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ON GENERATORS OF THE GROUP OF PROJECTIVE TRANSFORMATIONS

The problem of composing of projectivities by two cyclic projective collineations was considered in [1]. Now, we shall deal with the problem how to minimize the class of generators of the group of projectivities.

Firstly, we introduce some auxiliary notions and notations.

Notation. $P_n(F)$ - n-dimensional projective space over the field F. If F is arbitrary, we write simply P_n instead of $P_n(F)$. $GP_n(F)$ - the group of projective transformations (projectivities) of $P_n(F)$ onto itself.
$|F|$ - number of elements of the field F.
$Z(H_1,\dots,H_m)$ - the join of subspaces $H_1,\dots,H_m$ (the smallest subspace containing $H_1,\dots,H_m$).

We write $LJ^k(a_1,\dots,a_m)$, if every k of m points $a_1,\dots,a_m$ are linearly independent $(k\leq m)$.

If $f: X\to X$ and $x_0\in X$, then we denote the set $\{x_0, fx_0,\dots,f^m x_0\}$ by the symbol $(x_0)_f^m$.

Definitions:

Definition 1. A transformation $f:X\to X$ is called k-cyclic (k-periodic) if $f^k = e$, where e is the identity.

Definition 2. We shall say that a projective transformation $\varphi: P_n\to P_n$ has property 1, if every point of P_n lies in 1-dimensional subspace of P_n invariant under φ. Assume that projectivity φ has property m, and at the same time φ has not property k for all $0 \leq k \leq m-1$. Then m will be called the characteristic of φ (in symbols, $m = \text{char}\,\varphi$).

Notice that every projectivity of P_n has property n.

Definition 3. A normal cyclic collineation of P_n is an n+1 - cyclic projectivity of P_n having characteristic n.

We shall write $NC_n(F)$ for the set of all normal cyclic collineations of $P_n(F)$.

Definition 4. If for each x belonging to field F there exists $y \in F$ such that $y^k = x$ ($y^k = x$ or $y^k = -x$), then F will be called a k-closed (k-semiclosed) field. The set of all k-closed (k-semiclosed) fields will be denoted by $C_k(SC_k)$. Clearly, $C_k \subset SC_k$ and $C_k = SC_k$ if k is odd. Note also that if k is even and $F \in SC_k \setminus C_k$, then F can be ordered. Hence char F = 0.

First, we shall strenghten Lemma 2 ([1]). The notation from the proof of the lemma will be preserved. Choose an allowable coordinate system in α in such a way that the points p_0, p_1, p_2 have the coordinates $p_{ij} = \delta_i^j$ $i,j = 0,1,2$. For each point $x \in Z(p_0,p_1)$ there exists the point $y = Z(x,x') \cap l_\alpha$. Conversely, if y is such a point on l_α, then there may exist the second point $z \neq x$ such that $z \in Z(p_0,p_1)$ and $y = Z(z,z') \cap l_\alpha$. This can be checked by easy calculations. We should exclude at most n+1 positions of such a point y on l_α. Similarly, we ought to exclude at most n+1 positions of analogous point y on l_β. Therefore if the line $Z(p_0,p_1)$ contains at least 4n+5 different points, then the lemma is true. Hence it is true, if $|F| \geq 4n+4$.

Similarly, Theorem 1 [1] is also true, if $|F| \geq 4n+4$. Let us consider Theorem 5 [1] now. The intention of the theorem was to find a decomposition of an arbitrary projectivity on two cyclic projectivities.

That is why the field F has been assumed to be algebraically closed.

Now we shall apply the proof of this theorem to receive an another result.

The matrix of an arbitrary projectivity of P_n may be written in the form as in the mentioned proof (see [2]). Choose numbers ϱ_i, $i \neq 0, r_1, \ldots, r_{s-1}$, in such a way that $\varrho_i \neq \varrho_j$ for $i \neq j$, and $\varrho_i^s \neq 1$, all i. Since $1 \leq s \leq n+1$, we can do it, if $|F| \geq n+2$. Then char $h = n$. Thus, we have obtained the following result:

C o r o l l a r y 1. Every projectivity of $P_n(F)$, where $|F| \geq n+2$, is a composition of a normal cyclic collineation g and a projectivity h of characteristic n.

The above corollary and Theorem 1 [1] strengthened here imply.

T h e o r e m 1. If $|F| \geq 4n+4$ and f is a projectivity of $P_n(F)$, then f is a composition of at most three normal cyclic collineations.

Similarly, with respect to Corollary 1 and Theorem 6 [1] we obtain

T h e o r e m 2. If $|F| \geq n+2$ and f is a projectivity of $P_n(F)$, then f is a composition of two normal cyclic collineations and one involutive collineation (being a harmonic homology, or an elation if char $F = 2$).

Theorem 1 states that the set $NC_n(F)$ generates $GP_n(F)$, when $|F| \geq 4n+4$. This result is analogous to the well-known property of the group of isometries in Euclidean space which is generated by symmetries.

It should be noticed here that in the case $n = 1$ normal cyclic collineations as well as symmetries are 2-cyclic transformations.

In this work we shall consecutively restrict the class of generators of $GP_n(F)$. Two distinct classes of generators will be obtained with respect to start from Theorem 1 or Theorem 2. We are begining from Theorem 2.

First of all, we shall divide the set $J_n(F)$ of all involutive projectivities of $P_n(F)$ onto two subsets $J_n^1(F)$

and $J_n^2(F)$. The first of them will contain the transformations having at least one united point, the second will contain the remaining elements of $J_n(F)$. The set of harmonic homologies we shall denote by $J_n^o(F)$. Evidently, $J_n^o(F) \subset \subset J_n^1(F)$. Although $e^2 = e$, we shall assume that $e \notin J_n(F)$.

Lemma 1. Let char $F \neq 2$, and let $f \in J_n^1(F)$. Then there exist exactly two fundamental subspaces F_1, F_2 of f such that $\dim F_1 + \dim F_2 = n-1$ (i.e. $Z(F_1,F_2) = P_n(F)$.

Explanation. Fundamental subspace of f is a subspace every point of which is a united point of f.

