DEMONSTRATIO MATHEMATICA
Vol. XIV No 4 1981

Izabela Foltyńska

ON THE OSCILLATORY SOLUTION OF CERTAIN NONLINEAR INTEGRAL EQUATION

In this paper we will consider the oscillatory properties of a solution of the integral equation

$$(1)\qquad x(t) - \int_{t_0}^{t} K(t,s)x^{\alpha}(s)ds = f(t,x(t)),$$

where

$$f : \langle t_0, \infty) \times R \longrightarrow R$$

$$K : \langle t_0, \infty)^2 \longrightarrow R$$

are continuous functions and $K(t,s) > 0$ for $t \geq s \geq t_0$, and $K(t,s) \leq K(r,s)$ for $t \geq r$, $\alpha = \frac{m}{n}$; $m,n = 1,3,5,\ldots$

By a regular solution of the integral equation (1) we understand every solution $x : \langle t_0, \infty) \longrightarrow R$ for which

$$\operatorname{mes}\left\{t : x(t) = 0\right\} = 0.$$

We will consider only the regular solutions.

A regular solution of the integral equation (1) we call:

a. nonoscillatory, when it has a constant sign for $t \geq t_0$,

b. oscillatory, when it has at least one zero point for $t \geq t_0$,

c. bounded, if there is such a constant c, that

$$|x(t)| \leq c \quad \text{for} \quad t \geq t_0,$$

d. tending to zero when $t \to \infty$, if $\lim_{t\to\infty} x(t) = 0$.

Furthermore we introduce the following assumptions:

$$(2) \qquad \begin{cases} x\, f(t,x) > 0 & \text{for} \quad x \neq 0 \\ f(t,0) = 0. \end{cases}$$

$$(3) \qquad \text{If} \quad x_1 < x_2 \quad \text{then} \quad f(t,x_1) < f(t,x_2).$$

There exists a sequence of numbers L_n such that

$$(4) \qquad L_n = \min_{t \geq t_0} |f(t, L_{n-1})| > 0 \quad \text{for} \quad n \geq 1,$$

where

$$L_0 = \pm 1$$

and

$$\varliminf_{n\to\infty} L_n > 0.$$

There exists a positive constant N, such that

$$(5) \qquad |f(t,x') - f(t,x'')| \leq N|x' - x''| .$$

Under these assumptions we get the following theorem.

T h e o r e m . The divergence of the integral

$$\lim_{t\to\infty} \int_{t_0}^{t} K(t,s)ds = \infty$$

is a necessary and sufficient condition of oscillation of a bounded solution of the integral equation (1).

P r o o f of sufficiency. Let us suppose that the solution $x(t)$ of the equation (1) is nonoscillatory. For the proof we accept that $x(t) > 0$ for $t \geq t_0$.

From the assumption that the solution is bounded it follows that

$$c > x(t) = \int_{t_0}^{t} K(t,s)x^{\alpha}(s)ds + f(t,x(t)) \geq \int_{t_0}^{t} K(t,s)x^{\alpha}(s)ds$$

that is, that

$$x(t) \geq \int_{t_0}^{t} K(t,s)x^{\alpha}(s)ds$$

and

$$x^{\alpha}(s) \geq \left\{ \int_{t_0}^{s} K(s,v)x^{\alpha}(v)dv \right\}^{\alpha}.$$

We multiply both sides of the last inequality by

$$\frac{K(t,s)}{\left\{ \int_{t_0}^{s} K(s,v)x^{\alpha}(v)dv \right\}^{\alpha}}$$

and next we integrate it over $\langle t_0, t \rangle$. Hence for $\alpha = \frac{m}{n} \neq 1$

$$\int_{t_0}^{t} \frac{K(t,s)x^{\alpha}(s)ds}{\left\{ \int_{t_0}^{s} K(s,v)x^{\alpha}(v)dv \right\}^{\alpha}} \geq \int_{t_0}^{t} K(t,s)ds.$$

Because $K(t,s)$ is a nondecreasing function of the first variable, then

$$\int_{t_0}^{t} \frac{K(t,s)x^{\alpha}(s)ds}{\left\{ \int_{t_0}^{s} K(t,v)x^{\alpha}(v)dv \right\}^{\alpha}} \geq \int_{t_0}^{t} K(t,s)ds$$

and

$$\frac{1}{(1-\alpha)\left\{\int_{t_0}^{s} K(t,v)x^{\alpha}(v)dv\right\}^{\alpha-1}}\Bigg|_{t_0}^{t} \geqslant \int_{t_0}^{t} K(t,s)ds.$$

The left side of this inequality is finite, when $t \to \infty$. However, the right side is, from the assumption, divergent. Hence, we get a contradiction.

