DEMONSTRATIO MATHEMATICA
Vol. XIV No 4 1981

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ON DECOMPOSED RECURRENT FINSLER SPACE

In this paper, we have decomposed the relative curvature tensor $\tilde{K}^{i}_{jhk}$, in recurrent Finsler space as $\tilde{K}^{i}_{jhk} = B^{i}\alpha_{jhk}$, where B^{i} is a contravariant vector field and α_{jhk} is a suitable tensor field. The necessary and sufficient condition for a decomposed recurrent Finsler space (or briefly as DR-F_n) to be non flat are obtained.

1. Preliminaries

Let $\tilde{K}^{i}_{jhk}$ be a relative curvature tensor of the n-dimensional Finsler space F_n ([1]), with symmetric metric tensor g_{ij} and symmetric connection coefficients Γ^{i}_{jk}.

Let the relative curvature tensor $\tilde{K}^{i}_{hjk}$ be recurrent so that

$$(1.1) \qquad \tilde{K}^{i}_{hjk;m} = A_m \tilde{K}^{i}_{hjk} ,$$

where A_m is a non vanishing covariant vector.

Further, let us decompose the relative curvature tensor as

$$(1.2) \qquad \tilde{K}^{i}_{jhk} \overset{def}{=\!=} B^{i}\alpha_{jhk},$$

where α_{jhk} represents a suitable tensor field which is skew symmetric in h and k and B^i is a contravariant vector field such that

$$B^i_{;m} \stackrel{\text{def}}{=\!=} \alpha\delta^i_m + A_m B^i, \tag{1.3}$$

where α is assumed to be some scalar function and m followed by notation ";" means corresponding covariant differentiation with respect to x^m as given by

$$X^i_{;m} = \partial_m X^i - (\partial_h X^i)(\partial_m \xi^h) + X^s \Gamma^{\ i}_{sm}. \tag{1.4}$$

D e f i n i t i o n . A Finsler space is said to be decomposed recurrent Finsler space, if the relative curvature tensor $\tilde{K}^{\ i}_{hjk}$ satisfies (1.1), (1.2) and (1.3). For brevity it will be denoted by DR-F_n.

2. <u>The tensor field α_{jhk}</u>

The tensor field α_{jhk} satisfies following relations:

$$(2.1)\quad \begin{cases} \text{(a)} & \alpha_{hjk} + \alpha_{jkh} + \alpha_{khj} = 0, \\ \text{(b)} & \alpha_{hjk;m} + \alpha_{hkm;j} + \alpha_{hmj;k} = 0, \\ \text{(c)} & \alpha_{hjk} + \alpha_{hjk} = 0. \end{cases}$$

Verification of these identities can be done with the help of (1.1), (1.2) and Bianchi identities for relative curvature tensor $\tilde{K}^{\ i}_{hjk}$.

Differentiating (1.2) covariantly with respect to x^m and in view of (1.1) and (1.3), we get

$$\alpha_{jhk;m} = C_m \alpha_{jhk}, \tag{2.2}$$

where we define C_m as follows

$$(2.3)\quad \begin{cases} (a) & \beta C_m + \alpha B_m = 0 \\ (b) & B_i B^h \overset{\text{def}}{=\!=} \beta \delta^h_i. \end{cases}$$

Covariant differentiation of (1.3) with respect to x^l and application of commutation

$$(2.4)\qquad X^i_{;ml} - X^i_{;lm} = X^r \tilde{K}^{\ i}_{rml},$$

yields

$$(2.5)\qquad B^r \tilde{K}^{\ i}_{rml} = A_{ml} B^i + (\delta^i_m \phi_l - \delta^i_l \phi_m),$$

where

$$(2.5)\quad \begin{cases} (a) & A_{ml} \overset{\text{def}}{=\!=} A_{m;l} - A_{l;m}, \\ (b) & \phi_l \overset{\text{def}}{=\!=} \alpha_{;l} - A_l \alpha. \end{cases}$$

Contracting (2.5) by putting $i = l$, we have

$$(2.6)\qquad B^r (\tilde{K}_{rl} + A_{rl}) = 0.$$

Multiplying (2.6) by B_h and using (2.5)(b), we get

$$(2.7)\qquad \beta(\tilde{K}_{hl} + A_{hl}) = 0,$$

which leads to

$$(2.8)\qquad \tilde{K}_{hl} + A_{hl} = 0.$$

Thus we have following theorem.