Proof. The lemma is trivial, when $n = 1$. Assume its genuineness for $n = k$. Consider $n = k+1$. Let a be a united point of f. We may assume that H is not a fundamental hyperplane of f. In fact, if H were a fundamental hyperplane, then f would be a harmonic homology, since char $F \neq 2$. Therefore the lemma would be true. Obviously, f/H has a united point. In fact, if $b \notin H$ is a such point that $fb \neq b$, then the point $c = Z(b,fb) \cap H$ is a united point of f. According to the inductive assumption, there exist in H two fundamental subspaces F_1 and F_2 such that $\dim F_1 + \dim F_2 = n-2$. The point c must belong to F_1 or F_2. But on the line $Z(b,fb)$ there is a second united point $d \notin H$, since $f/Z(b.fb)$ is an involution and char $F \neq 2$. Assume that e.g. $\dim F_1 \geqslant \dim F_2$. Let next F_1^* be a subspace contained in F_1, and let $\dim F_1^* = \dim F_1 - 1$. Obviously, $\dim H_1 = n-1$, where $H_1 = Z(d,F_1^*,F_2)$. It is immediately seen that H_1 is invariant under f. Hence, by the inductive assumption, H_1 contains exactly two fundamental subspaces of f. So, one of subspaces $Z(d,F_1^*)$ or $Z(d,F_2)$ must be a fundamental subspace of f. Thence, $Z(d,F_1^*)$ or $Z(d,F_2)$ is a fundamental subspace of f. Since $\dim Z(d,F_1) = \dim F_1+1$ and $\dim Z(d,F_2) = \dim F_2+1$, the lemma is true, q.e.d.

Remark. The assumption char $F \neq 2$, is necessary. If char $F = 2$, then there are involutive elations in $P_n(F)$.

L e m m a 2. If n is even, then every involutive collineation of P_n belongs to J_n^1.

P r o o f . Suppose that $a \neq fa$ for every point $a \in P_n$. Then each line $Z(a,fa)$ is invariant under f. Let $a_1,\dots,a_{\frac{n}{2}}$ be such points that $a_2 \notin Z(a_1,fa_1)$, $a_3 \notin Z(Z(a_1,fa_1), Z(a_2,fa_2))$, etc. Then $H = Z(a_1,\dots,a_{\frac{n}{2}}, fa_1,\dots,fa_{\frac{n}{2}})$ is a hyperplane invariant under f. Let b be a point not belonging to H. Then the point $Z(b,fb) \cap H$ is a united point of f, contrary to our supposition.

L e m m a 3. Let n be even. If every normal cyclic collineation of $P_n(F)$ is a composition of a finite number of involutive collineations, then F must belong to C_{n+1}.

P r o o f . Let $g \in NC_n(F)$. The matrix of g has, in some coordinate system, the form

$$(1) \qquad G = \begin{bmatrix} 0 & 0 & \dots & 0 & c_n \\ 1 & 0 & \dots & \cdot & 0 \\ 0 & c_1 & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & c_{n-1} & 0 \end{bmatrix}, \quad 0 \neq c_i \in F$$

(see [1]).

Note that $c = \det G = (-1)^n \prod_{i=1}^{n} c_i$. Assume that $g = f_1 f_2 \dots f_k$, where $f_i \in J_n^1(F)$. Denote by F_i the matrices of f_i, $i=1,\dots,k$. In view of Lemmas 1 and 2, it follows that $\det F_i = \varepsilon_i a_i^{n+1}$, where $a_i \in F$, and $\varepsilon_i^2 = 1$, $i = 1,\dots,k$. The equality $\lambda G = F_1 \cdot F_2 \cdot \dots \cdot F_k$ implies $\lambda^{n+1} c = \prod_{i=1}^{k} a_i^{n+1} \varepsilon_i$. This ends the proof.

L e m m a 4. Let n be odd. If every normal cyclic collineation of $P_n(F)$ is a composition a finite number involutive collineations, then $F \in SC_{\frac{n+1}{2}}$.

P r o o f . If $f \in J_n^2(F)$, then the determinant of its matrix is equal to $a^{\frac{n+1}{2}}$, where $a \in F$. Let $g \in NC_n(F)$, and let $g = f_1, f_2, \ldots, f_k$, where $f_1, \ldots, f_l \in J_n^1(F)$ and $f_{l+1}, \ldots, f_k \in J_n^2(F)$. As in the proof of previous lemma we obtain the equality $\lambda^{n+1} c = a_1^{n+1} \cdot \ldots \cdot a_l^{n+1} \cdot a_{l+1}^{\frac{n+1}{2}} \cdot \ldots \cdot a_k^{\frac{n+1}{2}}$, q.e.d.

Remark. It is not necessary to be $l < k$ in the above.

L e m m a 5. Let n be even. If $F \in C_{n+1}$, then every normal cyclic collineation of $P_n(F)$ is a composition of two involutive collineations.

P r o o f . Let $f \in NC_n(F)$. Let $(a_o)_f^n$, b_I, b_{II} be such points that $fa_n = a_o$, $fb_I = b_{II}$ and $LJ^{n+1}(a_o, \ldots, a_n, b_I)$. Thus f is determined by the following point system:

$$f : \begin{bmatrix} a_o & \ldots & a_{n-1} & a_n & b_I \\ a_1 & \ldots & a_n & a_o & b_{II} \end{bmatrix}.$$

The points in the second row of the above matrix are images of the respective points of the first row. In the sequel, we shall simplify the notation by writing the indexes only. Then

$$f : \begin{bmatrix} 0 & \ldots & n-1 & n & I \\ 1 & \ldots & n & 0 & II \end{bmatrix}.$$

