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ON CERTAIN CLASSES OF β-STARLIKE AND QUASI-β-STARLIKE k-SYMMETRICAL HOLOMORPHIC FUNCTIONS

1. In paper [7] the author has introduced the family $\mathcal{P}(\beta)$ composed of functions P defined by the formula

$$(1.1) \qquad P(z) = 1 + p_1 z + p_2 z^2 + \dots$$

holomorphic in the disc $K = \{z : |z| < 1\}$ and satisfying the condition

$$(1.2) \qquad \left|\frac{P(z) - 1}{P(z) + 1}\right| < \beta, \quad z \in K,$$

where β is a fixed number of the interval $(0,1]$. Let us observe that $\mathcal{P}(\beta) \subset \mathcal{P}$ and $\mathcal{P}(1) \equiv \mathcal{P}$, where $\mathcal{P}$ is the well-known class of Carathéodory functions.

Let $\mathcal{P}(\beta,k)$, $k \in N$, be the subclass composed of k-symmetrical functions of the class $\mathcal{P}(\beta)$, that is of functions of the form

$$(1.3) \qquad P(z) = 1 + p_k z^k + p_{2k} z^{2k} + \dots$$

which satisfy condition (1.2) in the disc K. By the definition of the family $\mathcal{P}(\beta,k)$ we have $\mathcal{P}(1,k) \equiv \mathcal{P}_k$, where $\mathcal{P}_k$ is the subclass of k-symmetrical functions of the class $\mathcal{P}$. It is easy to verify that if $P \in \mathcal{P}(\beta,k)$ and $0 < \beta < 1$, then

$$(1.4)\qquad \left|P(z) - \frac{1+\beta^2}{1-\beta^2}\right| < \frac{2\beta}{1-\beta^2}, \quad z \in K.$$

In the same manner as in paper [5] one may establish the following lemmas:

L e m m a 1. The function P belongs to $\mathcal{P}(\beta,k)$ if and only if there exists a function ω holomorphic in the disc K and satisfying the conditions $\omega(0) = 0$, $|\omega(z)| \le |z|^k$, and

$$(1.5)\qquad P(z) = \frac{1+\beta\omega(z)}{1-\beta\omega(z)}, \qquad z \in K.$$

L e m m a 2. The function P belongs to $\mathcal{P}(\beta,k)$ if and only if there exists a function p of the class $\mathcal{P}_k$ such that

$$(1.6)\qquad P(z) = \frac{(1+\beta)p(z) + 1 - \beta}{(1-\beta)p(z) + 1 + \beta}, \qquad z \in K.$$

Denote by z_0 an arbitrarily fixed point of the disc K, and define the functional

$$(1.7)\qquad H : \mathcal{P}(\beta,k) \ni P \longrightarrow H(P) = P(z_0).$$

L e m m a 3. The range of the functional (1.7) is the closed disc with center c and radius ϱ, where

$$(1.8)\qquad c = \frac{1+\beta^2 r^{2k}}{1-\beta^2 r^{2k}}, \quad \varrho = \frac{2\beta r^k}{1-\beta^2 r^{2k}}, \quad r = |z_0|.$$

Let $\mathcal{P}_2(\beta,k)$ denote the subclass of the family $\mathcal{P}(\beta,k)$ composed of functions P of the form (1.6) such that the function $p \in \mathcal{P}_k$ is defined by the formula

$$(1.9)\qquad p(z) = \frac{1+\lambda}{2} p_1(z) + \frac{1-\lambda}{2} p_2(z),$$

where

(1.10) $$p_m(z) = \frac{1+\varepsilon_m z^k}{1-\varepsilon_m z^k}, \quad |\varepsilon_m| = 1, \quad m = 1,2,$$

L e m m a 4. If $P \in \mathcal{P}_2(\beta,k)$, then for $z = re^{i\varphi}$, $0 \leq r < 1$, $0 \leq \varphi < 2\pi$, we have

(1.11) $$P(z) = c + \tilde{k}\gamma,$$

where $|\gamma| = 1$, $0 \leq \tilde{k} \leq \rho$, c and ρ being defined by formulas (1.8).