Similarly for $\alpha = \frac{m}{n} = 1$, it follows from the assumption that the solution of the equation (1) is bounded, that

$$c > x(t) = \int_{t_0}^{t} K(t,s)x(s)ds + f(t,x(t)) \geqslant \int_{t_0}^{t} K(t,s)x(s)ds$$

that is, that

$$x(s) \geqslant \int_{t_0}^{s} K(s,v)x(v)dv.$$

Next we multiply both sides of the above inequality by

$$\frac{K(t,s)}{\int_{t_0}^{s} K(s,v)x(v)dv}$$

and we integrate it from t_0 to t. Hence

$$\int_{t_0}^{t} \frac{K(t,s)x(s)ds}{\int_{t_0}^{s} K(t,v)x(v)dv} \geqslant \int_{t_0}^{t} \frac{K(t,s)x(s)ds}{\int_{t_0}^{s} K(s,v)x(v)dv} \geqslant \int_{t_0}^{t} K(t,s)ds$$

and

$$\ln\left|\int_{t_0}^{s} K(t,v)x(v)dv\right|\Bigg|_{t_0}^{t} \geqslant \int_{t_0}^{t} K(t,s)ds,$$

leads also to a contradiction for $t \to \infty$.

In this way, we show that in the case when $\alpha \neq 1$ and when $\alpha = 1$, the bounded solution of the integral equation (1) is oscillatory.

P r o o f of necessity. We show, that when

$$\lim_{t\to\infty} \int_{t_0}^{t} K(t,s)ds < \infty ,$$

then the equation (1) has a bounded nonoscillatory solution.

The existence of a bounded continuous solution is established by the method of succesive approximation. We consider the sequence $\{x_n(t)\}$ defined in the following way

$$x_0(t) = L_0 = 1 \text{ for } t \in \langle t_0, \infty)$$

$$x_n(t) = f(t,x_{n-1}(t)) + \int_{t_0}^{t} K(t,s)x_{n-1}^{\alpha}(s)ds \quad \text{for} \quad n \geq 1.$$

The elements of the sequence $\{x_n(t)\}$ may be estimated from below. By induction we get

$$x_1(t) \geq f(t,1) \geq \min_{t \geq t_0} f(t,1) = L_1$$

....

$$x_n(t) = f(t,x_{n-1}(t)) + \int_{t_0}^{t} K(t,s)x_{n-1}^{\alpha}(s)ds \geq$$

$$\geq f(t,L_{n-1}) \geq \min_{t \geq t_0} f(t,L_{n-1}) = L_n.$$

Hence

$$x_n(t) \geq L_n \quad \text{for} \quad n = 1,2,\ldots$$

what proves that the elements of the sequence of the succesive approximation are nonoscillatory.

We have now to show that there is a limit of the sequence $\{x_n(t)\}$

(6) $$x(t) = \lim_{n\to\infty} x_n(t) \quad \text{for} \quad t \in \langle t_0, \infty)$$

and that this limit satisfies the equation (1).

To show the existence of the limit (6) of the sequence $\{x_n(t)\}$ it is enough to prove the convergence of the series

(7) $$x_0(t)+\left[x_1(t)-x_0(t)\right] + \left[x_2(t)-x_1(t)\right]+\dots+\left[x_n(t)-x_{n-1}(t)\right]+\dots$$

We estimate the absolute values of terms (7)

$$\left|x_1(t)-x_0(t)\right| \leqslant \left|f(t,x_0(t))-x_0(t)\right| + \left|\int_{t_0}^{t} K(t,s)x_0^{\alpha}(s)ds\right| \leqslant M$$

for a sufficient large t.