Assume that $a_i = (\delta_i^j, \ldots, \delta_i^j)$, $b_I = (1, \ldots, 1)$, $b_{II} = (c_n, 1, c_1, \ldots, c_{n-1})$, where δ_i is Kronecker δ.
Then (1) is a matrix of f. Take into account the following matrices:

$$A = \begin{bmatrix} 0 & 0 & p_1^2 & 0 & \cdots & & & 0 \\ 0 & p_1 & 0 & \cdot & \cdots & & & 0 \\ 1 & 0 & \cdot & \cdot & \cdots & & & 0 \\ 0 & \cdot & \cdot & \cdot & \cdots & & & p_n \\ \cdot & \cdot & \cdot & \cdot & \cdots & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & p_{\frac{n}{2}+1} & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & p_1^2 p_{\frac{n}{2}+1}^{-1} & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & 0 & p_1^2 p_{n-1}^{-1} & & & \cdot \\ 0 & 0 & 0 & p_1^2 p_n^{-1} & 0 & \cdots & & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & q_{\frac{n}{2}+1}^2 & 0 & \cdots & & & 0 \\ 1 & 0 & \cdots & & & & 0 \\ 0 & \cdot & \cdots & & & & q_n \\ \cdot & \cdot & & & & & \cdot \\ \cdot & \cdot & & & & q_{\frac{n}{2}+1} & \cdot \\ \cdot & \cdot & & & q_{\frac{n}{2}+1}^2 \cdot q_{\frac{n}{2}+2}^{-1} & & \\ 0 & 0 & q_{\frac{n}{2}+1}^2 & q_n^{-1} & 0 & \cdots & 0 \end{bmatrix}$$

$p_i \neq 0$ for $i = 1, \frac{n}{2}+1, \ldots, n$; $q_j \neq 0$ for $j = \frac{n}{2}+1, \ldots, n$. One can see easily that $A^2 = p_1^2 J$ and $B^2 = q_{\frac{n}{2}+1}^2 J$, where J is the unit matrix. Hence the projectivities $y = Ax$ and $y = Bx$ are involutive.

We require non-zero values for p_i, q_j which satisfy the equation

(3) $$\lambda G = A\,B.$$

(3) is equivalent to the following equation system:

$$\begin{aligned}
q^2_{\frac{n}{2}+1} &= p_1 c_1\\
q^2_{\frac{n}{2}+1} p_n &= p_1 q_n c_2\\
\vdots \quad & \quad \vdots\\
q^2_{\frac{n}{2}+1} p_{\frac{n}{2}+2} &= p_1 q_{\frac{n}{2}+2} c_{\frac{n}{2}}\\
p_1 q_{\frac{n}{2}+1} &= p_{\frac{n}{2}+2} c_{\frac{n}{2}+1}\\
\vdots \quad & \\
p_1 q_{n-1} &= p_n c_{n-1}\\
p_1 q_n &= c_n .
\end{aligned}$$

One can see easily that all values of the indeterminates p_i, q_j, except $q_{\frac{n}{2}+1}$, can be found with the help of powers of $q_{\frac{n}{2}+1}$. However $q^{n+1}_{\frac{n}{2}+1} = c_1 \cdot c_2 \cdot \ldots \cdot c_n$, q.e.d.

L e m m a 6. Let n be odd. If $F \in SC_{\frac{n+1}{2}}$ then every normal cyclic collineation of $P_n(F)$ is a composition of two involutive collineations.

P r o o f . Let the matrices A, B be as follows:

$$A = \begin{bmatrix} 0 & 0 & p_1^2 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & p_1 & 0 & \cdot & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & \cdot & \cdot & \cdots & \cdots & \cdots & 0 \\ 0 & \cdot & \cdot & \cdot & \cdots & \cdots & \cdots & p_n \\ \vdots & \vdots & \vdots & \vdots & & & p_{\frac{n+5}{2}} & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \pm p_1 & & \vdots \\ \vdots & \vdots & \vdots & \vdots & p_1^2 p^{-1}_{\frac{n+5}{2}} & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & \vdots \\ 0 & 0 & 0 & p_1^2 p_n^{-1} & \cdots & \cdots & \cdots & \cdot \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & q_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & \cdot & \cdots & \cdots & \cdots & 0 \\ 0 & \cdot & \cdot & \cdots & \cdots & \cdots & q_n \\ \vdots & \vdots & \vdots & & & \cdot & \vdots \\ \vdots & \vdots & \vdots & & \cdot & & \vdots \\ \vdots & \vdots & \vdots & & q_{\frac{n+3}{2}} & & \vdots \\ \vdots & \vdots & \vdots & q_1 q^{-1}_{\frac{n+3}{2}} & & & \vdots \\ \vdots & \vdots & \vdots & & & & \vdots \\ 0 & 0 & q_1 q_n^{-1} & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

$p_i, q_j \neq 0$ for $i = 1, \frac{n+5}{2}, \ldots, n$, $j = 1, \frac{n+3}{2}, \ldots, n$.

The equation $\lambda G = A B$ leads us to

$$q_1 = p_1 c_1$$

$$q_1 p_n = p_1 q_n c_2$$

$$\vdots \qquad \vdots$$

$$q_1 p_{\frac{n+5}{2}} = p_1 q_{\frac{n+5}{2}} c_{\frac{n-1}{2}}$$

$$\pm q_1 = q_{\frac{n+3}{2}} c_{\frac{n+1}{2}}$$

$$p_1 q_{\frac{n+3}{2}} = p_{\frac{n+5}{2}} c_{\frac{n+3}{2}}$$

$$\vdots \qquad \vdots$$

$$p_1 q_n = c_n.$$

As previously, all p_i, q_j, except q_n, can be found with the help of powers of q_n. But $q_n^{\frac{n+1}{2}} = \pm c_1^{\frac{n-1}{2}} \cdot c_2^{\frac{n-1}{2}} \cdot \prod_{i=2}^{n-1} c_i^{-1}$.

This ends the proof.

From Lemmas 3 and 5 it follows

T h e o r e m 3. Let n be even. The following are equivalent:

(i) $F \in C_{n+1}$;

(ii) every normal cyclic collineation of $P_n(F)$ is a composition of two involutive collineations.

Similarly, from Lemmas 4 and 6 it follows

T h e o r e m 4. Let n be odd. The following are equivalent:

(i) $F \in SC_{\frac{n+1}{2}}$;

(ii) every normal cyclic collineation of $P_n(F)$ is a composition of two involutive collineations.

With respect to Lemmas 5, 6 and Theorem 2 we arrive at the following theorem.

T h e o r e m 5. If $|F| \geq SC_{n+1}$ and $|F| \geq n+1$, then every projectivity of $P_n(F)$ is a composition of at most five involutive collineations.

Remark. If n is odd, then it suffices to assume that $F \in SC_{\frac{n+1}{2}}$.

