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ON THE SINGULAR POINTS OF PRINGSHEIM-DU BOIS REYMOND OF A FUNCTION OF TWO VARIABLES

Introduction

Let $F(x,y)$ be a function of class C^{∞} in R^2. Then, for any point $(x,y) \in R^2$ we can write the equality

$$(*)\quad T_F(x,y;h,k) = F(x,y) + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{r=0}^{n} \binom{n}{r} \frac{\partial^n F(x,y)}{\partial x^r \partial y^{n-r}} h^r k^{n-r}.$$

The following cases are possible:

a) There exists a number $R(x,y) > 0$ such that for $|h| < R(x,y)$, $|k| < R(x,y)$ the series $(*)$ is absolutely convergent and there exists a number $\delta(x,y) \in (0,R(x,y)>$ such that for $|h| < \delta(x,y)$, $|k| < \delta(x,y)$ we have

$$T_F(x,y;h,k) = F(x+h,\ y+k).$$

Then the point (x,y) is said to be a regular point of the function F.

b) The number $R(x,y)$ quoted under a) exists, but the number $\delta(x,y)$ does not, i.e. any open disk of center (x,y) contains a point $(x+h',y+k')$ such that $T_F(x,y;h',k') \neq$ $\neq F(x+h',y+k')$. In this case the point (x,y) is called a singular point of Cauchy (C-singular point).

c) The number $R(x,y)$ quoted under a) and b) does not exist. Then the point (x,y) is said to be a singular point of Pringsheim-Du Bois Reymond (P-singular point).

As is well known [1], the point (x,y) is a P-singular point of the function F if and only if

$$\lambda(x,y) = \limsup_{m\to\infty} \sqrt[m]{\frac{1}{m!} \sum_{r=0}^{m} \binom{m}{r} \frac{\partial^m F(x,y)}{\partial x^r \partial y^{m-r}}} = +\infty.$$

The singular points of a function of one or several variables are defined in a similar manner. Zahorski [2] proved that for a function of one variable of class C^∞ the set $P \subset R$ is the set of all its P-singular points and the set $C \subset R$ is the set of all its C-singular points if and only if

(I) P is of class G_δ and C is of class F_σ and of first category,

(II) $P \cup C = \overline{P \cup C}$, $P \cap C = \emptyset$.

On the other hand, it is known [3] that the conditions (I) and (II) are necessary for a function of several variables of class C^∞, and that if P is a void set, then they are also sufficient. A full characterization of the pair of sets P and C is unknown till now, even for $m = 2$, that is we do not know whether the conditions (I) and (II) are sufficient (when both sets P and C are non-void). The method used by Zahorski in constructing the desired function of one variable is long and difficult and, moreover, it cannot be generalized to functions of several variables. There are two reasons for this being so: 1) closed sets have even in R^2 a more composed structure than on the straight line, so that lemmas which give a topological basis for the construction of the needed function are false in R^2; 2) Zahorski based his construction on the approximation theorem of Runge, to which there is no analogue for the multidimensional space. Making use of a theorem of Whitney [4] and of the quoted result of Zahorski it can be proved that for any closed set P lying in R^m there exists a function $F(x_1,\dots,x_m)$ of class C^∞ in R^m for which any point of the set P is P-singular and any point of the set $R^m \setminus P$ is regular. However, such proof for the sufficiency of conditions (I) and (II) in the case where C is a void set does not point any way to be followed in the general case (when both sets P and C are non-void).

In this paper we shall prove that for $m = 2$ closedness gives a full characterization of the set of P-singularities in the class of functions which have no C-singularities; the method used to this purpose

- is almost elementary,
- does not make use of the theorems of Whitney, Zahorski and Runge,
- includes the analogous result of Zahorski and may be considered as a simplification in the one-dimensional case,
- can be extended to the case R^m $(m \geqslant 3)$,
- may possibly be applied to the general construction (when both sets P and C are non-void).