L e m m a 5. If $P \in \mathcal{P}_2(\beta,k)$, then for $|z| = r < 1$ we have

(1.12) $$zP'(z) = k\,\frac{P^2(z)-1}{2} - k\,\frac{\rho^*}{2\rho}\left[\rho^2 - |P(z)-c|^2\right]\eta^*,$$

where c, ρ are defined by formulas (1.8) and

(1.13) $$\rho^* = \frac{2r^k}{1-r^{2k}}, \quad |\eta^*| = 1.$$

2. Denote by $S^*(\beta,k)$ the class of functions f defined by the formula

(2.1) $$f(z) = z + \sum_{n=1}^{\infty} a_{nk+1} z^{nk+1}$$

holomorphic in the disc K and such that

(2.2) $$\frac{zf'(z)}{f(z)} = P(z), \quad P \in \mathcal{P}(\beta,k).$$

Let us observe that $S^*(\beta,1) = S^*(\beta)$, where $S^*(\beta)$ is the family considered in paper [7]; $S^*(1,1) = S^*$, S^* being the well-known class of starlike functions.

T h e o r e m 1. If $f \in S^*(\beta,k)$, $0 < \beta < 1$, then

(2.3) $$\left|a_{nk+1}\right| \leq (-1)^n \binom{-2/k}{n} \beta^n, \quad n=0,1,\ldots,p,$$

(2.3) $$\left|a_{nk+1}\right| \leq (-1)^p \frac{p}{n} \binom{-2/k}{n} \beta^p, \quad n=p+1,p+2,\ldots,$$

where p is a natural number from the interval

(2.4) $$\left[\frac{2\beta}{k(1-\beta)}, \frac{2\beta + k(1-\beta)}{k(1-\beta)}\right).$$

P r o o f . Let $f \in S^*(\beta,k)$. By Lemma 1 and by definition of the family $S^*(\beta,k)$ we have

$$\frac{zf'(z)}{f(z)} = \frac{1+\beta\omega(z)}{1-\beta\omega(z)}, \quad \omega(0) = 0 \quad \text{and} \quad |\omega(z)| \leq |z|^k.$$

Hence, taking account of (2.1), we get

(2.5) $$\sum_{m=0}^{\infty} mk\, a_{mk+1} z^{mk+1} = \beta\omega(z) \sum_{m=0}^{\infty} (mk+2)\, a_{mk+1} z^{mk+1}.$$

Since $a_1 = 1$, the inequality (2.1) holds for $n = 0$. Let $n > 0$; then, making use of the method given in paper [2], we obtain

(2.6) $$(nk)^2 \left|a_{nk+1}\right|^2 \leq \sum_{m=0}^{n-1} \left[(mk+2)^2\beta^2 - (mk)^2\right] \left|a_{nk+1}\right|^2.$$

Let us note that

(2.7) $$\left[(n-1)k + 2\right]^2\beta^2 - (n-1)^2k^2 > 0$$

if and only if $n \leq p$, where p is a natural number from the interval (2.4). Making use of (2.6) we get by induction (2.3) and (2.3'). The estimation (2.3) is sharp. The function which realizes the equality is of the form

$$f(z) = \frac{z}{(1-\varepsilon\beta z^k)^{2/k}}, \qquad |\varepsilon| = 1.$$

For $k = 1$ we get the estimate given in paper [9]. Observe, moreover, that for $\beta = 1$ the inequality (2.7) holds for every natural number n, hence the estimation (2.3) holds for $n=0,1,2,\ldots$ In the case where $k = 1$ we get the well-known result for starlike functions.

T h e o r e m 2. If $f \in S^*(\beta,k)$, then for $|z| = r$, $0<r<1$, we have

$$(2.8) \qquad \frac{r}{(1+\beta r^k)^{2/k}} \le |f(z)| \le \frac{r}{(1-\beta r^k)^{2/k}}.$$

The estimates are sharp. Equalities hold at the point $z = re^{i\varphi}$ for the functions

$$(2.9) \qquad f^*(z) = \frac{z}{(1+\beta e^{-ik\varphi_z k})^{2/k}}, \quad f^{**}(z) = \frac{z}{(1-\beta e^{-ik\varphi_z k})^{2/k}},$$

respectively.