Thus, we have shown that $GP_n(F)$ is generated by the class of involutive collineations. Next we shall show that it suffices to use a small subset of $J_n(F)$ in order to generate $GP_n(F)$.

L e m m a 7. Let n be odd. If every element of $J_n^2(F)$ is a composition of a finite number of elements of $J_n^1(F)$, then F must belong to SC_2.

P r o o f . Let $g \in J_n^2(F)$, let $f_1,\ldots,f_k \in J_n^1(F)$, and let $g = f_1 \ldots f_k$. Write $G, F_1, \ldots, F_k$ for the respective matrices of $g, f_1, \ldots, f_k$. By the assumption, there is $a \in F$ such that $\det G = a^{\frac{n+1}{2}} \neq b^{n+1}$ for each $b \in F$. Since $\det F_i = \varepsilon_i b_i^{n+1}$, where $b_i \in F$ and $\varepsilon_i^2 = 1$, the equality $\lambda G = \prod_{i=1}^{k} F_i$ implies $\lambda^{n+1} a^{\frac{n+1}{2}} = \prod_{i=1}^{k} b_i^{n+1} \cdot \varepsilon_i$, q.e.d.

L e m m a 8. If n is odd and $F \in SC_2$, then every element of $J_n^2(F)$ is a composition of two elements of $J_n^1(F)$.

P r o o f . Assume that $f \in J_n^2(F)$. Hence we can choose $\frac{n+1}{2}$ straight lines $l_1, \ldots, l_{\frac{n+1}{2}}$ in such a way that $l_1 \cap l_2 = \emptyset$, $l_3 \cap Z(l_1, l_2) = \emptyset, \ldots, l_{\frac{n+1}{2}} \cap Z(l_1, \ldots, l_{\frac{n-1}{2}}) = \emptyset$, and $fl_i = l_i$ for $i = 1, \ldots, \frac{n+1}{2}$. Let a_i be a point belonging to l_i, all i. Call $b_i = fa_i$ for $i = 1, \ldots, \frac{n+1}{2}$. Obviously, $LJ^{n+1}(a_1, \ldots, a_{\frac{n+1}{2}}, b_1, \ldots, b_{\frac{n+1}{2}})$. Assume that

$a_1 = (1,0,\dots,0)$, $b_1 = (0,1,0,\dots,0),\dots,b_{\frac{n+1}{2}} = (0,\dots,0,1)$.

Then f has the matrix of the form

$$C = \operatorname{diag}(C_1, C_2, \dots, C_{\frac{n+1}{2}}),$$

where $C_1 = \begin{bmatrix} 0 & 1 \\ c_2c_3 & 0 \end{bmatrix}$, $C_2 = \begin{bmatrix} 0 & c_2 \\ c_3 & 0 \end{bmatrix}$, $C_3 = \begin{bmatrix} 0 & c_2c_4 \\ c_3c_4^{-1} & 0 \end{bmatrix}, \dots$

$$C_{\frac{n+1}{2}} = \begin{bmatrix} 0 & c_2c_{\frac{n+3}{2}} \\ c_3c_{\frac{n+3}{2}}^{-1} & 0 \end{bmatrix};$$

$0 \neq c_i \in F$, and $c_2c_3 \neq c^2$ for each $c \in F$.

Take into account the $n+1 \times n+1$ matrices

$$P = \operatorname{diag}(P_1, P_2, \dots, P_{\frac{n+1}{2}}) \quad \text{and} \quad Q = \operatorname{diag}(Q_1, Q_2, \dots, Q_{\frac{n+1}{2}}),$$

where $P_1 = \begin{bmatrix} 0 & 1 \\ p_1^2 & 0 \end{bmatrix}$, $P_2 = \begin{bmatrix} p_1 & 0 \\ 0 & -p_1 \end{bmatrix} = P_4 = P_6 =$

$$= \dots,\ P_i = \begin{bmatrix} 0 & p_i \\ p_1^2p_i^{-1} & 0 \end{bmatrix} \quad \text{for } i = 3,5,\dots;$$

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = Q_3 = Q_5 = \dots,\quad Q_j = \begin{bmatrix} 0 & q_j \\ q_j^{-1} & 0 \end{bmatrix}$$

for $j = 2,4,\dots, 0 \neq p_i, q_j \in F$, all i,j.

It is easily seen that $P^2 = p_1^2J$ and $Q^2 = J$. Hence the projectivities $y = Px$ and $y = Qx$ are involutive. As in the proof of Lemma 5, we require non-zero values for p_i, q_j which satisfy the equation

$$\lambda C = P \cdot Q.$$

This is equivalent to the following system of equations:

$$(1') \qquad p_1^2 = -c_2c_3$$

$$(2') \qquad p_1q_2 = -c_2$$

$$(3') \qquad p_1q_2^{-1} = c_3$$

$$(4') \qquad p_2 = c_2c_4$$

$$(5') \qquad p_1^2p_2^{-1} = -c_3c_4^{-1}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$(n-1') \qquad p_1q_{\frac{n+1}{2}} = -c_2c_{\frac{n+3}{2}}$$

$$(n') \qquad p_1q_{\frac{n+1}{2}}^{-1} = c_3c_{\frac{n+3}{2}}^{-1} .$$

Note that $(2')\cdot(3') = (4')\cdot(5') = \ldots = (n-1')\cdot(n') = (1')$. Hence we may eliminate the equations $(3'),(5'),\ldots,(n')$ from the system. By the assumption $c_2c_3 \neq c^2$, we can count p_1 from $(1')$. From $(2')$ we count q_1, etc.

From Lemmas 6, 8 and Theorems 2, 5 it follows

T h e o r e m 6. If $|F| \geq n+2$, $F \in SC_{n+1}$, then any projectivity of $P_n(F)$ is a composition of at most seven projectivities from $J_n^1(F)$.

More precisely, if n is even, then every projectivity of $P_n(F)$ is a composition of at most five elements of $J_n^1(F)$. If however n is odd, then every projectivity of $P_n(F)$ is a composition of at most seven elements of $J_n^1(F)$, since the transformation $y = Ax$ from Lemma 6 belongs to $J_n^1(F)$.

L e m m a 9. If char $F \neq 2$ and $f \in J_n^1(F)$, then f is a composition of at most $E(\frac{n}{2})$ of harmonic homologies.

