DEMONSTRATIO MATHEMATICA
Vol. XV No 2 1982

G.D. Awasthi, A.K. Shukla

ON SOME CURVE IN A SUBSPACE OF A FINSLER SPACE

1. Introduction

Consider a subspace F_m with coordinates u^α $(\alpha=1,\dots,m)$ and metric tensor $g_{\alpha\beta}(u,u')$ imbedded in a Finsler space F_n with coordinates $x^i(i=1,\dots,n)$ and metric tensor $g_{ij}(x,x')$, then we have [5]:

$$g_{\alpha\beta}(u,u') = g_{ij}(x,x')X^i_\alpha X^j_\beta; \tag{1.1}$$

where

$$X^i_\alpha = \partial x^i/\partial u^\alpha.$$

In a Finsler space there exist two sets of (n-m) dimensional normals, one $n^i_{(\mu)}$ $(\mu = m+1,\dots,n)$[1] independent of x'^μ and the other $\overset{*}{n}{}^i_{(\mu)}$ depending on x'^i. The secondary normals $\overset{*}{n}{}^i_{(\mu)}$ satisfy the following relations [2]:

$$g_{ij}\, \overset{*}{n}{}^j_{(\mu)}\, X^i_\alpha = \overset{*}{n}_{(\mu)i}\, X^i_\alpha = 0, \tag{1.2}$$

$$g_{ij}\, \overset{*}{n}{}^i_{(\mu)}\, \overset{*}{n}{}^j_{(\mu)} = 1, \tag{1.3}$$

---

[1] Early Greek indices α, β, γ etc., take the values from $1,\dots,m$; later Greek indices μ, ν, τ etc., take the values from $m+1,\dots,n$ and Latin indices i,j,k, etc., take the values from $1,\dots,n$.

(1.4) $$g_{ij}\, n^{*i}_{(\mu)}\, n^{*j}_{(\nu)} = \Phi_{(\mu)}\, \delta^{\mu}_{\nu},$$

and

(1.5) $$X^{i}_{\alpha\beta} = \sum_{\mu} \Omega^{*}{}_{(\mu)\alpha\beta}\, n^{*i}_{(\mu)}$$

where $\Omega^{*}{}_{(\mu)\alpha\beta}$ are the secondary second fundamental tensors of the subspace.

Consider now a set of (n-m) congruences $\lambda^{*i}_{(\mu)}$, of curves such that one curve of each congruence passes through each point of the subspace F_m. If $\lambda^{*i}_{(\mu)}$ can be expressed as:

(1.6) $$\lambda^{*i}_{(\mu)} = t^{*\alpha}_{(\mu)} X^{i}_{\alpha} + \sum_{\nu} C^{*}_{(\mu\nu)} n^{*i}_{(\nu)},$$

then we have

(1.7) $$\cos\theta^{*}{}_{(\mu\nu)} = C^{*}_{(\mu\nu)}$$

and

(1.8) $$1 - t^{*\alpha}_{(\mu)} t^{*}_{(\mu)\alpha} = \cos^{2}\theta^{*}{}_{(\mu\nu)},$$

where $\theta^{*}{}_{(\mu\nu)}$ is the angle between the vectors whose contravariant components are $\lambda^{*i}_{(\mu)}$ and $n^{*i}_{(\nu)}$.

A curve C in F_m which is imbedded in F_n has m-1 curvatures and m-1 normals relative to F_m and n-1 curvatures and n-1 normals relative to F_n. Let curvatures and unit normals relative to F_m be denoted by $K'_{(r)}$ $(r = 1,\dots,m-1)$ and $\xi^{\alpha}_{(r)}$ $(r = 2,\dots,m)$ and those relative to F_n by $K_{(r)}$ $(r = 1,\dots,n-1)$ and $\eta^{i}_{(r)}$, $(r = 2,\dots,n)$ respectively. The unit vector tangent to the curve C is denoted by $\eta^{i}_{(1)}$ or $\xi^{\alpha}_{(1)}$ according to that, if it is regarded as a vector in F_n or in F_m.

