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AN APPLICATION OF DIAGONAL OPERATION: DIRECT DECOMPOSABILITY OF HOMOMORPHISMS

The decomposition of homomorphisms of direct products of algebras was investigated in a number of papers dealing with the necessary or sufficient conditions for the direct decomposability of homomorphisms (see, e.g., [1], [2], [3] and [10]). However, as far as we know, the complete characterization of this phenomen appears comparatively rarely in the mentioned bibliography. The aim of the present paper is to fill this gap by giving necessary and sufficient conditions for the direct decomposability of homomorphisms (section 2, Theorem 1). Further, by applying this theorem, some concrete conditions for directly decomposable homomorphisms are derived.

1. Basic notions and preliminaries

(i) Directly decomposable homomorphisms

Let $\mathcal{A}_1, \mathcal{B}_1, \mathcal{A}_2, \mathcal{B}_2$ be algebras of the same type. It is a trivial fact that the direct product of homomorphisms $f_A : \mathcal{A}_1 \to \mathcal{A}_2$ and $f_B : \mathcal{B}_1 \to \mathcal{B}_2$, i.e. the mapping $f_A \times f_B : \mathcal{A}_1 \times \mathcal{B}_1 \to \mathcal{A}_2 \times \mathcal{B}_2$ defined by $f_A \times f_B(a,b) = (f_A(a), f_B(b))$ for every $(a,b) \in A_1 \times B_1$, is the homomorphism of algebra $\mathcal{A}_1 \times \mathcal{B}_1$ into the algebra $\mathcal{A}_2 \times \mathcal{B}_2$.

Conversely, we introduce the concept of directly decomposable homomorphisms as follows:

D e f i n i t i o n 1. The homomorphism $f: \mathcal{A}_1 \times \mathcal{B}_1 \to \mathcal{A}_2 \times \mathcal{B}_2$ is called directly decomposable if there are homomorphisms $f_A: \mathcal{A}_1 \to \mathcal{A}_2$ and $f_B: \mathcal{B}_1 \to \mathcal{B}_2$ such that $f = f_A \times f_B$.

Let $\mathfrak{A}$, $\mathfrak{B}$ be algebras of the same type. Then the symbol $Hom(\mathfrak{A}, \mathfrak{B})$ denotes the set of all homomorphisms of $\mathfrak{A}$ into $\mathfrak{B}$.

(ii) Diagonal operation

D e f i n i t i o n 2 (J.Płonka [9]). An algebra $\langle A,d\rangle$ is called an n-dimensional diagonal algebra if the n-ary operation $d: A^n \longrightarrow A$ satisfies

(D1) $$d(a,\dots,a) = a \quad \text{for every} \quad a \in A;$$

(D2) $$d\,(d(a_{11},\dots,a_{1n}),\dots,d(a_{n1},\dots,a_{nn})) = d(a_{11},\dots,a_{nn})$$

for every $a_{ij} \in A$, $1 \leq i,j \leq n$.

Every operation satisfying conditions (D1) and (D2) will be called a diagonal operation (see [5], [9] and also [4]). It is well-known that on the Cartesian product $A \times B$ of non-empty sets A,B a diagonal operation $d: (A\times B)^2 \longrightarrow A\times B$ can be defined by $d((a,b),(a',b')) = (a,b')$ for every (a,b), $(a',b') \in A \times B$. Following S.Fajtlowicz [5], the operation d will be called the canonical diagonal operation and the 2-dimensional diagonal algebra $\langle A \times B,d\rangle$ will be called the canonical diagonal algebra. One easily checks that $x = d(y,x)$ if and only if $pr_A x = pr_A y$ and, dually, $x = d(x,y)$ if and only if $pr_B x = pr_B y$ hold for every elements x,y of the canonical diagonal algebra $\langle A \times B,d\rangle$.

(iii) Nonindexed product

The concept of nonindexed product of algebras (of varying similarity type) was introduced by W.Narkiewicz (unpublished), and investigated or used by S.Fajtlowicz, A.Goetz [7], G.Grätzner, W.D.Neumann, W.Taylor and others. For the sake of completeness, we recall

D e f i n i t i o n 3. By the nonindexed product $\mathfrak{A} \otimes \mathfrak{B}$ of algebras $\mathfrak{A}$, $\mathfrak{B}$ is meant the algebra whose universe is the Cartesian product of the sets A,B, and which has an n-ary operation g corresponding to each pair of n-ary polynomials p and q of $\mathfrak{A}$ and $\mathfrak{B}$, respectively, where g is defined by

$$g((a_1,b_1),\dots,(a_n,b_n)) = (p(a_1,\dots,a_n),\ q(b_1,\dots,b_n))$$

for every n-tuple $(a_i,b_i)_{i\le n} \in (A\times B)^n$.