P r o o f . Since $f \in J_n^1(F)$, the matrix of f may be written, in some coordinate system, in the form

$$A = \mathrm{diag}(\overbrace{-1,\dots,-1}^{k},1,\dots,1).$$

Evidently, $A = B_1 \cdot B_2 \cdot \ldots \cdot B_k$, where $B_i = \mathrm{diag}(1,\dots,1,\overset{i}{-1},1,\dots,1)$

q.e.d.

From Lemma 9 and Theorem 6 we obtain

Theorem 7. If $\mathrm{char}\, F \not| 2$, $F \in SC_{n+1}$ and $|F| \geq n+2$, then any projectivity of $P_n(F)$ is a composition of at most $6E\left(\frac{n+1}{2}\right)+1$ transformations from $J_n^o(F)$.

Let now $a_1,\dots,a_n$ be fixed linearly independent points in P_n. Write Φ_i for the set of harmonic homologies having the center a_i, $i=1,2$. Similarly, we denote by Ψ_1 the set of all harmonic homologies the fundamental hyperplanes of which belong to the pencil $Z(a_2,\dots,a_n)$ (i.e. all these hyperplanes contain $Z(a_2,\dots,a_n)$). Call $\Phi = \Phi_1 \cup \Phi_2$, $\Psi = \Phi_1 \cup \Psi_1$.

In what follows, we shall write $f_{a,H}$ for the harmonic homology with the center a and the fundamental hyperplane H.

Lemma 10. Every harmonic homology of P_n $(n \geq 2)$ is a composition of at most seven (three distinct) transformations from Ψ.

Proof. Assume that $f_{a,H} \notin \Psi$. We distinguish two cases:

1^o $a_1 \notin H$. Let a_{n+1} be such a point that $LJ^{n+1}(a_1,\dots,a_{n+1})$ and $a_{n+1} \in H$. Call $a_1' = fa_1$, $H_2 = Z(a_2,\dots,a_{n+1})$, $b_i = H \cap Z(a_1,a_i)$ for $i = 2,\dots,n+1$. Let us define $g_{a_1,\bar{H}}$ and $h_{\bar{a},H_2}$ as follows

$$g_{a_1,\bar{H}} : \begin{bmatrix} a_1 & a_2 & b_2 & \dots & a_n & b_n & a_{n+1} \\ a_1 & b_2 & a_2 & \dots & b_n & a_n & a_{n+1} \end{bmatrix}, \quad h_{\bar{a},H_2} : \begin{bmatrix} a_2 & \dots & a_{n+1} & b & a_1 \\ a_2 & \dots & a_{n+1} & a_1 & b \end{bmatrix},$$

where $b = g_{a_1,\bar{H}}a_1'$. Certainly, $f = ghg$.

2^o $a_1 \in H$. Then it suffices to take such points c_1, a_{n+1} that $a \neq c_1 \notin H$, $a_{n+1} \notin H$, and $LJ^{n+1}(a_1,\dots,a_{n+1})$. Write H_2 for $Z(a_2,\dots,a_{n+1})$. As in 1^o, we can construct $g_{c_1,\bar{H}}$ and $h_{\bar{a},H_2}$ such that $f = ghg$. It is obvious that $a_1 \notin \bar{H}$.

Hence, by 1°, there exist such $g_1 \in \Phi_1$ and $h_1 \in \Psi_1$ that $g = g_1 h_1 g_1$. This ends the proof.

From the last lemma and Theorem 7 it follows

L e m m a 11. If char $F \neq 2$, $F \in SC_{n+1}$ and $|F| \geq n+2$, then the group $GP_n(F)$ is generated by Ψ.

L e m m a 12. If the center of a harmonic homology f lies on $Z(a_1, a_2)$, then f is a composition of at most three transformations from Φ.

P r o o f . Assume that $f_{a,H} \notin \Phi$. Evidently, $a_1 \notin H$ or $a_2 \notin H$, say $a_1 \notin H$. Call $a'_1 = fa_1$. Let $a_3, \dots, a_{n+1}$ be such points that $LJ^{n+1}(a_1, \dots, a_{n+1})$ and $a_i \in H$ for $i = 3, \dots, n+1$. Let next H_1 be such a hyperplane that $g_{a_1,H_1} a_2 = a$ and $a_i \in H_1$ for $i = 3, \dots, n+1$. Let us define $h_{a_2,\bar{H}}$ as follows

$$h_{a_2,\bar{H}} : \begin{bmatrix} a_2 & a_3 & \dots & a_{n+1} & a_1 & b \\ a_2 & a_3 & \dots & a_{n+1} & b & a_1 \end{bmatrix},$$

where $b = g_{a_1,H_1} a'_1$. One easily checks that $f = ghg$, q.e.d.

With respect to Lemma 12, the question arises whether it is possible to generate the set of all harmonic homologies with the help of harmonic homologies with given centers. The answer will be contained in two following lemmas.

L e m m a 13. Let Φ_i be the set of all harmonic homologies with the center a_i, $i = 1, \dots, k$. If every harmonic homology of P_n is a composition of a finite number elements of $\Phi_{1k} = \bigcup_{i=1}^{k} \Phi_i$, then there must be $k \geq n+1$.

P r o o f . Suppose $k < n+1$, and suppose $f_{a,H} = f_1 f_2 \dots f_m$, where $a \notin Z(a_1, \dots, a_k) \not\subset H$ and $f_i \in \Phi_{1k}$, $i = 1, \dots, m$. Clearly, $Z(a_1, \dots, a_k)$ is invariant under $f_1 f_2 \dots f_k$, and consequently it is invariant under $f_{a,H}$. This is a contradiction with the assumption $a \notin Z(a_1, \dots, a_k) \not\subset H$.

L e m m a 14. Let $a_0,\dots,a_n$ be fixed linearly independent points in P_n, and let $\Phi_0,\dots,\Phi_n$ be sets of all harmonic homologies having as centers $a_0,\dots,a_n$, respectively. Then every transformation $f_{a,H}$ of P_n is a composition of a finite number transformations from $\Phi_{on} = \bigcup_{i=0}^{n} \Phi_i$.