1. Let

$$K_1, K_2, \ldots, K_n, \ldots \tag{1}$$

be the sequence, defined in paper [5], § 4, of closed squares corresponding to the closed set $P \subset R^2$. Let (a_n, b_n) be the center and $2l_n$ the length of the side of the square K_n. Consider an arbitrary sequence $\{M_n\}$ of real numbers $M_n \geqslant 1$. Assume, as in paper [5] (formulas (17)), that

$$r_n = 20^n, \quad C_n = M_n^2 \cdot 10^{2n+1}/l_n^2, \; B_n = M_n \cdot 10^n/l_n^2, \; L_n = l_n + l_n^2 \tag{2}$$

and consider the entire functions $f_n(x,y)$, related to the squares K_n, defined by formula (18) of paper [5]:

$$f_n(x,y) =$$
$$= (M_n^2/C_n^{8r_n})\tilde{U}(x-a_n, B_n, L_n)\tilde{U}(y-b_n, B_n, L_n)\cos C_n(x-a_n)\cos C_n(y-b_n)$$

and let

(3) $$F(x,y) = \sum_{n=1}^{\infty} f_n(x,y).$$

At last we set

(4) $$a = 1 - (2/100\sqrt{\pi})\exp(-100), \quad b = 2e^{-20}/100\sqrt{\pi}.$$

L e m m a 1. If $k \geqslant 2$, then the following inequalities hold at any point $(x,y) \in K_k$:

(5) $$\left|\frac{\partial^{8r_k} f_k(x,y)}{\partial x^{4r_k}\partial y^{4r_k}}\right| > M_k^2(|\cos C_k(x-a_k)|\, a-b)(|\cos C_k(y-b_k)|\, a-b)$$

for $|\cos C_k(x-a_k)| \geqslant 1/\sqrt{2}$ and $|\cos C_k(y-b_k)| \geqslant 1/\sqrt{2}$,

(6) $$\left|\frac{\partial^{8r_k+1} f_k(x,y)}{\partial x^{4r_k+1}\partial y^{4r_k}}\right| > M_k^2 C_k(|-\sin C_k(x-a_k)|\, a-b)(|\cos C_k(y-b_k)|\, a-b)$$

for $|-\sin C_k(x-a_k)| \geqslant 1/\sqrt{2}$ and $|\cos C_k(y-b_k)| \geqslant 1/\sqrt{2}$,

(7) $$\left|\frac{\partial^{8r_k+1} f_k(x,y)}{\partial x^{4r_k}\partial y^{4r_k+1}}\right| > M_k^2 C_k(|\cos C_k(x-a_k)|\, a-b)(|-\sin C_k(y-b_k)|\, a-b)$$

for $|\cos C_k(x-a_k)| \geqslant 1/\sqrt{2}$ and $|-\sin C_k(y-b_k)| \geqslant 1/\sqrt{2}$,

(8) $$\left|\frac{\partial^{8r_k+2} f_k(x,y)}{\partial x^{4r_k+1}\partial y^{4r_k+1}}\right| > M_k^2 C_k^2(|-\sin C_k(x-a_k)|\, a-b)(|-\sin C_k(y-b_k)|\, a-b)$$

for $|-\sin C_k(x-a_k)| \geqslant 1/\sqrt{2}$ and $|-\sin C_k(y-b_k)| \geqslant 1/\sqrt{2}$.

P r o o f . We start by observing that for any natural number $k \geqslant 2$ we have the inequality

(9) $$(2\cdot 20^k+1)\ln(4\cdot 20^k+1) + 4\cdot 20^k \ln 1{,}3 - 10^{2k} < -20^k,$$

easy to verify by induction. If $(x,y) \in K_k$, then $(x-a_k) \in \in \langle -l_k, l_k \rangle = \langle -L_k+h_k, L_k-h_k \rangle$, $(h_k=l_k^2)$ and, similarly, $(y-b_k) \in \langle -L_k+h_k, L_k-h_k \rangle$; hence, according to the inequality (9) of [5], we have

$$\tilde{U}(x-a_k,B_k,L_k) \geqslant 1 - (2/\sqrt{\pi}B_k h_k)\exp(-B_k^2 h_k^2) =$$

$$= 1 - (2/M_k\cdot 10^k\sqrt{\pi})\exp(-M_k^2\cdot 10^{2k}) > a.$$

Therefore

(10) $$\tilde{U}(x-a_k,B_k,L_k) > a \quad \text{and} \quad \tilde{U}(y-b_k,B_k,L_k) > a.$$