The following two statements are easily verified:

If $\mathcal{A}_1$, $\mathcal{B}_1$, $\mathcal{A}_2$, $\mathcal{B}_2$ are algebras of the same type then $\mathcal{A}_1 \otimes \mathcal{B}_1$ is of the same type as $\mathcal{A}_2 \otimes \mathcal{B}_2$.

If e_i^2 denotes the i-th trivial operation, $i = 1,2$, then the operation d of $\mathcal{A}\otimes\mathcal{B}$ corresponding to the pair (e_1^2,e_2^2) is the canonical diagonal operation on the set $A\times B$.

(iv) For any mapping $f: A \to B$ and every equivalence relation θ on A the binary relation $\{(f(a_1),f(a_2));\ (a_1,a_2)\in\theta\} \subseteq B^2$ is denoted by $f\times f(\theta)$. Further details of $f\times f(\theta)$ can be found in [12].

2. The decomposition theorem

T h e o r e m 1. Let $\mathcal{A}_1$, $\mathcal{B}_1$, $\mathcal{A}_2$, $\mathcal{B}_2$ be algebras of the same type. Then for any homomorphism $f\in \mathrm{Hom}(\mathcal{A}_1\times\mathcal{B}_1, \mathcal{A}_2\times\mathcal{B}_2)$ the following conditions are equivalent:

(1) f is directly decomposable;

(2) $f\in \mathrm{Hom}(\mathcal{A}_1\otimes\mathcal{B}_1, \mathcal{A}_2\otimes\mathcal{B}_2)$;

(3) $f\in \mathrm{Hom}(\langle A_1\times B_1,d\rangle, \langle A_2\times B_2,d\rangle)$;

(4) $f\times f\,(\mathrm{Ker}\ pr_{A_1}) \subseteq \mathrm{Ker}\ pr_{A_2}$ and $f\times f(\mathrm{Ker}\ pr_{B_1}) \subseteq \mathrm{Ker}\ pr_{B_2}$.

P r o o f . (1)$\Rightarrow$(2). Assume $f = f_A\times f_B$ for homomorphisms $f_A : \mathcal{A}_1\to\mathcal{A}_2$ and $f_B : \mathcal{B}_1\to\mathcal{B}_2$ (notice that f_A and f_B are uniquely determined whenever they exist). Obviously, f_A preserves every polynomial of algebra $\mathcal{A}_1$ and, independently, f_B preserves every polynomial of $\mathcal{B}_1$. Summarizing, the mapping $f = f_A\times f_B$ preserves every operation of the nonindexed product $\mathcal{A}_1\otimes\mathcal{B}_1$.

(2)$\Rightarrow$(3) It follows directly from the remark in section 1(iii).

(3)$\Rightarrow$(4) Apparently, it suffices to prove the first inclusion. Thus, assume $(a,b)\in f\times f\,(\mathrm{Ker}\ pr_{A_1})$. Then there are elements $u,v\in A_1\times B_1$ such that $a = f(u)$, $b = f(v)$ and $(u,v)\in \mathrm{Ker}\ pr_{A_1}$. By section 1 (ii), we get $u = d(v,u)$

and thus, using the hypothesis, $f(u) = d(f(v),f(u))$. This means that $a = d(b,a)$ and hence $pr_{A_2} a = pr_{A_2} b$ or, equivalently, $(a,b) \in \operatorname{Ker} pr_{A_2}$, which was to be proved.

(4) $\Longrightarrow$ (1) Choose arbitrary element $(r,s) \in A_1 \times B_1$. By applying the hypothesis, one easily checks that the following two assignements $a \longmapsto pr_{A_2} f(a,s)$, for every $a \in A_1$, and $b \longmapsto pr_{B_2} f(r,b)$, for every $b \in B_1$, represent mappings $f_A(a) = pr_{A_2} f(a,s)$ and $f_B(b) = pr_{B_2} f(r,b)$. Since elements r,s were chosen arbitrarily, we have also $f_A(a) = pr_{A_2} f(a,b)$ and $f_B(b) = pr_{B_2} f(a,b)$ for any $(a,b) \in A_1 \times B_1$, proving that $f(a,b) = (f_A(a), f_B(b))$, i.e. $f = f_A \times f_B$. It remains to verify that f_A and f_B are homomorphisms; we omit the easy proof.