2. C_B-curves in a Finsler space

First of all we shall define C_B-curves as the curves of the subspace having the property that at every point, its normal geodesic surface (formed by first and second normal) contains the tangent vector to a curve of the congruence of curves through that point. Next we define C_B-curves of order p as the curves of the subspace having the property that at every point, its normal geodesic surface (formed by first and p+1-th normal) contains the tangent vector to a curve of the congruence of curves through that point. If the vector with contravariant components $\lambda^{*i}_{(\mu)}$ lies in this geodesic surface, then we have

$$\lambda^{*i}_{(\mu)} = a_{(\mu)}\eta^{i}_{(2)} + b_{(\mu)}\eta^{i}_{(p+2)}, \tag{2.1}$$

where $a_{(\mu)}$ and $b_{(\mu)}$ are to be determined.
From (1.6) and (2.1), we obtain

$$t^{*\alpha}_{(\mu)}X^{i}_{\alpha} + \sum_{\nu} C^{*}_{(\mu\nu)}n^{*i}_{(\nu)} = a_{(\mu)}\eta^{i}_{(2)} + b_{(\mu)}\eta^{i}_{(p+2)}. \tag{2.2}$$

If we assume that the curve C is an asymptotic curve of order p in a Finsler space [4], then we have

$$\eta^{i}_{(p+1)} = \xi^{\alpha}_{(p+1)}X^{i}_{\alpha}. \tag{2.3}$$

Differentiating (2.3) w.r.t. the arc length of the curve C we obtain the equation

$$K_{(p+1)}\eta^{i}_{(p+2)} = K'_{(p+1)}\xi^{\alpha}_{(p+2)}X^{i}_{\alpha} + \sum_{\mu} \Omega^{*}_{(\mu)\alpha\beta}\xi^{\alpha}_{(1)}\xi^{\beta}_{(p+1)}\eta^{*i}_{(\mu)} \tag{2.4}$$

which can be written in the form

$$K_{(p+1)}\eta^{i}_{(p+2)} = K'_{(p+1)}\xi^{\alpha}_{(p+2)}X^{i}_{\alpha} + \sum_{\mu} K_{(\mu)(p+1)}n^{*i}_{(\mu)}, \tag{2.5}$$

where

$$K_{(\mu)(p+1)} = \Omega^*_{(\mu)\alpha\beta}\xi^{\alpha}_{(1)}\xi^{\beta}_{(p+1)}. \tag{2.6}$$

By virtue of (2.3) and (2.5), the equation (2.2) takes the form

$$t^{*\alpha}_{(\mu)}X^i_\alpha + \sum_\nu C^*_{(\mu\nu)}n^{*i}_{(\nu)} = a_{(\mu)}\xi^\alpha_{(2)}X^i_\alpha + \tag{2.7}$$
$$+ C_{(\mu)}(K'_{(p+1)}\xi^\alpha_{(p+2)}X^i_\alpha + \sum_\nu K_{(\nu)(p+1)}n^{*i}_{(\nu)}),$$

where $C_{(\mu)} = b_{(\mu)}/K_{(p+1)}$. Multiplying (2.7) by $g_{ij}X^j_\beta$ and using (1.1) and (1.2) we obtain

$$g_{\alpha\beta}t^{*\alpha}_{(\mu)} = a_{(\mu)}g_{\alpha\beta}\xi^\alpha_{(2)} + C_{(\mu)}K'_{(p+1)}g_{\alpha\beta}\xi^\alpha_{(p+2)}. \tag{2.8}$$

Again multiplying (2.7) by $g_{ij}(x,x')n^{*j}_{(2)}$ and summing on i, we get

$$C^*_{(\mu\nu)} = C_{(\mu)}K_{(\nu)(p+1)}. \tag{2.9}$$

Now multiplying (2.8) by $\xi^\beta_{(2)}$ summing on β, and using the relations

$$\begin{cases} g_{\alpha\beta}\xi^\alpha_{(2)}\xi^\beta_{(2)} = 1, \\ g_{\alpha\beta}\xi^\alpha_{(2)}\xi^\beta_{(p+2)} = 0 \end{cases} \tag{2.10}$$

we get

$$g_{\alpha\beta}t^{*\alpha}_{(\mu)}\xi^\beta_{(2)} = a_{(\mu)}. \tag{2.11}$$

From (2.8), (2.9) and (2.11) we now obtain the equation

$$(2.12)\qquad g_{\alpha\beta}\overset{*}{t}{}^{\alpha}_{(\mu)}K_{(\nu)(p+1)} =$$