P r o o f . We may assume that $a \neq a_i$ for $i=0,\dots,n$. If we had $a \in Z(a_i,a_j)$, then, by Lemma 12, we would obtain the thesis. Hence we may assume that $a \notin Z(a_i,a_j)$ for all $i \neq j$. Write b_0 for $Z(a_0,a) \cap Z(a_1,\dots,a_n)$. If $b_0 \in Z(a_i,a_j)$ $i,j = 1,\dots,n$, then we can construct, as in Lemma 12, the set of all harmonic homologies having the center b_0. Next, since $a \in Z(b_0,a_1)$, we can construct $f_{a,H}$. If $b_0 \notin Z(a_i,a_j)$, then we take into account $b_1 = Z(a_1,b_0) \cap Z(a_2,\dots,a_n)$ etc.

Thus the following theorem is true.

T h e o r e m 8. If char $F \neq 2$, $F \in SC_{n+1}$ and $|F| \geq n+2$, then $GP_n(F)$ is generated by Φ_{on}.

Now we shall, still restrict the set Ψ of generators of $GP_n(F)$. Let $H_1,\dots,H_{n+1}$ be linearly independent hyperplanes such that $H_n \supset Z(a_2,\dots,a_n) = G_1 \subset H_{n+1}$ and $a_1 \notin H_i$, all i. Write Φ_1^* for the set $\{f_{a_1,H_1},\dots,f_{a_1H_{n-1}}\}$. Let us denote $\Psi_1 \cup \Phi_1^*$ by Φ^*.

L e m m a 15. Every element of J_n^o is a composition of a finite number elements of the set Φ^*.

P r o o f . 1^o Let us call $H_o = Z(a_1,\dots,a_n)$, $f_1 = f_{a_1,H_1}$, $G_2 = f_1G_1$. Naturally, $H_o = Z(G_1,G_2)$. Take into account $g_{a_1,Z(G_1,y)} = g \in \Psi_1 \cup \Phi_1$, where $y \in H_1 \setminus G_1$. Clearly, the transformation f_1gf_1 belongs to Φ_1 and its fundamental hyperplane is $Z(G_2,y)$. Thus we have obtained the set Ψ_2 of all elements of Φ_1 in fundamental hyperplanes of which is contained G_2.

2^o Let G_o be a hyperplane containing $G_3 = G_1 \cap G_2$. Take into account $f_{a,G_o} = f_o$. We shall show that f_o can be obtained as a composition of elements of $\Psi_1 \cup \Psi_2$. Clearly, we may assume that $G_1 \not\subset G_0 \neq H_0$. Let p be such a point that $p \in G_0 \setminus H_0$. Call $G^* = f_{a_1,Z(G_2,p)}G_1$. Let $F \ni p$ be a hyperplane containing G^*. Then $g_{b,F} = fhf$, where $f = f_{a_1,Z(G_2,p)}$, $c = fb$, and $h = h_{c,Z(G_1,p)}$. Clearly, $h \in \Psi_1$, $f \in \Psi_2$. Thus we can construct, with the help of elements of $\Psi_1 \cup \Psi_2$, any harmonic homology with the fundamental hyperplane containing G^*. Let H be the harmonic conjugate of $Z(G_1,p)$ with respect to G_0 and $Z(G^*,p)$, and let $x \neq a$ be a point belonging to $H \setminus Z(G_3,p)$. Let next y be the harmonic conjugate of a with respect to x and $Z(G_1,p) \cap Z(a,x)$. One can check easily that $f_0 = f_{x,Z(G_1,p)}h_{y,Z(G^*,p)}f_{x,Z(G_1,p)}$.

Thus we have constructed the set Ψ_3 of all harmonic homologies with fundamental hyperplanes passing through G_3.

3^0 Call $G_4 = f_{a_1,H_2}G_3$. We can construct, as in 1^0, the set $\Psi_4 \subset \Phi_1$ of all harmonic homologies with fundamental hyperplanes containing G_4 and having a_1 as a center.

4^0 As in 2^0, we can obtain the set Ψ_5 of all harmonic homologies containing $G_5 = G_3 \cap G_4$ in their fundamental hyperplanes.

Since $\dim G_1 = n-2$, $\dim G_3 = n-3, \dots, \dim G_{2n-1} = -1$, we arrive at the set $\Psi_{2n-1} = J_n^0$, q.e.d.

Hence we have obtained the following lemma:

L e m m a 16. If $\operatorname{char} F \neq 2$, $\|F\| \geq n+2$, and $F \in SC_{n+1}$, then the set Φ^* generates $GP_n(F)$.

Let now $b_1, b_2, \dots, b_{n+1}$ be fixed linearly independent points in P_n such that $b_1 = a_1$ and $Z(b_i,b_j) \cap G_1 = \phi$, all i, j. Let Γ_i denote the set of all elements of Ψ_1 having b_i as a center, $i = 1,\dots,n+1$. Call $\Gamma_0 = \bigcup_{i=1}^{n+1} \Gamma_i$ and $\Gamma = \Gamma_0 \cup \Phi_1^*$.

L e m m a 17. Every element of Ψ_1 is a composition of a finite number elements of Γ_0.

P r o o f . Assume that $f_{b,B} \in \Psi_1$ and at the same time $f_{b,B} \notin \Gamma_0$. First will be considered the case $b \in Z(b_i,b_j)$ for some i, j. Let e.g. $b \in Z(b_1,b_2)$. Let b' denote the harmonic conjugate of b_1 with respect to b and b_2. Similarly, let d be the harmonic conjugate of c with respect to b_1 and b', where $c = Z(b_1,b_2) \cap B$. Take into account harmonic homologies $g_{b_1,Z(G_1,b')} = g$ and $h_{b_2,Z(G_1,d} = h$.

It is easy to see that $f_{b,B} = ghg$. If $b \notin Z(b_i,b_j)$ for all i, j, then we take into account the point $\bar{b}_1 = Z(b,b_1) \cap \cap Z(b_2,\ldots,b_{n+1})$. If now $\bar{b}_1 \in Z(b_i,b_j)$ for some i, j, then we can construct, as above, any harmonic homology belonging to Ψ_1 and having $\bar{b}_1$ as a center. Next, also as above, we can construct $f_{b,B}$. If $\bar{b}_1 \notin Z(b_i,b_j)$ for all i, j, then we take $\bar{b}_2 = Z(\bar{b}_1,b_2) \cap Z(b_3,\ldots,b_{n+1})$ etc, q.e.d.

