DEMONSTRATIO MATHEMATICA
Vol. XIV No 4 1981

Henryk Gacki

ON THE EXISTENCE AND UNIQUENESS OF A SOLUTION OF THE RANDOM INTEGRAL EQUATION WITH ADVANCING ARGUMENT

1. Introduction

In this paper we shall prove two theorems on the existence of a unique random solution of the following random integral equation with advancing argument:

$$(1) \qquad x(t,\omega) = h(t,\omega) + \int_0^{t+\delta(t)} K(t,\tau,\omega)f(\tau,x(\tau,\omega))d\tau$$

in the class of all continuous and bounded functions defined on R^+ with values in $L_2(\Omega,A,P)$, where (Ω,A,P) denotes the probability space.

Nonrandom differential equations with advancing argument have been investigated by other authors (see [1], [2], [3]).

The problem of the existence of a solution for random integral equation of the Volterra type with advancing argument has been considered in the class of all mappings $x:R^+_x\Omega \longrightarrow R$ such that, for every $t, x(t,\cdot)$ is a random variable (see [5]).

The fundamental theorems of this paper generalize some results of Christ. P.Tsokos and W.J.Padgett [6], by making use of certain ideas of that paper.

2. Preliminaries

We start by introducing the following definitions:

D e f i n i t i o n 1. The function $x: R^+ \times \Omega \to R$ is called a stochastic process if for each $t \in R^+$, $x(t,\cdot)$ is a random variable defined on Ω.

D e f i n i t i o n 2. A stochastic process $x(t,\omega), t \in R^+$, is said to belong to the space $L_2 = L_2(\Omega, A, P)$ or to be a second order if for each $t \in R^+$, we have

$$E(|x(t,\cdot)|^2) = \int_\Omega x(t,\omega)^2 dP(\omega) < \infty$$

We introduce a norm in the following way:

$$\|x(t,\cdot)\|_{L_2} = (E|x(t,\cdot)|^2)^{\frac{1}{2}}.$$

Thus, for each $t \in R^+$, L_2 is a Banach space.

D e f i n i t i o n 3. The function $x: \Omega \to R$ is called P-essentially bounded if there is a constant $\alpha > 0$ such that,

$$P\{\omega : |x(\omega)| > \alpha\} = 0.$$

The space of all such functions will be denoted by $L_\infty = L_\infty(\Omega, A, P)$, the norm in this space is given by the formula:

$$|||x||| = \inf_{\Omega_0} \sup_{\Omega \setminus \Omega_0 \ni \omega} |x(\omega)|,$$

where $P(\Omega_0) = 0$.

D e f i n i t i o n 4. By $C(R^+, L_2(\Omega, A, P))$ we denote the space of all continuous and bounded functions x defined on R^+ with values in $L_2(\Omega, A, P)$. The norm in this space is defined by the formula

$$\|x\|_C = \sup_{t \geq 0} \|x(t,\omega)\|_{L_2(\Omega, A, P)}.$$

D e f i n i t i o n 5. We shall denote by $C_g(R^+,L_2)$ the space of all continuous functions x from R^+ into L_2 such that there exists a positive number z and a positive continuous function g defined on R^+, satisfying the inequality

$$\left(\int_{\Omega}|x(t,\omega)|^2 dP(\omega)\right)^{\frac{1}{2}} \leq z\, g(t) \quad \text{for every} \quad t\in R^+.$$

In C_g the norm will be defined as follows

$$\|x\|_{C_g} = \sup_{t\geq 0} \frac{\|x(t,\cdot)\|}{g(t)} L_2.$$

D e f i n i t i o n 6. Let $C_c(R^+,L_2)$ denote the space of all continuous functions x on R^+ into L_2. On this space is defined a family of seminorms:

$$\|x\|_n = \sup_{0\leq t\leq n}\left(\int_{\Omega} x(t,\omega)^2 dP(\omega)\right)^{\frac{1}{2}} \quad n=1,2,\ldots$$

The topology on C_c is defined by the following distance function

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n}\frac{\|x-y\|_n}{1+\|x-y\|_n}.$$

Having the topology defined above, we may treat $C_c(R^+,L_2)$ as a complete metric space. Hence, $C(R^+,L_2)$ is a subspace of $C_c(R^+,L_2)$.