An immediate consequence of the preceding theorem is the following

C o r o l l a r y 1. Let $\mathcal{A}$, $\mathcal{B}$ be algebras of the same type. Then the following two conditions are equivalent:

(1) Every endomorphism of the product $\mathcal{A} \times \mathcal{B}$ is directly decomposable;

(2) The congruence relations $\operatorname{Ker} pr_A$ and $\operatorname{Ker} pr_B$ are fully invariant.

Combining Theorem 1(2) with, e.g., ([11]; Lemma 1.12), we obtain the following connection between directly decomposable homomorphisms and directly decomposable congruence relations (see [6] for this concept).

C o r o l l a r y 2. The kernel of directly decomposable homomorphism is a directly decomposable congruence.

Notice that the converse is false, see the example below.

3. Applications to some concrete classes of algebras

Firstly, we apply Theorem 1 to the class of algebras investigated by G.A.Fraser and A.Horn [6] and, independently, by H.Werner [12].

C o r o l l a r y 3. Let $\mathcal{A}_1, \mathcal{B}_1, \mathcal{A}_2, \mathcal{B}_2$ be algebras of the same type with two binary polynomials $+$ and $\cdot$, and constants 0 and 1 such that the identities $x \cdot 1 = x+0 = 0+x = x$ and $x \cdot 0 = 0$ hold in $\mathcal{A}_1, \mathcal{B}_1, \mathcal{A}_2, \mathcal{B}_2$. Then a homomorphism

$f \in Hom(\mathfrak{A}_1 \times \mathfrak{B}_1, \mathfrak{A}_2 \times \mathfrak{B}_2)$ is directly decomposable if and only if $f(1,0) = (1,0)$ and $f(0,1) = (0,1)$.

P r o o f . The "only if" part being trivial, assume now that $f(1,0) = (1,0)$ and $f(0,1) = (0,1)$ hold. Then, thanks to the expression of the canonical diagonal operation (see [12]), we get

$$\begin{aligned} f(d(x,y)) &= f((x\cdot(1,0))+(y\cdot(0,1)))=(f(x)\cdot f(1,0))+(f(y)\cdot f(0,1))= \\ &= (f(x)\cdot(1,0))+(f(y)\cdot(0,1))=(pr_{A_2}f(x),0)+(0,pr_{B_2}f(y))= \\ &= (pr_{A_2}f(x),pr_{B_2}f(y))=d(f(x),f(y)) \end{aligned}$$

for every $x,y \in A_1 \times B_1$. By applying Theorem 1(3), the conclusion follows.

R e m a r k . Apparently, Corollary 3 gives us necessary and sufficient conditions for the directly decomposable homomorphisms of

(i) lattices with 0 and 1 : A homomorphism $f \in Hom(\mathfrak{A}_1 \times \mathfrak{B}_1, \mathfrak{A}_2 \times \mathfrak{B}_2)$ is directly decomposable if and only if $f(1,0) = (1,0)$ and $f(0,1) = (0,1)$;

(ii) rings with 1 : A homomorphism $f \in Hom(\mathfrak{A}_1 \times \mathfrak{B}_1, \mathfrak{A}_2 \times \mathfrak{B}_2)$ is directly decomposable if and only if $f(1,0) = (1,0)$ or $f(0,1) = (0,1)$.

The following example shows that for arbitrary homomorphism of direct product of rings with 1 the condition "$f(1,0) = (1,0)$ or $f(0,1) = (0,1)$" need not be fulfilled. Consider the Galois field GF(2) with two elements 0 and 1, and let f be the mapping $f\colon GF(2) \times GF(2) \longrightarrow GF(2) \times GF(2)$ defined by $f(0,0) = (0,0)$, $f(1,0) = (0,0)$, $f(0,1) = (1,1)$, $f(1,1) = (1,1)$. Clearly, f is the homomorphism of $GF(2) \times GF(2)$ into $GF(2) \times GF(2)$ such that $f(1,0) \neq (1,0)$ and $f(0,1) \neq (0,1)$.