T h e o r e m 2.1. The Ricci tensor $\widetilde{K}_{ij}$ of the decomposed recurrent Finsler space is expressed in terms of the skew symmetric tensor field A_{ij} as

$$\widetilde{K}_{ij} = A_{ji}.$$

The above Theorem leads to following corollary

C o r o l l a r y 2.1. The necessary and sufficient condition for a DR-F_n, to have a non vanishing Ricci tensor is that the recurrence vector fields is non gradient.

3. The case $\alpha_{jhk} = A_j\alpha_{hk}$

In this section, we shall consider a case of decomposition $\alpha_{jhk} = A_j\alpha_{hk}$, where α_{hk} is a skew symmetric covariant tensor field and A_j is arbitrary covariant vector field.

We have

$$\alpha_{jhk} \overset{\text{def}}{=\!=} A_j\alpha_{hk}. \tag{3.1}$$

Differentiating (3.1) covariantly with respect to x^m and making use of (2.2) and arranging suitably we have

$$A_j\alpha_{hk;m} = (C_mA_j - A_{j;m})\alpha_{hk}.$$

If non vanishing vector field A_j is covariantly constant, the above result reduces to

$$\alpha_{hk;m} = C_m\alpha_{hk}. \tag{3.2}$$

Thus we have

T h e o r e m 3.1. The necessary and sufficient condition for the tensor field α_{hk}, to satisfy similar recurrence condition as that for α_{jhk}, is that the vector field A_j is covariantly constant.

Under decomposition (3.1), we have

$$\tilde{K}_{jhk}^{\ i} = B^i A_j \alpha_{hk}. \tag{3.3}$$

Let us differentiate (3.3) covariantly with respect to x^m and put $A_{j;m} = 0$. Hence we have

$$A_j \alpha_{hk}(B^i_{;m} - (A_m - C_m)B^i) = 0.$$

If DR-F_n is not flat, the last result yields

$$B^i_{;m} = d_m B^i, \tag{3.4}$$

where

$$d_m \overset{\text{def}}{=\!=} A_m - C_m. \tag{3.4)(a}$$

In view of (2.1)(b) and (2.2), we have

$$C_m \alpha_{jk} + C_j \alpha_{km} + C_k \alpha_{mj} = 0, \tag{3.5}$$

where we have omitted the non vanishing A_h.

Transvecting the equation (3.5) by B^m with $B^m C_m + \alpha = 0$, and (1.2), we get

$$\alpha_{hjk} = B_k \tilde{K}_{hj} - B_j \tilde{K}_{hk}. \tag{3.5}$$

The equation (2.5), in view of (1.2) reduces to

$$\tilde{K}_{jhk}^{\ i} = \beta(\delta^i_k \tilde{K}_{hj} - \delta^i_j \tilde{K}_{hk}),$$

which on transvection by A_i yields

$$A_i \tilde{K}_{hjk}^{\ i} = \beta(A_k \tilde{K}_{hj} - A_j \tilde{K}_{hk}). \tag{3.6}$$

Differentiating (1.1) covariantly with respect to x^l and applying the commutation formula (2.4), we get

$$\tilde{K}_{hjk}^{r}\tilde{K}_{rml}^{i} - \tilde{K}_{rjk}^{i}\tilde{K}_{hml}^{r} - \tilde{K}_{hrk}^{i}\tilde{K}_{jml}^{r} - \tilde{K}_{hjr}^{i}\tilde{K}_{kml}^{r} = A_{ml}\tilde{K}_{hjk}^{i}$$

which on transvection by A_i and in view of (3.6) yields

$$(\alpha_{ml} + A_{ml})A_i\tilde{K}_{hjk}^{i} = 0.$$

Let $A_i\tilde{K}_{hjk}^{i} \neq 0$, then we have

(3.7) $$\alpha_{ml} + A_{ml} = 0.$$

The equation (3.5) in view of (3.7) reduces to

(3.8) $$A_hA_{ml} + (B_l\tilde{K}_{hm} + B_m\tilde{K}_{hl}) = 0,$$

which yields the following theorem

T h e o r e m 3.2. The necessary conditions for the decomposition of the relative curvature tensor $\tilde{K}_{hjk}^{i}$ in the form (3.3) with $A_i\tilde{K}_{jhk}^{i} \neq 0$ are