D e f i n i t i o n 7. The random process x is a random solution of the equation (1) if for every fixed $t\in R^+$ $x(t,\cdot)\in L_2$ and $x(t,\cdot)$ satisfies (1) a.s.

Now, let (B,D) be a pair of Banach spaces such that $B,D\in C_c(R^+,L_2)$ and let T be a linear operator from $C_c(R^+,L_2)$ into itself.

D e f i n i t i o n 8. The pair of spaces (B,D) is called admissible with respect to the operator $T: C_c(R^+,L_2) \to C_c(R^+,L_2)$ if $T(B) \subset D$.

3. Existence theorems

The main result of the paper are the following theorems:

T h e o r e m 1. Suppose the following assumptions are satisfied:

1^o There exists a function $a: R^+ \to R^+$ and a constant $\Lambda, 0 < \Lambda < e^{-1}$ such that

$$|b(t+\delta(t))-b(t)| < \Lambda, \text{ for every } t \in R^+, \text{ where } b(t) = \int_0^t a(\tau)d\tau.$$

2^o The function $\delta(t)$ is nonnegative, continuous for $t \in R^+$.

3^o For every $t,\tau \in R^+$ such that $0 \leq \tau \leq t+\delta(t) < \infty$ the stochastic kernel $K(t,\tau,\cdot)$ is a random variable satisfying the condition:

$$|||K(t,\tau,\cdot)||| \leq a(\tau)e^{-b(t)}.$$

4^o The function $f: R^+ \times R \to R$ is continuous with respect to $t \in R^+$ uniformly in scalars $x \in R$, such that $f(t,0) \in C_g$ and f satisfies the following Lipschitz condition:

$$|f(t,x) - f(t,y)| \leq \lambda g(t)|x-y|,$$

where $0 \leq \lambda < e^{1-e\Lambda}$ and

$$g(t) = e^{eb(t)}.$$

5^o Let h be a continuous, bounded function from R^+, into L_2 thus $h \in C(R^+,L_2)$.

6^0 Moreover, for any fixed p the inequality

$$\|h\|_C + L\|f_0\|_{C_g} \leq p(1-\lambda e^{e\Lambda-1}),\ L = e^{e\Lambda-1},\ f_0 = f(t,0)$$

holds.

Then the equation (1) has a unique random solution, $x \in C$ such that

$$\|x\|_C \leq p$$

P r o o f . At first we shall prove that under the assumptions 1^0, 3^0 the pair (C_g, C) is admissible with respect to the operator T defined as follows:

$$(Tx)(t,\omega) = \int_0^{t+\delta(t)} K(t,\tau,\omega)x(\tau,\omega)d\tau, \quad \text{where} \quad x \in C_g.$$

In virtue of a generalized Holder's inequality (see [4], p.530) we have

$$\text{(2)} \qquad \|(Tx)(t,\cdot)\|_{L_2} = \left(\int_\Omega \left(\int_0^{t+\delta(t)} K(t,\tau,\omega)x(\tau,\omega)d\tau\right)^2 dP(\omega)\right)^{\frac{1}{2}} \leq$$

$$\leq \int_0^{t+\delta(t)} |||K(t,\tau,\cdot)||| \cdot \|x(\tau,\cdot)\|_{L_2}\, d\tau.$$

This inequality and the assumptions 1^0 and 3^0 imply

$$\text{(3)} \qquad \|(Tx)(t,\cdot)\|_{L_2} \leq \|x\|_{C_g}\, e^{-1} e^{e[b(t+\delta(t))-b(t)]} \leq \|x\|_{C_g}\, e^{e\Lambda-1}.$$

Hence, $\|(Tx)(t,\cdot)\|_{L_2}$ is bounded, so $Tx \in C$ for every $x \in C_g$. Therefore $T(C_g) \subset C$ and (C_g, C) is admissible with to T.