Nevertheless, in the case of integral domains, it can be easily seen that $f(1,0) = (1,0)$ or $f(1,0) = (0,1)$ is true for every homomorphism onto, i.e., every surjective homomorphism f is directly decomposable or the composition $\pi \circ f$, where π is the permutation $\pi(a_2,b_2) = (b_2,a_2)$ for every $(a_2,b_2) \in A_2 \times B_2$, is directly decomposable.

In the case of arbitrary lattices, the directly decomposable homomorphisms are characterized by the following

C o r o l l a r y 4. Let $\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{A}_2, \mathfrak{B}_2$ be arbitrary lattices. Then a homomorphism $f \in \mathrm{Hom}(\mathfrak{A}_1 \times \mathfrak{B}_1, \mathfrak{A}_2 \times \mathfrak{B}_2)$ is directly decomposable if and only if f preserves the operations $(\vee,\wedge)$ and $(\wedge,\vee)$.

P r o o f . It follows directly from the fact that the canonical diagonal operation d on the product $\mathfrak{A} \otimes \mathfrak{B}$ of lattices $\mathfrak{A}$, $\mathfrak{B}$ can be expressed as follows

$$d((a,b),(a',b')) = ((a,b)\wedge(a\vee a',b\wedge b'))\vee((a',b')\wedge(a\wedge a',b\vee b'))$$

for every elements $(a,b), (a',b') \in A \times B$.

Now we turn our attention to the directly decomposable homomorphisms of unitary R-modules. Let $\mathfrak{A}$ be a unitary R-module over a ring R with unit element 1, and let $\{\lambda_r : A \longrightarrow A; r \in R\}$ be the set of fundamental unary operations of $\mathfrak{A}$. Obviously, the nonindexed product $\mathfrak{A} \otimes \mathfrak{B}$ of two unitary R-modules $\mathfrak{A}$, $\mathfrak{B}$ has unary operations $\lambda_{r,s} : A \times B \longrightarrow A \times B$, $r,s \in R$, defined by $\lambda_{r,s}(a,b) = (\lambda_r(a),\lambda_s(b))$ for every $(a,b) \in A \times B$. Using these operations we characterize the directly decomposable homomorphisms of unitary R-modules.

C o r o l l a r y 5. Let $\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{A}_2, \mathfrak{B}_2$ be unitary R-modules. Then for any homomorphism $f \in \mathrm{Hom}(\mathfrak{A}_1 \times \mathfrak{B}_1, \mathfrak{A}_2 \times \mathfrak{B}_2)$ the following three conditions are equivalent:

(1) f is directly decomposable;

(2) f preserves the unary operations $\lambda_{r,s}$ for every $r,s \in R$;

(3) f preserves the unary operations $\lambda_{1,0}$ and $\lambda_{0,1}$.

P r o o f . The canonical diagonal operation d on the product of two unitary R-modules $\mathfrak{A}$, $\mathfrak{B}$ can be expressed in the form $d((a,b),(a',b')) = \lambda_{1,0}(a,b) + \lambda_{0,1}(a',b')$ for every $(a,b),(a',b') \in A \times B$; the rest of the proof is trivial. Finally, the following "classical" theorem can also be derived from Theorem 1 or, better to say, from the preceding assertion.

C o r o l l a r y 6. Let $\mathcal{A}_1$, $\mathcal{B}_1$, $\mathcal{A}_2$, $\mathcal{B}_2$ be vector spaces with bases $\{a_1^1,\dots,a_{n_1}^1\}$, $\{b_1^1,\dots,b_{m_1}^1\}$, $\{a_1^2,\dots,a_{n_2}^2\}$, $\{b_1^2,\dots,b_{m_2}^2\}$, respectively. Then a homomorphism $f\in \mathrm{Hom}(\mathcal{A}_1\times\mathcal{B}_1,\mathcal{A}_2\times\mathcal{B}_2)$ is directly decomposable if and only if the matrix representation (f_{jk}) of f in the bases $\{(a_1^i,0),\dots,(a_{n_i}^i,0),\ (0,b_1^i),\dots,(0,b_{m_i}^i)\}$, $i=1,2$, has two blocks, i.e. $f_{jk}=0$ if $j\le n_2$, $k>n_1$ or $j>n_2$, $k\le n_1$.

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