(3.9) $$\alpha_{ml} + A_{ml} = 0,$$

(3.9)(a) $$A_hA_{ml} + (B_l\tilde{K}_{hm} + B_m\tilde{K}_{hl}) = 0.$$

4. Decomposition of curvature tensor with conditions

In this section we shall consider the decomposition of relative curvature tensor $\tilde{K}_{hjk}^{i}$ as follows

(4.1) $$\tilde{K}_{hjk}^{i} = B^iA_h\alpha_{jk},$$

and we shall discuss the above decomposition with the condition

(4.2) $$B^i A_h \overset{\text{def}}{=\!=} P^i_h.$$

R e m a r k 4.1. In this section unlike the previous section we have taken $A_{h;m} \neq 0$.

Let

(4.3) $$\begin{cases} \text{(a)} & P_{j;m} \overset{\text{def}}{=\!=} a_m P^i_j, \\ \text{(b)} & A_{i;m} \overset{\text{def}}{=\!=} b_m A_i. \end{cases}$$

In view of (4.2), (4.1) takes the form

(4.4) $$\tilde{K}^{\ i}_{hjk} = P^i_h \alpha_{jk}.$$

Differentiating (4.4) covariantly with respect to x^l and x^m successively and after some simple calculations, we get

(4.5) $$(A_{l;m}+A_l A_m) = (a_{l;m}-b_{l;m}+c_{l;m}) + (a_l a_m-b_l b_m+c_l c_m).$$

Differentiating (4.2) covariantly with respect to x^m and using (4.3)(a),(b) and (2.3)(a),(b) we get

$$\beta(a_m-b_m) = \alpha B_m + \beta A_m = -\beta C_m + \beta A_m,$$

which easily reduces to

(4.6) $$a_m - b_m = -C_m + A_m.$$

The application of (4.6) in (4.5) yields

(4.7) $$a_l a_m - b_l b_m + C_l C_m - A_l A_m = 0.$$

Thus we have following theorem

T h e o r e m 4.1. Under the decomposition of relative curvature tensor in the form of (4.4) with $A_{i;m} \neq 0$ and (4.3)(a),(b), the covariant vector fields a_m and b_m satisfies the (4.6) and (4.7).

Let us now discuss the case with the possibility of being zero, so that

(4.8) $$B^i_{;j} = A_j B^i = P^i_j.$$

Transvecting (3.1) by B^j and in view of (3.5), we get

(4.9) $$\phi \alpha_{hk} = \beta(\delta^j_k \tilde{K}_{jh} - \delta^j_h \tilde{K}_{jk}),$$

where we have put

(4.10) $$\phi \overset{\text{def}}{=\!=} P^i_i = A_i B^i \quad \text{(a non zero scalar)}.$$

Covariant differentiation of (4.9) with respect to x^m with some calculations yields

(4.11) $$\phi\,(b_m + A_m) = 0,$$

In view of (4.11), (4.6) reduces to

(4.12) $$a_m + C_m = 0,$$

where we have omitted the non zero scalar ϕ.
The relation (4.12) in case $\alpha = 0$, reduces to

(4.13) $$a_m = 0.$$

Differentiating (4.8) covariantly with respect to x^m and commutating the indices j and m we get

$$\phi \alpha_{jm} = (a_m A_j - a_j A_m).$$

In view of (4.13) and (4.12), the last result reduces to

$$\alpha_{jm} = 0. \tag{4.14}$$

Thus we have following theorem

T h e o r e m 4.2. Under the decomposition of relative curvature tensor $\tilde{K}^{i}_{jhk}$ in the form (4.1) with α being zero, we have the results:

(i) The vector field P^i_j is covariantly constant.

(ii) DR-F_n is flat.

C o r o l l a r y 4.1. In order that the DR-F_n is not flat space it is necessary and sufficient that decomposed vector field B^i is non recurrent in the sense of same recurrence vector as that of $\tilde{K}^{i}_{jhk}$.

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Received May 7, 1980.