$$= g_{\alpha\beta}\xi^{\alpha}_{(2)}K_{(\nu)(p+1)}g_{\gamma\delta}\overset{*}{t}{}^{\gamma}_{(\mu)}\xi^{\delta}_{(2)} + \overset{*}{C}_{(\mu\nu)}K'_{(p+1)}g_{\alpha\beta}\xi^{\alpha}_{(p+2)}$$

which on multiplication by $g^{\beta\varepsilon}$ and summing on β yields

$$(2.13)\qquad \overset{*}{C}_{(\mu\nu)}K'_{(p+1)}\xi^{\varepsilon}_{(p+2)} -$$

$$- K_{(\nu)(p+1)}\left(\overset{*}{t}{}^{\varepsilon}_{(\mu)} - g_{\gamma\delta}\overset{*}{t}{}^{\gamma}_{(\mu)}\xi^{\delta}_{(2)}\xi^{\varepsilon}_{(2)}\right) = 0.$$

Equation (2.13) represents the differential equation of C_B-curves of order p in F_m. For a congruence specified by the parameters $\overset{*}{t}{}^{\alpha}_{(\mu)}$, the solution of (2.13) determines the C_B-curves of order p in F_m relative to that congruence. Denoting the L.H.S. of (2.13) by

$$(2.14)\qquad \overset{*}{\gamma}{}^{\varepsilon}_{(p)} = \overset{*}{C}_{(\mu\nu)}K'_{(p+1)}\xi^{\varepsilon}_{(p+2)} - K_{(\nu)(p+1)}\overset{*}{B}{}^{\varepsilon}_{(\mu)},$$

where

$$(2.15)\qquad \overset{*}{B}{}^{\varepsilon}_{(\mu)} = \left(\overset{*}{t}{}^{\varepsilon}_{(\mu)} - g_{\gamma\delta}\overset{*}{t}{}^{\gamma}_{(\mu)}\xi^{\delta}_{(2)}\xi^{\varepsilon}_{(2)}\right)$$

and calling $\overset{*}{\gamma}{}^{\varepsilon}_{(p)}$ the contravariant components of the curvature vector of C_B-curves of order p we define these curves as follows.

A C_B-curve of order p of F_m w.r.t. a congruence determined by the parameters $\overset{*}{t}{}^{\varepsilon}_{(\mu)}$ is a curve along which the curvature vector $\overset{*}{\gamma}{}^{\varepsilon}_{(p)}$ is a null vector.

If in particular $p = 1$, then (2.13) reduces to the equation

$$(2.16)\quad C^{*}_{(\mu\nu)}K'_{(2)}\xi^{\varepsilon}_{(3)} - K_{(\nu)(2)}\left(t^{*\varepsilon}_{(\mu)} - g_{\gamma\delta}\, t^{*\gamma}_{(\mu)}\xi^{\delta}_{(2)}\xi^{\varepsilon}_{(2)}\right) = 0$$

which we call the differential equation of C_B-curves in F_m.

3. Some special properties

From (2.14) it follows that if $K_{(\nu)\|(p+1)} = 0$ (i.e., the curve C is an asymptotic curve of order p+1, [4]) and $\gamma^{*\varepsilon}_{(p)} = 0$, then we have either $K'_{(p+1)} = 0$ (i.e., the curve C is a geodesic of order p+1, [4]) or the vector with components $\xi^{\varepsilon}_{(p+2)}$ is a null vector. Hence we have the following theorem.

Theorem 3.1. A necessary and sufficient condition for a C_B-curve of order p to be an asymptotic curve of order p+1 is one of the following:

(i) it is a geodesic of order p+1 in F_m;
(ii) the vector with components $\xi^{\varepsilon}_{(p+2)}$ is a null vector;
(iii) the congruence is formed of tangential vectors only.

In addition to this, if $K'_{(p+1)} = 0$ and $\gamma^{*\varepsilon}_{(p)} = 0$, we get either $K_{(\nu)(p+1)} = 0$ or $B^{*\varepsilon}_{(\mu)} = 0$. Hence we have

Theorem 3.2. A necessary and sufficient condition for a C_B-curve of order p to be a geodesic of order p+1 in F_m is one of the following:

(i) it is an asymptotic curve of order p+1,
(ii) the vector with components $B^{*\varepsilon}_{(\mu)}$ is a null vector.