From the condition $(x,y) \in K_k$ it follows that

(11) $$|x-a_k \pm L_k| \geqslant l_k^2 \quad \text{and} \quad |y-b_k \pm L_k| \geqslant l_k^2.$$

From [5], form.(14), we get

$$\frac{\partial^{8r_k} f_k(x,y)}{\partial x^{4r_k}\,\partial y^{4r_k}} =$$

$$=M_k^2(\cos C_k(x-a_k)\tilde{U}(x-a_k,B_k,L_k)+E_{1,4r_k})\cdot(\cos C_k(y-b_k)\cdot$$

$$\cdot\tilde{U}(y-b_k,B_k,L_k)+E_{2,4r_k}),$$

$$\frac{\partial^{8r_k+1} f_k(x,y)}{\partial x^{4r_k+1}\,\partial y^{4r_k}} = M_k^2(-C_k\sin C_k(x-a_k)\tilde{U}(x-a_k,B_k,L_k)+E_{1,4r_k+1})\cdot$$

$$\cdot\ (\cos C_k(y-b_k)\tilde{U}(y-b_k,B_k,L_k)+E_{2,4r_k}),$$

$$\frac{\partial^{8r_k+1} f_k(x,y)}{\partial x^{4r_k}\,\partial y^{4r_k+1}} = M_k^2(\cos C_k(x-a_k)\tilde{U}(x-a_k,B_k,L_k)+E_{1,4r_k})\cdot$$

$$\cdot\ (-C_k\sin C_k(y-b_k)\tilde{U}(y-b_k,B_k,L_k)+E_{2,4r_{k+1}}),$$

$$\frac{\partial^{8r_k+2} f_k(x,y)}{\partial x^{4r_k+1}\,\partial y^{4r_k+1}} = M_k^2(-C_k\sin C_k(x-a_k)\tilde{U}(x-a_k,B_k,L_k)+E_{1,4r_k+1})\cdot$$

$$\cdot\ (-C_k\sin C_k(y-b_k)\tilde{U}(y-b_k,B_k,L_k)+E_{2,4r_k+1});$$

According to inequality (14a) of [5] we obtain for $p = 4r_k+1$

$$|E_{1,4r_k+1}| \leq \frac{B_k}{\sqrt{\pi}}\cdot\frac{(4r_k+1)!}{(2r_k)!}\,((1+(B_k/C_k)+(2B_k^2/C_k)\,|x-a_k+L_k|)^{4r_k}\cdot$$

$$\cdot\ \exp(-B_k^2(x-a_k+L_k)^2)+(1+(B_k/C_k)+(2B_k^2/C_k)\,|x-a_k-L_k|)^{4r_k}\cdot$$

$$\cdot\ \exp(-B_k^2(x-a_k-L_k)^2)) := \bar{E}_{1,4r_k+1}.$$

From the same inequality we draw for $p = 4r_k$

$$\left|E_{1,4r_k}\right| \leq \frac{B_k}{\sqrt{\pi}}\cdot\frac{(4r_k)!\,C_k^{-1}}{\left[\frac{4r_k-1}{2}\right]!}\,((1+(B_k/C_k)+(2B_k^2/C_k)\,|x-a_k+L_k|)^{4r_k-1}\cdot$$

$$\cdot\ \exp(-B_k^2(x-a_k+L_k)^2)+(1+(B_k/C_k)+(2B_k^2/C_k)\,|x-a_k-L_k|)^{4r_k-1}\cdot$$