P r o o f . Let $f \in S^*(\beta,k)$. From the definition of the family $S^*(\beta,k)$ we draw directly

$$(2.10) \qquad \frac{zf'(z)}{f(z)} - \frac{1}{z} = \frac{P(z)-1}{z},$$

whence

$$f(z) = |z| \quad \exp\int_0^1 \mathrm{re}\, \frac{P(zt)-1}{t}\,dt, \quad t \in R,$$

Making use of Lemma 3 we get

$$\frac{1-\beta(rt)^k}{1+\beta(rt)^k} \le \mathrm{re}\; P(zt) \le \frac{1+\beta(rt)^k}{1-\beta(rt)^k}.$$

Taking account of these inequalities in (2.10) we obtain the assertion of the theorem.

It is easy to check that the equalities hold in (2.8) for the functions defined by (2.9), respectively.

Let us now consider the functional $\mathrm{re}\left[1+\frac{zf''(z)}{f'(z)}\right]$ defined in the class $S^*(\beta,k)$. We shall prove the following theorem.

T h e o r e m 3. If $f\in S^*(\beta,k)$, then for every fixed z, $|z|=r$, $r\in(0,1)$, we have the sharp estimation

$$\mathrm{re}\left[1+\frac{zf''(z)}{f'(z)}\right]\geqslant\begin{cases} m_1(r,\beta) & \text{for} \quad r\in(0,r_*)\\ m_2(r,\beta) & \text{for} \quad r\in(r_*,1),\end{cases}\tag{2.11}$$

where

$r_*\in(0,1)$ is the unique root of the equation

$$A(r,\beta)=0,\tag{2.12}$$

$$A(r,\beta)=(k+1)(1-r^k)^2(1-r^{2k})-2kr^k(1-\beta)(1+\beta r^{2k}),\tag{2.13}$$

$$m_1(r,\beta)=\frac{1-2(1+k)\beta r^k+\beta^2r^{2k}}{1-\beta^2r^{2k}},\tag{2.14}$$

$$m_2(r,\beta)=\sqrt{k(R-1)(2+k+kR)}-\frac{k(1+\beta^2r^{2k})}{\beta(1-r^{2k})},\quad R=\frac{\varrho^*}{\varrho}\tag{2.15}$$

for ϱ, ϱ^* defined by the formulas (1.8), (1.13).

In the estimation (2.11) equality holds for the following functions, respectively

$$f_1(z)=f_1(z,\beta)=z(1+\beta e^{-ik\varphi_z k})^{2/k},\tag{2.16}$$

$$f_2(z)=f_2(z,\beta)=z\exp\int_0^z\frac{2\beta e^{-i\varphi_\zeta k}(d-e^{-ik\varphi_\zeta k})}{1-(1+\beta)de^{-ik\varphi_\zeta k}+\beta e^{-2ik\varphi_\zeta 2k}}\,d\zeta,\tag{2.17}$$

(2.18) $$d(r;\beta) = \frac{1}{r}\,\frac{(1+\beta r^2)s_1 - (1-\beta r^2)}{(1+\beta)s_1 - (1-\beta)},$$

(2.19) $$s_1 = \sqrt{\frac{k(\varrho^* - \varrho)}{2\varrho + k(\varrho^* + \varrho)}}.$$

P r o o f . Let $f \in S^*(\beta,k)$. Then, by definition of the family $S^*(\beta;k)$, we have

$$1 + \frac{zf''(z)}{f'(z)} = P(z) + z\,\frac{P'(z)}{P(z)}, \qquad P \in \mathcal{P}(\beta,k).$$

Making use of a theorem of Zmorovič [10] and of Lemmas 3-5 we get

$$\operatorname{re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geqslant \min_{(s,t)} G(s,t),$$

where the function

(2.20) $$G(s,t) = \left[\frac{(2+k)s^2 - k}{2s} - \frac{ck\varrho^*}{\varrho}\right]\cos t + \frac{(s^2+1)k}{2s\varrho}$$

is defined in the domain

(2.21) $$D = \left\{(s,t) : c - \varrho \leqslant s \leqslant c + \varrho,\ -\psi(s) \leqslant t \leqslant \psi(s)\right\},$$

$$\psi(s) = \arccos\frac{s^2+1}{2cs}, \qquad 0 \leqslant \psi(s) \leqslant \psi(1),$$