Let us denote by $K_{B(p)}$ the curvature of the C_B-curve of order p, then we have from (2.14) the following relation

$$(3.1)\quad K_{B(p)} = K'_{(p+1)}C^{*}_{(\mu\nu)} \pm K_{(\nu)(p+1)}\left(1 - \sum_{\nu} \cos^2\theta^{*}_{(\mu\nu)}\right)^{1/2} \cdot$$

$$\cdot \sin\beta_{(\mu)},$$

where $\beta_{(\mu)}$ is the angle between vectors $t^{*\alpha}_{(\mu)}$ and $\xi^{\beta}_{(2)}$. The positive and negative signs are to be taken according as the angle between $\overset{*}{t}{}^{\alpha}_{(\mu)}$ and $\xi^{\beta}_{(p+2)}$ is $90+\beta_{(\mu)}$ or $90-\beta_{(\mu)}$ respectively. From (3.1) we can easily obtain:

T h e o r e m 3.3. A necessary and sufficient condition for a C_B-curve of order p to be a geodesic of order p+1 in F_m is one of the following:

(i) the curve is an asymptotic curve of order p+1,

(ii) the congruence is that of normals only,

(iii) the vectors $\overset{*}{t}{}^{\alpha}_{(\mu)}$ and $\xi^{\beta}_{(2)}$ are parallel.

In particular for a congruence of tangential vectors equation (3.1) takes the form

$$K_{B(p)} = \pm K_{(\nu)(p+1)} \sin B_{(\mu)} \tag{3.2}$$

which by virtue of the relation [4]:

$$K^2_{(p+1)} = K'^2_{(p+1)} + K^2_{(\nu)(p+1)} \tag{3.3}$$

yields

$$K^2_{(p+1)} = K'^2_{(p+1)} + K^2_{B(p)} \operatorname{cosec}^2\beta_{(\mu)}. \tag{3.4}$$

Hence we have the following theorem.

T h e o r e m 3.4. For the congruence of tangential vectors, C_B-curves of order p are geodesic of order p+1 in both F_m and F_n simultaneously, provided the vectors $\overset{*}{t}{}^{\alpha}_{(\mu)}$ and $\xi^{\beta}_{(2)}$ are not parallel.

Again equation (3.1) for a normal congruence reduces to

$$K_{B(p)} = K'_{(p+1)} \tag{3.5}$$

which by virtue of (3.3) yields

$$K^2_{(p+1)} = K^2_{B(p)} + K^2_{(\nu)(p+1)}. \tag{3.6}$$

Hence we have

T h e o r e m 3.5. For the congruence of normals, C_B-curves of order p are asymptotic curves of order p+1 and geodesic of order p+1 in F_n simultaneously.

Again if the vectors $t^{*\alpha}_{(\mu)}$ and $\xi^{\beta}_{(2)}$ are parallel we have

$$K^2_{(p+1)} = K^2_{(\nu)(p+1)} + K^2_{B(p)} \sec^2\theta^*_{(\mu\nu)}. \tag{3.7}$$

Thus we get

T h e o r e m 3.6. If the vectors $t^{*\alpha}_{(\mu)}$ and $\xi^{\beta}_{(2)}$ are parallel, then a necessary and sufficient condition for a C_B-curve of order p to be geodesic of order p+1 in F_n is that it has to be an asymptotic curve of order p+1, provided the congruence is formed not of tangential vectors only.

REFERENCES

[1] P.B. B h a t t a c h a r y a : Curves on the surface of reference whose normal plane at any point contains the ray of the congruence through that point., Ganita (1950).

[2] R.S. M i s h r a , R.S. S i n h a : Union and hyper-asymptotic curves of a Finsler subspace and hypersurface, Rend. Circ. Mat. Palermo (2) 14 (1965).

[3] S.C. R a s t o g i , H.K.N. T r i v e d i : On a curve in a subspace of a Riemannian space (accepted for publication in (1970) Comptes rendus de l'Academic Bulgare des Sciences).

[4] S.C R a s t o g i : Generalised curves in a Finsler space (accepted for publication in Univ. Nac. Tucuman, Argentina, 21 (1971).

[5] H. R u n d : The differential geometry of Finsler spaces, Berlin (1959).

DEPARTMENT OF MATHEMATICS, LUCKNOW UNIVERSITY, LUCKNOW, INDIA
Received January 28, 1980, revised version January 12. 1982.