From the above and Lemma 16 there follows

L e m m a 18. If char $F \neq 2$, $|F| \geqslant n+2$, and $F \in \in SC_{n+1}$, then $GP_n(F)$ is generated by Γ.

Let now H^* be such a fixed hyperplane that $G_1 \subset H^*$ and $b_i \notin H^*$, $i = 1,\ldots,n+1$. Call $f_i = f_{b_i,H^*}$, $\Psi_1^* = \{f_1,\ldots,f_n\}$ and $\Gamma^* = \Phi_1^* \cup \Psi_1^* \cup \Gamma_{n+1}$, $i = 1,\ldots,n$.

L e m m a 19. Every element of ψ_1 is a composition of a finite number elements of $\Psi_1^* \cup \Gamma_{n+1}$.

P r o o f . Assume that $h = h_{b_{n+1},H} \in \Gamma_{n+1}$. Cal $c_i = H^* \cap Z(b_{n+1},b_i)$, $i = 1,\ldots,n$. Let b_i^* be the harmonic conjugate of b_{n+1} with respect to b_i and c_i, $i=1,\ldots,n$. Let next hyperplane T_i be the harmonic conjugate of H with respect to H^* and $Z(G_1,b_i)$, $i = 1,\ldots,n$. Naturally, the harmonic homology $f_i h f_i$ has the center b_i and the fundamental hyperplane T_i, $i = 1,\ldots,n$. Hence, for given $T_i \supset G_1$ there exists a hyperplane $H \supset G_1$ such that $f_i h_{b^*_{n+1},H} f_i = g_{b_i,T_i}$. Thus we can obtain each harmonic homology from Ψ_1 having the center b_i^*. Notice that if $Z(b_i,b_j) \cap G_1 = \emptyset$, then $Z(b_i^*,b_j^*) \cap G_1 = \emptyset$. Hence, by Lemma 17, we can construct each element of Ψ_1.

According to Theorem 7 and the above lemma we obtain

T h e o r e m 9. If char $F \neq 2$, $|F| \geq n+2$, and $F \in SC_{n+1}$, then $GP_n(F)$ is generated by Γ^*.

This is the last theorem from the series. The set $\Gamma^{*'}$ of generators of the projective group $GP_n(F)$ is a one-parameters set only.

Now, we shall construct another set of generators of $GP_n(F)$, with the help of normal cyclic collineations.

By $NC_n^o(F)$ we will denote the set of all normal cyclic collineations which have at least one united point.

L e m m a 20. If each element of $NC_n(F)$ is a composition of a finite number elements of $NC_n^o(F)$, then F must belong to SC_{n+1}.

P r o o f . Let us assume that $f \in NC_n(F)$, $f = f_1 f_2 \dots f_k$, $f_i \in NC_n^o(F)$, $i = 1,\dots,k$. Write $A, A_1,\dots,A_k$ for the matrices of $f, f_1,\dots,f_k$, respectively. By (1), the characteristic polynomial of f is $(-x)^{n+1} + \det A$. We distinguish two cases:

1) n even. Then $\det A_i = a_i^{n+1}$, where $a_i \in F$, $i = 1,\dots,k$. By $\varrho A = A_1 \cdot \ldots \cdot A_k$, it follows that $\varrho^{n+1} \det A = \prod_{i=1}^{k} a_i^{n+1}$.

2) n odd. Then $\det A_i = -a_i^{n+1}$, $a_i \in F$, $i = 1,\dots,k$. Hence $\varrho^{n+1} \det A = (-1)^k \prod_{i=1}^{k} a_i^{n+1}$, q.e.d.

L e m m a 21. Let $f \in NC_n(F)$. Then a coordinate system can be chosen such that f has the matrix of the form (1) satisfying the following equalities: $c_1 = c_2 \dots = 1$, $c_n = c$.

P r o o f . We may assume that (1) is the matrix of f. Take into account the matrix $T = \mathrm{diag}(t_o,\dots,t_n)$, where $t_o = t_1$, $t_2 = t_1 c_1^{-1}$, $t_3 = t_2 c_2^{-1}, \dots, t_n = t_{n-1} c_{n-1}^{-1}$, $t_1 \neq 0$. It is immediately seen that the matrix TGT^{-1} satisfies the required conditions.

L e m m a 22. If $F \in SC_{n+1}$, then every element of $NC_n(F)$ is a composition of at most two elements of $NC_n^o(F)$.

P r o o f . First of all, notice that if n is even, then $NC_n^o(F) = NC_n(F)$, since $F \in SC_{n+1}$. If however n is odd and $F \in C_{n+1}$, then also $NC_n^o(F) = NC_n(F)$. Hence we may assume that n is odd and $F \in SC_{n+1}$, but $F \notin C_{n+1}$. Then F may be ordered by a choice of the set of positive elements (x is positive if $x = y^{n+1}$, $y \in F$). Let $f \in NC_n(F) \setminus NC_n^o(F)$, and let A denote the matrix of f. By Lemma 22, A may be written in the from (1) with the additional conditions $c_1 = \dots = c_{n-1} = 1 \neq c = c_n$.

Since $f \notin NC_n^o(F)$, $\det A = -c > 0$. Denote $a_0 = (1,0,\dots,0),\dots,a_n = (0,\dots,0,1)$, $b_I = (1,\dots,1)$, $b_{II} = (c,1,\dots,1)$. Then f is determined as follows

$$f : \begin{bmatrix} 0 & \dots & n & I \\ 1 & \dots & 0 & II \end{bmatrix} .$$

Let us define

$$h : \begin{bmatrix} 0 & 1 & 2 & 3 & \dots & n-1 & n & I \\ II & n & 1 & 2 & \dots & n-2 & I & n-1 \end{bmatrix},$$

$$g : \begin{bmatrix} II & n & 1 & 2 & \dots & n-2 & I & n-1 \\ 1 & 2 & 3 & 4 & \dots & n & 0 & II \end{bmatrix} .$$

Clearly, $g,h \in NC_n(F)$ and $f = gh$. Hence $\varrho A = G \cdot H$, where G, H are the matrices of g, h. Note that

$$H = \begin{bmatrix} c & 0 & 0 & 0 & \dots & & -c \\ 1 & 0 & c-1 & 0 & & & \cdot \\ \cdot & \cdot & 0 & c-1 & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \\ & \cdot & \cdot & & & c-1 & -c \\ & & \cdot & & & 0 & -c \\ 1 & c-1 & 0 & \dots & & 0 & -c \end{bmatrix}$$

Since $c<0$, $\det H = -c(c-1)^n < 0$. Hence, since $\det A > 0$, $\det G < 0$. Thus we have shown that $g, h \in NC_n^0(F)$, q.e.d.