c, ϱ, ϱ^* being defined by the formulas (1.8), (1.13), respectively. A direct calculation shows that the function $G(s,t)$ can attain a minimum in the domain D only on the diameter $t = 0$. Hence the problem of determining the minimum of the function $G(s,t)$ in the considered domain D comes to find the minimum of the function $G_1(s) = G(s,0)$ in the interval $[c-\varrho,\ c+\varrho]$. The function $G_1(s)$ attains at the point s_1, given by formula (2.19), its absolute minimum in the interval $[c-\varrho, c+\varrho]$ if $c-\varrho < s_1$, whereas if $s_1 \leqslant \leqslant c-\varrho$, then $G_1(s)$ attains its absolute minimum in $[c-\varrho, c+\varrho]$ at the point $c-\varrho$. From the above considerations follows the desired conclusion. It is easy to verify that the functions (2.16),(2.17) realize the equality in the estimation (2.11).

A similar argument leads to the following theorem.

T h e o r e m 4. If $f \in S^*(\beta,k)$, then for any fixed z, $|z| = r$, $r \in (0,1)$, we have the sharp estimation

$$(2.22) \qquad \operatorname{re}\left[1 + \frac{zf''(z)}{f'(z)}\right] \leqslant \begin{cases} m_1(r_* - \beta) & \text{for } r \in (0, r_{**}) \\ m_2(r_* - \beta) & \text{for } r \in (r_{**}, 1), \end{cases}$$

where $r_{**} \in (0,1)$ is the unique root of the equation

$$(2.23) \qquad A(r,-\beta) = 0.$$

The equalities in the estimation (2.22) hold for the function of the form

$$(2.24) \qquad f_1^*(z) = f_1(z,-\beta)$$

$$(2.25) \qquad f_2^*(z) = f_2(z,-\beta),$$

respectively, where $f_1(z,\beta)$, $f_2(z,\beta)$ are defined by (2.16), (2.17). Set

$$r(f) = \sup\left\{r : \operatorname{re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > 0, \quad |z| < r\right\}.$$

As known, the number

$$\text{r.c. } S^*(\beta,k) = \inf\left\{r(f) : f \in S^*(\beta,k)\right\}$$

is called the radius of convexity of the considered family. Since the family $S^*(\beta,k)$ is compact, we draw from Theorem 3 the following one.

T h e o r e m 5. The radius of convexity of the family $S^*(\beta,k)$ is given by the formula

$$\text{r.c. } S^*(\beta,k) = \begin{cases} r_0 & \text{for} \quad \beta \in (0,\beta_*\rangle \\ \left[\dfrac{1+k-\sqrt{2k+k^2}}{\beta}\right]^{1/k} & \text{for} \quad \beta \in (\beta_*,1), \end{cases}$$

where $r_0 \in (0,1)$ is the unique root of the equation

$$(1-\beta)(1+\beta r^{2k})\left[(2+k)\beta(1-r^{2k}) + k(1-\beta^2 r^{2k})\right] = k(1+\beta^2 r^{2k})^2$$

and

$$\beta_* = \frac{(1+k-\sqrt{2k+k^2})(\sqrt{2k+k^2}-\sqrt{4+6k+2k^2})}{2+k}.$$

For k = 1 we get the result established in paper [7].

T h e o r e m 6. If $f \in S^*(\beta,k)$, then for $|z| = r$, $0<r<1$, we have

$$n(r) \leq |f'(z)| \leq N(r), \tag{2.26}$$

where

$$n(r) = \begin{cases} (1-\beta r^k)(1+\beta r^k)^{-(2+k)/k} & \text{for} \quad r \in (0, r_*\rangle \\ (1-\beta r_*^k)(1+\beta r_*^k)^{-(2+k)/k} \cdot \exp R(r) & \text{for} \quad r \in (r_*, 1), \end{cases}$$

$$R(r) = \int_{r_*}^{r} \frac{m_1(r,\beta)-1}{r}\, dr,$$

and

$$N(r) = \begin{cases} (1+\beta r^k)(1-\beta r^k)^{-(2+k)/k} & \text{for} \quad r \in (0,r_{**}\rangle \\ (1+\beta r_{**}^k)(1-\beta r_{**}^k)^{-(2+k)/k} \exp S(r) & \text{for} \quad r \in (r_{**},1), \end{cases}$$

$$S(r) = \int_{r_{**}}^{r} \frac{m_1(r,-\beta) - 1}{r}\, dr,$$

r_*, $r_{**} \in (0,1)$ being the unique roots of equations (2.12) and (2.23), respectively.