$$\cdot\ \exp(-B_k^2(x-a_k-L_k)^2)) := \bar{E}_{1,4r_k}.$$

Since $p!/[(p-1)/2]!$ increases with p, we have

$$(1+(B_k/C_k)+(2B_k^2/C_k)|x-a_k\pm L_k|)^{4r_k-1} <$$

$$< (1+(B_k/C_k)+(2B_k^2/C_k)|x-a_k\pm L_k|)^{4r_k},$$

whence

$$\bar{E}_{1,4r_k} < \frac{1}{C_k}\bar{E}_{1,4r_k+1}. \tag{12}$$

We shall now find upper estimates for the expressions

$$|E_{1,4r_k}|, \quad |E_{1,4r_k+1}|, \quad |E_{2,4r_k}|, \quad |E_{2,4r_k+1}|.$$

By (12) it is clearly enough to find an upper estimate for $\bar{E}_{1,4r_k+1}$, from which the estimates for $E_{2,4r_k}$ and $E_{2,4r_k+1}$ will follow by replacing x by y and a_k by b_k. Let $p = 1+B_k/C_k$, $q = 2B_k^2/C_k$, $c = 4r_k$, $d = B_k^2$; then the positive function $g(t) = (p+qt)^c \cdot \exp(-dt^2)$, $t \geqslant 0$, takes an absolute maximum at the point

$$t=t_0 = \frac{-2pd+2(p^2d^2+2q^2cd)^{1/2}}{4qd} = (-p/2q)+(1/2)((p^2/q^2)+(2c/d))^{1/2} =$$

$$= -\frac{B_k+C_k}{4B_k^2} + \frac{1}{2}\left(\frac{(B_k+C_k)^2}{4B_k^4} + \frac{8r_k}{B_k^2}\right)^{1/2} \leqslant \frac{1}{B_k}(2r_k)^{1/2} =$$

$$= \frac{l_k^2}{M_k\cdot 10^k}(2\cdot 20^k)^{1/2} \leqslant l_k^2.$$

Since $g(t)$ decreases for $t > t_0$, we have $g(t) \leqslant g(l_k^2)$ for $t \geqslant l_k^2$. Hence, taking into account the inequalities (11), we can write

$$(1+(B_k/C_k)+(2B_k^2/C_k)\,|x-a_k\pm L_k|)^{4r_k}\exp(-B_k^2(x-a_k\pm L_k)^2)\leqslant$$

$$\leqslant(1+(B_k/C_k)+(2B_k^2l_k^2/C_k)^{4r_k}\cdot\exp(-B_k^2l_k^4)=$$

$$=(1+(1/M_k\cdot 10^{k+1})+(2/10))^{4\cdot 20^k}\cdot\exp(-10^{2k}\cdot M_k^2)<(1,3)^{4\cdot 20^k}\cdot e^{-10^{2k}}.$$

From this estimate and from inequality (9) it follows that

$$\bar{E}_{1,4r_k+1}<\frac{2B_k}{\sqrt{\pi}}\cdot\frac{(4r_k+1)!}{(2r_k)!}(1,3)^{4\cdot 20^k}\cdot e^{-10^{2k}}\doteq$$

$$=2C_k(4\cdot 20^k+1)!(1,3)^{4\cdot 20^k}\cdot e^{-10^{2k}}/\sqrt{\pi}M_k\cdot 10^{k+1}(2\cdot 20)^k)!<$$

$$<2C_k(4\cdot 20^k+1)!(1,3)^{4\cdot 20^k}\cdot e^{-10^{2k}}/100\sqrt{\pi}(2\cdot 20^k)!=$$

$$=\frac{2C_k}{100\sqrt{\pi}}(2\cdot 20^k+1)(2\cdot 20^k+2)\dots(4\cdot 20^k+1)(1,3)^{4\cdot 20^k}\cdot e^{-10^{2k}}<$$

$$<\frac{2C_k}{100\sqrt{\pi}}(4\cdot 20^k+1)^{2\cdot 20^k+1}(1,3)^{4\cdot 20^k}\cdot e^{-10^{2k}}=$$

