f}_i^1[\bar{x},\varphi])\| < 2\omega(\varrho(\hat{f}_i^1[x,\varphi_p],\ \hat{f}_i^1[\bar{x},\varphi])),$$

which is true for $x \in X$, $\bar{x} \in X_\varepsilon$, $i = 1,\dots,n$.

From relations (19) and (24) we conclude, by virtue of assumption 3^0, that for $p > \max(N_{\bar{\varepsilon}},\ N'_{\bar{\varepsilon}},\ N''_{\bar{\varepsilon}})$ the following inequality holds

$$\|h(x,\varphi_p(x),\varphi_p(\hat{f}_1^1[x,\varphi_p]),\dots,\varphi_p(\hat{f}_n^1[x,\varphi_p]))-h(\bar{x},\varphi(\bar{x}),\varphi(\hat{f}_1^1[\bar{x},\varphi]),$$

$$,\dots,\varphi(\hat{f}_n^1[\bar{x},\varphi]))\| \leqslant \omega(\varrho(x,\bar{x})),\qquad x \in X,\qquad \bar{x} \in X_\varepsilon,$$

whence, taking into account (13), we get the following estimate

$$(25)\quad \|h(x,\varphi_p(x),\varphi_p(\hat{f}_1^1[x,\varphi_p]),\dots,\varphi_p(\hat{f}_n^1[x,\varphi_p])) -$$

$$- h(\bar{x},\varphi(\bar{x}),\varphi(\hat{f}_1^1[\bar{x},\varphi]),\dots,\ \varphi(\hat{f}_n^1[\bar{x},\varphi]))\| < \frac{\bar{\varepsilon}}{2}\,.$$

Finally, by inequalities (12), (15), (16) and (25) we obtain

$$\|\psi_p(x) - \psi(x)\| < \varepsilon \qquad \text{for} \quad x \in X,\quad p > \max(N_{\bar{\varepsilon}},N'_{\bar{\varepsilon}},N''_{\bar{\varepsilon}}).$$

Thus the inequality $|||\psi_p - \psi||| < \varepsilon$ holds for any $\varepsilon > 0$ and for $p > \max(N_{\bar{\varepsilon}},N'_{\bar{\varepsilon}},N''_{\bar{\varepsilon}})$; that is to say, the operation T defined by equality (9) is continuous.

To prove that the assumptions of Schauder s theorem are satisfied, we have merely to show that the set A is compact.

To this purpose we have to show that for any $x \in X$ the set $B_x := \bigcup_{\varphi \in A} \{\varphi(x)\}$ is compact in Y. Let $y_1, y_2 \in B_x$. From the definition of the set B_x it follows that

$$y_1 = \varphi_1(x), \quad y_2 = \varphi_2(x), \quad \varphi_1, \varphi_2 \in A.$$

By inequality (7) and assumption 1^0 we have

$$\| y_1 - y_2 \| = \| \varphi_1(x) - \varphi_2(x) \| \leqslant 2\omega(\varrho(x, \xi)).$$

From this inequality it follows that the set B_x is bounded. Hence the set B_x is compact in Y. By Theorem 10.1 of [7], p.164, the set A is compact in $\mathcal{F}$. The set A, being a closed subset of the set $\mathcal{F}$, is compact.

Since all the assumptions of Schauder's theorem are satisfied, there exists at least one solution $\varphi : X \to Y$ of equation (1) satisfying conditions (7), which completes the proof of our theorem.

E x a m p l e . Consider the equation

$$\varphi(x) = \frac{251}{512} x - \frac{3}{512} \sin \frac{1}{2} x + \frac{1}{64} \varphi(x) + \frac{1}{32} \varphi\left(\frac{1}{8} x + \frac{3}{8} \sin \varphi(x)\right),$$

where φ is the unknown function, $x \in X := \langle -1, 1 \rangle$. Assume $Y := (-\infty, \infty)$, $\omega : \langle 0,2 \rangle \to \langle 0,2 \rangle$, $\omega(t) = t$. The given functions involved in the equation: $h : X \times Y^2 \to Y$, $f_1^1 : X \times Y \to X$ are defined as follows

$$h(x, y_0, y_1) = \frac{251}{512} x - \frac{3}{512} \sin \frac{1}{2} x + \frac{1}{64} y_0 + \frac{1}{32} y_1, \quad x \in X, \ y_0, y_1 \in Y,$$

$$f_1^1(x, u) = \frac{1}{8} x + \frac{3}{8} \sin u, \quad x \in X, \ u \in Y$$

and are continuous in their domains of definition and satisfy assumptions 1^o, 2^o of our theorem.

Let us show that the remaining assumptions of the theorem are also satisfied. Now, for all $x,\bar{x} \in X$, $y_i,\bar{y}_i \in Y$, $i=0,1$ such that

$$|y_i - \bar{y}_i| \leqslant 2\,|x - \bar{x}|, \qquad (i=0,1)$$

we have the inequality

$$|h(x,y_0,y_1) - h(\bar{x},\bar{y}_0,\bar{y}_1)| \leqslant \frac{601}{1024}\,|x - \bar{x}| .$$

Hence the function h satisfies the conditions of assumption 3^o of the theorem. Moreover, let us note that for arbitrary $x,\bar{x} \in X$, $u,\bar{u} \in Y$ the following estimate holds

$$|f_1^1(x,u) - f_1^1(\bar{x},\bar{u})| \leqslant \frac{1}{8}\,|x - \bar{x}| + \frac{3}{8}\,|u - \bar{u}|.$$

Hence, using the condition $|u - \bar{u}| \leqslant 2\,|x - \bar{x}|$, we get

$$|f_1^1(x,u) - f_1^1(\bar{x},\bar{u})| \leqslant \frac{7}{8}\,|x - \bar{x}|.$$

The function f_1^1 satisfies the conditions of assumption 4^o(a).

Let us prove that assumption 4^o(b) is also satisfied. To any real number $\zeta > 0$ there exists a positive number k_ζ (say $k_\zeta = \frac{5}{8}\zeta$) such that for all $x,\bar{x} \in X$, $u,\bar{u} \in Y$ such that

$$|x - \bar{x}| \geqslant \zeta, \quad |u - \bar{u}| \leqslant 2\,|x - \bar{x}|$$

the following inequality holds

$$|f_1^1(x,u) - f_1^1(\bar{x},\bar{u})| \geqslant \left|\frac{1}{8}\,|x - \bar{x}| - \frac{3}{8}\,|u - \bar{u}|\right| \geqslant \frac{5}{8}\,|x - \bar{x}| \geqslant \frac{5}{8}\zeta.$$

Hence the given functions f_1^1, h, involved in the considered equation, satisfy the assumptions of the theorem.

It is easy to see that the function $\varphi : X \to Y$ defined by the formula $\varphi(x) = \frac{1}{2}x$ is in the set X a solution of the considered equation and that it satisfies conditions (7).

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