In view of Theorem 1 and the above lemmas it follows

T h e o r e m 10. Let $f \in GP_n(F)$, let $F \in SC_{n+1}$, and let $|F| \geqslant 4n+4$. Then f is a composition of at most six (three if n is even) elements of $NC_n^0(F)$.

Let us denote by $\mathcal{F}_n$ the set of all fields satisfying the following condition:

(i) $P_n(F)$ contains such a point system $a_0, \dots, a_{n^2+n}$ that $LI^{n+1}(a_0, \dots, a_{n^2+n})$.

Assume now that $F \in \mathcal{F}_n$. Let $a_0, b_1, \dots, b_{n^2+n}$ be fixed points of $P_n(F)$ such that $LI^{n+1}(a_0, b_1, \dots, b_{n^2+n})$. Let Σ_0 (Σ_i) denote the set of all elements of $NC_n^0(F)$ having $a_0(b_i)$ as a united point, $i = 1, \dots, n^2+n$. Call $\Sigma = \bigcup_{i=0}^{n^2+n} \Sigma_i$.

L e m m a 23. Every element of $NC_n^0(F)$ is a composition of at most three elements of Σ

P r o o f . Assume that $f \in NC_n^0(F) \setminus \Sigma$. Obviously, at least one from relations $LI^{n+1}((a_0)_f^n)$, $LI^{n+1}((b_1)_f^{n+1}), \dots$ $\dots, LI^{n+1}((b_{n^2+n})_f^{n^2+2n})$ holds. Let e.g. $LI^{n+1}((a_0)_f^n)$. Call $a_i = fa_{i-1}$, $i = 1, \dots, n$. Then

$$f : \begin{bmatrix} 0 & \dots & n & n+1 \\ 1 & \dots & 0 & n+1 \end{bmatrix}.$$

Take into account the following transformations:

$$i : \begin{bmatrix} 0 & 1 & 2 & \dots & n-2 & n-1 & n & n+1 \\ n & 1 & 2 & \dots & n-2 & n+1 & 0 & n-1 \end{bmatrix},$$

$$g_0 : \begin{bmatrix} n & 1 & 2 & \dots & n-2 & n+1 & 0 & n-1 \\ 1 & 2 & 3 & \dots & n-1 & n & 0 & n+1 \end{bmatrix}.$$

It is easy to see that $f = g_0 i$, $g_0 \in \sum_0$ and $i \in I_n^1(F)$.
Now, we distinguish two cases:

1^o The relation $LI^4(a_0, a_n, b_1, ib_1)$ holds. Then we may assume that also $LI^{n+1}(a_0, a_n, b_1, ib_1, a_1, \dots, a_{n-1})$, since otherwise points $a_1, \dots, a_{n-2}$ may be replaced by other united points $a_1'', \dots, a_{n-2}'$ of i satisfying this condition. Call $b_{1'} = b_1$, $b_{2'} = ib_1$. Let us define

$$h_1 : \begin{bmatrix} 0 & 1 & 2 & \dots & n-3 & n-2 & n & 1' & 2' \\ 2' & 2 & 3 & \dots & n-2 & n & 0 & 1' & 1 \end{bmatrix},$$

$$h_0 : \begin{bmatrix} 2' & 2 & 3 & \dots & n-2 & n & 0 & 1' & 1 \\ n & 1 & 2 & \dots & n-3 & n-2 & 0 & 2' & 1' \end{bmatrix}.$$

One can easily see that $h_0 \in \sum_0$, $h_1 \in \sum_1$, $i = h_0 h_1$, and consequently $f = g_0 h_0 h_1$.

2^o The points a_0, a_n, b_1, ib_1 are linearly dependent. Then we take $b_2 = (q_0, \dots, q_n)$, $b_3 = (r_0, \dots, r_n), \dots$. If, for some j, $LI^4(a_0, a_n, b_j, ib_j)$, then, as in 1^o, we can decompose i onto elements of $\sum$.

Suppose the relation $LI^4(a_0, a_n, b_j, ib_j)$ does not hold for $j = 1, \dots, n^2+n$. Put $a_0 = (1, 0, \dots, 0), \dots, a_n = (0, \dots, 0, 1)$, $a_{n+1} = (1, \dots, 1)$ and $b_1 = (p_0, \dots, p_n)$. It is easily seen that the matrix B of i is as follows:

$$B = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 & -1 \\ 0 & -1 & \dots & 0 & 1 & 0 \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ & & & -1 & 1 & 0 \\ 0 & & & 0 & 1 & 0 \\ -1 & 0 & \dots & 0 & 1 & 0 \end{bmatrix}.$$

Then, by the supposition, we arrive at the following equalities:

$$\begin{aligned} p_{n-1} &= 0 \quad \text{or} \quad p_j = p_k = \tfrac{1}{2}\, p_{n-1}; \\ q_{n-1} &= 0 \quad \text{or} \quad q_j = q_k = \tfrac{1}{2}\, q_{n-1}; \\ r_{n-1} &= 0 \quad \text{or} \quad r_j = r_k = \tfrac{1}{2}\, r_{n-1};, \qquad j,k=1,2,\dots,n-2, \\ &\vdots \qquad\qquad\qquad \vdots \end{aligned}$$

which contradict the assumption $LI^{n+1}(b_1,\dots,b_{n^2+n})$, q.e.d.

Thus we have obtained the following theorem.

Theorem 11. If $F \in \mathcal{F}_n$, $F \in SC_{n+1}$, $|F| \geqslant 4n+4$, and $f \in GP_n(F)$, then f is a composition of at most eighteen (nine when n is even) transformations from Σ.

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Contents

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