$$=\frac{2C_k}{100\sqrt{\pi}}e^{(2\cdot 20^k+1)\ln(4+20^k+1)+4\cdot 20^k\cdot\ln 1,3-10^{2k}}<$$

$$<\frac{2C_k}{100\sqrt{\pi}}e^{-20^k}<C_k\cdot b\ .$$

Finally, we have $|E_{1,4r_k+1}|<C_k\cdot b$ and, by (12), $|E_{1,4r_k}|<b$. Since $|ab+c|\geqslant|ab|-|c|$, from the last inequalities, from the analogous inequalities for $E_{2,4r_k}$ and $E_{2,4r_k+1}$ and from (10) we get

$$\left|\frac{\partial^{8r_k} f_k(x,y)}{\partial x^{4r_k}\partial y^{4r_k}}\right| \geqslant M_k^2(|\cos C_k(x-a_k)|\,\tilde{U}(x-a_k,B_k,L_k) - |E_{1,4r_k}|)\,\cdot$$

$$\cdot\,(|\cos C_k(y-b_k)|\,\tilde{U}(y-b_k,B_k,L_k) - |E_{2,4r_k}|) >$$

$$> M_k^2(|\cos C_k(x-a_k)|\,a-b)(|\cos C_k(y-b_k)|\,a-b),$$

and the inequality (5) is thus established. In a similar manner we get

$$\left|\frac{\partial^{8r_k+1} f_k(x,y)}{\partial x^{4r_k+1}\partial y^{4r_k}}\right| > M_k^2(C_k|-\sin C_k(x-a_k)|\,a-bC_k)(|\cos C_k(y-b_k)|\,a-b),$$

whence the inequality (6) follows. The same argument leads to inequalities (7) and (8).

2. Lemma 2. For any sequence $\{A_n\}$ of real numbers $A_n \geqslant 1$ there exists a sequence $\{M_n\}$ of real numbers $M_n \geqslant 1$ such that at any point $(x,y) \in K_n$ the function $F(x,y)$ defined by formulas (2)-(4) satisfies for $n \geqslant 2$ at least one of the inequalities

(13a) $$\left|\frac{\partial^{8r_n} F(x,y)}{\partial x^{4r_n}\partial y^{4r_n}}\right| > A_n,$$ (13b) $$\left|\frac{\partial^{8r_n+1} F(x,y)}{\partial x^{4r_n+1}\partial y^{4r_n}}\right| > A_n,$$

(13c) $$\left|\frac{\partial^{8r_n+1} F(x,y)}{\partial x^{4r_n}\partial y^{4r_n+1}}\right| > A_n,$$ (13d) $$\left|\frac{\partial^{8r_n+2} F(x,y)}{\partial x^{4r_n+1}\partial y^{4r_{n+1}}}\right| > A_n.$$

Proof. Let $M_1 = 1$. Assume that we have already defined the numbers $M_1, M_2, \ldots, M_{k-1}$, all of them $\geqslant 1$. Then, according to the notations (2), the numbers

$$C_1,\ C_2,\dots,C_{k-1} \quad \text{and} \quad B_1,\ B_2,\dots,B_{k-1}$$

are also known, and so are the entire functions $f_1(x,y)$, $f_2(x,y),\dots,f_{k-1}(x,y)$, which are the terms of the series (4), $(L_j = l_j + l_j^2,\ r_j = 20^j)$. We shall show that it is possible to define the next term M_k.