$$\left|x_2(t)-x_1(t)\right| \leqslant N \max_{t\geqslant t_0} \left|x_1(t)-x_0(t)\right| +$$

$$+\alpha c^{\alpha-1} \max_{t\geqslant t_0} \left|x_1(t) - x_0(t)\right| \int_{t_0}^{t} K(t,s)ds \leqslant$$

$$\leqslant M(N + \alpha c^{\alpha-1}K) \quad \text{for} \quad t\to\infty$$

....

$$\left|x_n(t)-x_{n-1}(t)\right| \leqslant M(N + \alpha c^{\alpha-1}K)^{n-1}.$$

In the case when $\alpha = 1$ we get the estimation of the series (7) in the form

$$\left|x_1(t)-x_0(t)\right| \leqslant M, \quad \left|x_2(t)-x_1(t)\right| \leqslant M(N+K), \quad \dots,$$

$$\left|x_n(t)-x_{n-1}(t)\right| \leqslant M(N+K)^{n-1}, \quad \dots$$

Every element of the series (7) (without the first element $x_0(t)$) is in absolute value smaller than or equal to the adequate element of the numerical series with positive elements:

$$(8)\quad M + M(N+\alpha c^{\alpha-1}K) + M(N+\alpha c^{\alpha-1}K)^2 + \ldots + M(N+\alpha c^{\alpha-1}K)^{n-1} + \ldots$$

It is a geometrical series where the quatient $q = N + \alpha c^{\alpha-1}K$. If now $N + \alpha c^{\alpha-1}K < 1$, then the numerical series (8) is convergent.

Similarly in the case when $\alpha = 1$, every element of the series (7) is in absolute value smaller than or equal to the elements of the geometrical series

$$(9)\quad M + M(N+K) + M(N+K)^2 + \ldots + M(N+K)^{n-1} + \ldots,$$

where the quatient $q = N+K$. If in this case $N+K < 1$, then the numerical series (9) is also convergent.

The elements of the series (7) are smaller than or equal to, in absolute value, the elements of the numerical convergent series, hence the series (7) is on the basis of Weierstrass's criterion uniformly convergent.

Hence when $N+\alpha c^{\alpha-1}K < 1$ and $N+K < 1$ there is a limit (6). Every element of series (7) is a continuous function of the variable t. Therefore the limit (6) is also a continuous function of the variable t.

We shall prove that the function $x(t)$ satisfies the integral equation (1). From the uniform continuity of the function $f(t,x(t))$ it follows, that for arbitrary $\varepsilon > 0$

$$\left| f(t,x_{n-1}(t)) - f(t,x(t)) \right| < \varepsilon .$$

for sufficient great n.

Furthermore

$$\left| x_n(t) - f(t,x(t)) - \int_{t_0}^{t} K(t,s)x^{\alpha}(s)ds \right| \leq$$

$$\leq \left| f(t,x_{n-1}(t))-f(t,x(t)) \right| + \left| \int_{t_0}^{t} K(t,s)\left[x_{n-1}^{\alpha}(s)-x^{\alpha}(s)\right] ds \right| \leq$$

$$\leq \varepsilon + \left| \int_{t_0}^{t} K(t,s)\left[x_{n-1}^{\alpha}(s)-x^{\alpha}(s)\right] ds \right| .$$

Now if $n \to \infty$, we get

$$\left| x(t) - f(t,x(t)) - \int_{t_0}^{t} K(t,s)x^{\alpha}(s)ds \right| \leq \varepsilon .$$

From the arbitrariness of the number ε it follows that $x(t)$ is the solution of the integral equation (1).

The proof of our theorem in the case when $x(t)<0$ is analogous.

We furthermore remark that in the special case when $K(t,s) = t - s$, and $\frac{\partial^2 f}{\partial t^2} = 0$ we get Atkinson's results [1] for the differential equation $x'' + f\, x^{2n-1} = 0$.

Similar results were obtained by Izjumowa [2] for the differential equation $u'' + f(t,u) = 0$.

REFERENCES

[1] F.V. A t k i n s o n : On second-order non-linear oscillations, Pacific J. Math. 5 (1955) 643-648.

[2] D.W. I z j u m o w a : On the conditions of oscillation and nonoscillation of the solutions of the nonlinear, second-order differential equations, Diff. Urawn. 2, 12 (1966) 1572-1583.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY, POZNAŃ
Received April 2nd, 1980.