P r o o f . Let $f \in S^*(\beta,k)$. Then

$$\log f'(z) = \log |f'(z)| + i \arg(f'(z)).$$

Setting $z = re^{i\varphi}$ we get

$$r \frac{\partial}{\partial r} \log |f'(z)| = \mathrm{re} \frac{zf''(z)}{f'(z)} .$$

Taking account of Theorems 3 and 4 we get the stated conclusion. The functions realizing the equalities in (2.26) are given by formulas (2.16), (2.17), (2.24), (2.25).

3. Let $G^M(\beta,k)$ denote the family of quasi-β-starlike holomorphic k-symmetrical functions g defined by the equation

$$f(g) = \frac{1}{M} f(z), \quad z \in K, \tag{3.1}$$

where $f \in S^*(\beta,k)$ and M is a fixed number from the interval $[1,\infty)$. By the adopted definition it is seen that $G^M(1,1) = G^M$, where G^M is the well-known class of holomorphic quasi-starlike functions.

Set $M = e^t$, $0 \leq t < \infty$ and let $g(z,t)$ be a holomorphic quasi-β-starlike function defined by the equation

$$f(g) = e^{-t} f(z), \quad z \in K, \quad f \in S^*(\beta,k), \quad 0 \leq t < \infty. \tag{3.2}$$

Then, by an argument similar to that of paper [4], it can be proved that

$$(3.3) \qquad \lim_{t\to\infty} e^t g(z,t) = f(z).$$

From the definition (1.1) it follows that if $P \in \mathcal{P}(\beta,k)$, then $\frac{1}{P}$ is also a function of the class $\mathcal{P}(\beta,k)$. Using this remark and the definition of the family $G^M(\beta,k)$ an argument analogous to that of paper [1] leads to the following theorem.

T h e o r e m 7. The function g belongs to the family $G^M(\beta,k)$ if and only if $g(z) = g(z,T)$, where $g(z,t)$ is a solution of the equation

$$(3.4) \qquad \frac{\partial g(z,t)}{\partial t} = -g(z,t)P(g(z,t)), \quad 0 \leqslant t \leqslant T, \quad P \in \mathcal{P}(\beta,k)$$

satisfying the initial condition $g(z,0) = z$.

We shall now give an application of this theorem. To this aim observe that the equation (3.4) is equivalent to the system of equations

$$(3.5) \qquad d \log |g(z,t)| = -\mathrm{re}\, P(g(z,t))\, dt,$$

$$(3.6) \qquad d \arg g(z,t) = -\mathrm{im}\, P(g(z,t))dt.$$

Making use of these equations we shall prove the following theorem.

T h e o r e m 8. If $g \in G^M(\beta,k)$, then for $|z| = r$ we have the sharp estimation

$$(3.7) \qquad x(r) \leqslant |g(z)| \leqslant \chi(r),$$

where

$$(3.8) \qquad \chi(r) = \left[\frac{M^k(1+\beta r^k)}{2\beta^2 r^k}\left(1+\beta r^k - \sqrt{(1+\beta r^k)^2 - \frac{4\beta r^k}{M^k}}\right) - \frac{1}{\beta}\right]^{1/k}$$

$$(3.9) \qquad \chi(r) = \frac{M^k(1-\beta r^k)}{2\beta^2 r^k}\left(1-\beta r^k + \sqrt{(1-\beta r^k)^2 - \frac{4\beta r^k}{M^k}}\right) + \frac{1}{\beta}\Bigg]^{1/k}.$$

P r o o f . From Lemma 3 we draw

$$\frac{1-\beta r^k}{1+\beta r^k} \leq \operatorname{re} P(z) \leq \frac{1+\beta r^k}{1-\beta r^k}. \tag{3.10}$$