Since the square K_k is closed, the numbers

$$J_{k,1} = \max_{(x,y)\in K_k} \left| \sum_{i=1}^{k-1} \frac{\partial^{8r_k} f_i(x,y)}{\partial x^{4r_k}\,\partial y^{4r_k}} \right|,$$

$$J_{k,2} = \max_{(x,y)\in K_k} \left| \sum_{i=1}^{k-1} \frac{\partial^{8r_k+1} f_i(x,y)}{\partial x^{4r_k+1}\,\partial y^{4r_k}} \right|,$$

$$J_{k,3} = \max_{(x,y)\in K_k} \left| \sum_{i=1}^{k-1} \frac{\partial^{8r_k+1} f_i(x,y)}{\partial x^{4r_k}\,\partial y^{4r_k+1}} \right|,$$

$$J_{k,4} = \max_{(x,y)\in K_k} \left| \sum_{i=1}^{k-1} \frac{\partial^{8r_k+2} f_i(x,y)}{\partial x^{4r_k+1}\,\partial y^{4r_k+1}} \right|$$

exist and are finite. If

$$\begin{cases} J_k := \max(J_{k,1},\ J_{k,2},\ J_{k,3},\ J_{k,4}), \\ H := \left(\dfrac{a}{\sqrt{2}} - b\right)^{-2} > 1, \end{cases}$$

where a, b are the numbers defined by formula (4), then the number

$$M_k = ((A_k + J_k + 1)H)^{1/2} > 1$$

satisfies the imposed condition. Indeed, by virtue of theorem 1 of paper [5] we have

$$(14) \qquad \frac{\partial^{p+q} F(x,y)}{\partial x^p \partial y^q} = \sum_{i=1}^{k-1} \frac{\partial^{p+q} f_i(x,y)}{\partial x^p \partial y^q} + \frac{\partial^{p+q} f_k(x,y)}{\partial x^p \partial y^q} + \sum_{i=k+1}^{\infty} \frac{\partial^{p+q} f_i(x,y)}{\partial x^p \partial y^q} .$$

It makes no harm that, here, only $f_1(x,y),\ldots,f_k(x,y)$ are exactly defined (fixed), whereas $f_s(x,y)$ for $s > k$ depend on the free parameter M_s, since Theorem 1 of [5] provides such a distribution independently of M_s under the condition that $M_s \geqslant 1$; this condition is indeed satisfied while the sequence $\{M_k\}$ is being defined by induction. For the same reason, according to Theorem 2 of [5], the last term in (14) satisfies at any point $(x,y) \in R^2$ the inequality

$$(15) \qquad \left| \sum_{i=k+1}^{\infty} \frac{\partial^{p+q} f_i(x,y)}{\partial x^p \partial y^q} \right| < 1$$

for each of the pairs: $p = 4r_k$, $q = 4r_k$; $p = 4r_k+1$, $q = 4r_k$; $p = 4r_k$, $q = 4r_k+1$; $p = 4r_k+1$, $q = 4r_k+1$. From the inequality $|a+b+c| \geqslant |b| - |a| - |c|$ and from (14) and (15) it follows that for any of these pairs we have

$$(16) \qquad \left| \frac{\partial^{p+q} F(x,y)}{\partial x^p \partial y^q} \right| \geqslant \left| \frac{\partial^{p+q} f_k(x,y)}{\partial x^p \partial y^q} \right| - J_k - 1$$

at any point $(x,y) \in R^2$. From the identity $\sin^2\alpha + \cos^2\alpha = 1$ it follows that for any real number α we have

$$|\sin\alpha| \geqslant 1/\sqrt{2} \quad \text{or} \quad |\cos\alpha| \geqslant 1/\sqrt{2}.$$

This means that at any point $(x,y) \in R^2$ at least one of the following conditions must be satisfied:

(a) $|\cos C_k(x-a_k)| \geqslant 1/\sqrt{2}$ and $|\cos C_k(y-b_k)| \geqslant 1/\sqrt{2}$,

(b) $|-\sin C_k(x-a_k)| \geqslant 1/\sqrt{2}$ and $|\cos C_k(y-b_k)| \geqslant 1/\sqrt{2}$,

(c) $|\cos C_k(x-a_k)| \geqslant 1/\sqrt{2}$ and $|-\sin C_k(y-b_k)| \geqslant 1/\sqrt{2}$,

(d) $|-\sin C_k(x-a_k)| \geqslant 1/\sqrt{2}$ and $|-\sin C_k(y-b_k)| \geqslant 1/\sqrt{2}$.