Chris. P. Tsokos and W.J. Padgett proved in [6], p.36 that under the assumption 4^0 if $x \in C$ then $f(t,x(t,\omega)) \in C_g$ and the following inequality holds

$$\|f(t,x)-f(t,y)\|_{C_g} \leq \lambda \|x-y\|_C \quad \text{for every} \quad x,y \in S, \tag{4}$$

where

$$S = \left\{ x \in C: \|x\|_C \leq p \right\}.$$

Now, define the operator $U: S \longrightarrow C$ by

$$(Ux)(t,\omega) = h(t,\omega) + \int_0^{t+\delta(t)} K(t,\tau,\omega) f(\tau, x(\tau,\omega))\, d\tau .$$

The operator U has the following properties:

(i) $U(S) \subset S$

(ii) U is a contraction on S.

We shall prove (i). Let $x \in S$. Applying assumptions 1^0, 3^0, 4^0 and (4), analogously as in (2) we obtain

$$\|U(x(t,\cdot))\|_{L_2} \leq \int_0^{t+\delta(t)} |||K(t,\tau,\cdot)||| \cdot \|f(\tau,\cdot)\|_{L_2} d\tau + \|h(t,\cdot)\|_{L_2} \leq$$

$$\leq \int_0^{t+\delta(t)} a(\tau) e^{-b(t)} e^{eb(\tau)} d\tau \|f\|_{C_g} + \|h(t,\cdot)\|_{L_2} \leq$$

$$\leq \|h(t,\cdot)\|_{L_2} + e^{e\Lambda-1} \|f\|_{C_g} \leq$$

$$\leq \|h(t,\cdot)\|_{L_2} + e^{e\Lambda-1} \|f-f_0\|_{C_g} + e^{e\Lambda-1} \|f_0\|_{C_g} \leq$$

$$\leq \|h(t,\cdot)\|_{L_2} + e^{e\Lambda-1} \lambda \|x\|_C + \|f_0\|_{C_g} \cdot e^{e\Lambda-1}.$$

This implies using 6^0

$$\|Ux\|_C \leq \|h\|_C + e^{e\Lambda-1}\lambda p + e^{e\Lambda-1}\|f_0\|_{C_g} \leq e^{e\Lambda-1}\lambda p + p(1-e^{e\Lambda-1}\lambda) = p.$$

Hence, $Ux \in S$ for all $x \in S$, thus $U(S) \subset S$.

Now, we shall prove (ii). Let $x, y \in S$; applying the generalized Holder's inequality, assumptions 1^0, 2^0, 3^0 and the inequality (3) we have

$$\|(Ux)(t,\cdot)-(Uy)(t,\cdot)\|_{L_2} \leq \left(\int_\Omega \left(\int_0^{t+\delta(t)} K(t,\tau,\omega)|f(\tau,x)-f(\tau,y)|d\tau\right)^2 dP(\omega)\right)^{\frac{1}{2}} \leq$$

$$\leq \int_0^{t+\delta(t)} |||K(t,\tau,\cdot)|||\cdot\|f(\tau,x)-f(\tau,y)\|_{L_2} d\tau \leq$$

$$\leq e^{e\Lambda-1}\|f(\tau,x)-f(\tau,y)\|_{C_g} \leq e^{e\Lambda-1}\lambda\|x-y\|_C.$$

This inequality implies that:

$$\|Ux-Uy\|_C \leq \|k\|x-y\|_C, \quad \text{where} \quad k = \lambda e^{e\Lambda-1},$$

so, by 4^0, U is a contraction operator on S.

Now, we may notice, that from the Banach fixed point theorem, there exists a unique fixed point of U in S, which is a unique random solution of the equation (1).