Using equation (3.5) and inequality (3.10) we get

$$-\frac{1+\beta x^k}{1-\beta x^k}\,dt \leq d\log x \leq -\frac{1-\beta x^k}{1+\beta x^k}\,dt, \tag{3.11}$$

where $x = |f(z,t)|$, $z \in K$. Integrating (3.11) from 0 to $T = \log M$ and taking account of the initial condition $f(z,0) = z$ we find

$$\begin{cases} \dfrac{|f(z)|^k}{(1+\beta|f(z)|^k)^2} \geq \dfrac{1}{M^k}\,\dfrac{r^k}{(1+\beta r^k)^2} \\[2ex] \dfrac{|f(z)|^k}{(1-\beta|f(z)|^k)^2} \leq \dfrac{1}{M^k}\,\dfrac{r^k}{(1-\beta r^k)^2}, \end{cases} \tag{3.12}$$

where $f(z) = f(z,T)$, $|z| = r$.
The inequalities (3.12) imply the assertion of the theorem. The estimation (3.7) is sharp. In fact, the equalities hold in (3.10) for the functions

$$P_1(z) = \frac{1-\beta\varepsilon z^k}{1+\beta\varepsilon z^k}, \qquad P_2(z) = \frac{1+\beta\varepsilon z^k}{1-\beta\varepsilon z^k},$$

respectively, with $\varepsilon = e^{-i\varphi}$. From (3.11) and from the definition of the family $G^M(\beta,k)$ we infer that the maximum of modulus of a quasi-β-starlike function is attained for the function satisfying the equation

$$\frac{f^k}{(1-\beta f^k)^2} = \frac{1}{M^k}\,\frac{z^k}{(1-\beta z^k)^2},$$

whereas the minimum is attained for the function satisfying the equation

$$\frac{f^k}{(1+\beta f^k)^2} = \frac{1}{M^k}\,\frac{z^k}{(1+\beta z^k)^2}\,.$$

For $k = 1$ and $\beta = 1$ we obtain the estimation of the modulus of a quasi-starlike function ([1], [3]). In the case where $k = 1$ and $\beta \in (0,1)$ we get the result established in [6].

Theorem 9. If $g \in G^M(\beta,k)$, then

$$(3.13)\qquad \left|\arg\frac{g(z)}{z}\right| \le \frac{1}{k}\log\frac{1-\beta|g|^k}{1+\beta|g|^k}\cdot\frac{1+\beta r^k}{1-\beta r^k}\,,\qquad |z| = r,$$

where the equality holds for the functions g defined by the equations

$$(3.14)\qquad \frac{g}{(1-\beta g^k)^{2/k}} = \frac{1}{M}\,\frac{z}{(1-i\beta z^k)^{2/k}}\,,$$

$$(3.15)\qquad \frac{g}{(1+\beta g^k)^{2/k}} = \frac{1}{M}\,\frac{z}{(1+i\beta z^k)^{2/k}}\,.$$

Proof. From equations (3.5) and (3.6) we obtain

$$(3.16)\qquad d\arg g(z,t) = \frac{\operatorname{im} P(g(z,t))}{\operatorname{re} P(g(z,t))}\, d\left|\log g(z,t)\right|\,,$$

where $P \in \mathcal{P}(\beta,k)$.

From Lemmas 1-5 we get

$$(3.17)\qquad \frac{-2\beta r^k}{1-\beta^2 r^{2k}} \le \frac{\operatorname{im} P(z)}{\operatorname{re} P(z)} \le \frac{2\beta r^k}{1-\beta^2 r^{2k}}\,,\qquad |z| = r.$$

The estimation (3.5) is sharp; the maximum is realized by the function

(3.18) $$P_1(z) = \frac{1+\beta iz^k}{1-\beta iz^k},$$

whereas the minimum is attained for the function

(3.19) $$P_2(z) = \frac{1-\beta iz^k}{1+\beta iz^k} \quad \text{for} \quad |z| = r.$$

From (3.16) and (3.17) we get

(3.20) $$\frac{2\beta x^k}{1-\beta^2 x^{2k}}\, d \log x \leq d \arg g(z,t) \leq \frac{-2\beta x^k}{1-\beta^2 x^{2k}}\, d \log x,$$

with $x = |g(z,t)|$.

Integrating (3.20) from 0 to $T = \log M$ and taking into account the condition $g(z,0) = z$ we get (3.13).

From (3.18) and (3.19) and from the definition of the family $G^M(\beta,k)$ we conclude that the functions g realizing the equalities in (3.13) are of the form (3.14) and (3.15), respectively, which completes the proof of the theorem. For $k = 1$ we obtain the result established in [6]. Taking $k = 1$ and $\beta = 1$ we get the case considered in [3].

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