Let $(x,y) \in K_k$. If (a) holds, then from inequality (5) we obtain

$$\left|\frac{\partial^{8r_k} f_k(x,y)}{\partial x^{4r_k}\,\partial y^{4r_k}}\right| > M_k^2\left(\frac{1}{\sqrt{2}}a - b\right)^2 = M_k^2 \cdot \frac{1}{H} = A_k + J_k + 1,$$

which, together with (16), implies (13a). If (b) holds, then (6), together with inequality $C_k > 1$ and (16), implies (13b). In the same manner we obtain (13c) if (c) holds and, if (d) holds, we get (13d) and the proof of the lemma is completed.

3. T h e o r e m . To any closed set $P \subset R^2$ there exists a function $F(x,y)$ such that

(i) $F(x,y)$ is of class C^∞ in R^2,

(ii) Any point $(x,y) \in R^2 \setminus P$ is a regular point of the function $F(x,y)$,

(iii) Any point $(x,y) \in P$ is a P-singular point of the function $F(x,y)$.

P r o o f . If the set P is void, then the function $F(x,y) \equiv 0$ satisfies the required conditions. Assume now that P is a non-void set. Consider the sequence of squares (15), [5], corresponding to the set P. Let

$$F(x,y) = \sum_{n=1}^{\infty} f_n(x,y)$$

be the function defined by formulas (2)-(4) and $\{M_n\}$, $M_n \geq 1$, the sequence selected according to lemma 2, with $A_n = n^n \cdot n!$. Then the assumptions of Theorems 1 and 3 of [5], which imply (i) and (ii), are satisfied. If $(x,y) \in P$, then (x,y) belongs to an infinity of squares K_n of the sequence (1); hence, by virtue of the lemma, at least one of the inequalities (13a)-(13d) holds for an infinity of n. This means that there exists a subsequence

$$\left\{ \max_{0 \leq j \leq m_k} \left| \frac{\partial^{m_k} F(x,y)}{\partial x^j \, \partial y^{m_k - j}} \right| \right\}$$

($m_k = 8r_k$ or $8r_k+1$ or $8r_k+2$, $r_k = 20^k$, $\lim_{k \to \infty} m_k = +\infty$) of the sequence $\left\{ \max_{0 \leq j \leq m} \left| \frac{\partial^m F(x,y)}{\partial x^j \, \partial y^{m-j}} \right| \right\}$ such that

$$\max_{0 \leq j \leq m_k} \left| \frac{\partial^{m_k} F(x,y)}{\partial x^j \, \partial y^{m_k - j}} \right| > A_{m_k} = (m_k)^{m_k} \cdot (m_k)! .$$

Hence it follows that

$$\lambda(x,y) = \limsup_{m \to \infty} \left(\frac{1}{m!} \sum_{j=0}^{m} \binom{m}{j} \left| \frac{\partial^m F(x,y)}{\partial x^j \, \partial y^{m-j}} \right| \right)^{1/m} \geq$$

$$\geq \limsup_{m \to \infty} \left(\frac{1}{m!} \max_{0 \leq j \leq m} \left| \frac{\partial^m F(x,y)}{\partial x^j \, \partial y^{m-j}} \right| \right)^{1/m} \geq$$

$$\limsup_{k\to\infty}\left(\frac{1}{(m_k)!}\max_{0\le j\le m_k}\left|\frac{\partial^{m_k}F(x,y)}{\partial x^j\,\partial y^{n_k-j}}\right|\right)^{1/m_k}\geqslant$$

$$\geqslant\limsup_{k\to\infty}\left(\frac{1}{(m_k)!}(m_k)^{m_k}\cdot(m_k)!\right)^{1/m_k}=\lim_{k\to\infty}m_k=+\infty,$$

which means that (x,y) is a P-singular point of the function $F(x,y)$ and the proof of the theorem is thus completed.

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