This fact completes the proof.

We shall close this paper with the analogous theorem dealing with the case when $g(t) = 1$, for all $t \in R^+$.

T h e o r e m 2. Let us assume that the following conditions hold for the random equation (1):

1^0 There exists a function $a: R^+ \to R^+$ and a constant Λ, $0 < \Lambda < e^{-1}$ such that

$$|b(t+\delta(t))-b(t)| < \Lambda, \quad \text{for every } t \in R^+, \text{ where } b(t) = \int_0^t a(\tau)d\tau.$$

2^0 The function $\delta(t)$ is nonnegative, continuous for $t \in R^+$.

3^0 For every $t, \tau \in R^+$ such that $0 \leq \tau \leq t+\delta(t) \leq \infty$ the stochastic kernel $K(t,\tau,\cdot)$ is a random variable satisfying the condition

$$|||K(t,\tau,\cdot)||| \leq a(\tau)e^{e[b(\tau)-b(t)]}$$

4^0 The function $f: R^+ \times R \longrightarrow R$ is continuous uniformly in x and satisfies the following condition

$$|f(t,x) - f(t,y)| \leq \lambda |(x-y)|, \quad \text{where} \quad 0<\lambda<e^{1-e\Lambda}.$$

5^0 Let h be a continuous and bounded function from R^+ into L_2 such that for fixed p the inequality

$$\|h\|_C + e^{1-e\Lambda}\|f_0\|_C \leq p(1-e^{e\Lambda-1})$$

holds.

Then the equation (1) has a unique random solution x, belonging to the class C and such that $\|x\|_C \leq p$.

P r o o f . We need to show that the pair (C,C) is admissible with respect to the integral operator T, defined as follows

$$(Tx)(t,\omega) = \int_0^{t+\delta(t)} K(t,\tau,\omega)x(\tau,\omega)d\tau, \text{ for } x \in C.$$

Indeed, let $x \in C$. Then by the condition 1^0 and the generalized Holder's inequality we have:

$$\|Tx(t,\cdot)\|_{L_2} = \int_0^{t+\delta(t)} |||K(t,\tau,\cdot)||| \cdot \|x(\tau,\cdot)\|_{L_2}\, d\tau \leq$$

$$\leq \|x\|_C \int_0^{t+\delta(t)} a(\tau)e^{e[b(\tau)-b(t)]}\,d\tau \leq e^{e\Lambda-1}\,\|x\|_C$$

Using the last inequality we obtain

$$\|Tx\|_C \leq e^{e\Lambda-1}\|x\|_C.$$

This inequality implies that $Tx \in C$ for every $x \in C$. Therefore the pair (C,C) is admissible with respect to T.

The remainder of the proof follows in a fashion similar to the proof of Theorem 1 and is omitted.

Acknowledgment. The author is indebted to J.Błaż for several useful remarks.

REFERENCES

[1] A. Bielecki, J. Błaż: Über eine Verallgemeinerung der Nicoletti - Aufgabe für Funktional - Differentialgleichungen mit voreilendem Argument, Monatsh. Math. 88 (1979) 287-291.

[2] J. Błaż: Über die Nicoletti - Aufgabe für Funktional - Differentialgleichungen mit voreilendem Argument, Arch. Math. 27 (1976) 530 - 534.

[3] J. Błaż, W. Walter: Über Funktional - Differentialgleichungen mit voreildendem Argument, Monatsch. Math. 82 (1976) 1-16.

[4] N. Dunford, J.T. Schwartz: Linear operators, Part I, New York 1958.

[5] H. Gacki: On the random integral equation with advancing argument, (to appear).

[6] Chris P. Tsokos, W.J. Padgett: Random integral equations with applications to stochastic systems, Berlin, Heidelberg, New York 1971.

INSTITUTE OF MATHEMATICS, SILESIAN UNIVERSITY, KATOWICE
Received May 3rd, 1980.