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# ON EINSTEIN-KÄHLERIAN SPACES WITH RECURRENT CURVATURE TENSOR

## 1. Introduction

In the present paper, the author has defined and studied Einstein-Kählerian conharmonic recurrent spaces and Einstein-Kählerian spaces with recurrent Bochner curvature tensor. Several theorems have been established. The necessary and sufficient condition for an Einstein-Kählerian conharmonic recurrent space to be Kählerian recurrent has been investigated.

An $n(=2m)$ dimensional Kählerian space is a Riemannian space which admits a tensor field $F_i^h$ satisfying

$$F_j^h \cdot F_h^i = -S_j^i, \tag{1.1}$$

$$F_{ij} = -F_{ji}, \quad (F_{ij} = F_i^\alpha g_{\alpha j}), \tag{1.2}$$

and

$$F_{i,j}^h = 0, \tag{1.3}$$

where the comma(,) followed by an index denotes the operation of covariant differentiation with respect to the metric tensor $g_{ij}$ of the Riemannian space. Let

$$R_{ijk}^h = \partial_i \begin{Bmatrix} j \\ j\ k \end{Bmatrix} \cdot \partial_j \begin{Bmatrix} h \\ i\ k \end{Bmatrix} + \begin{Bmatrix} h \\ i\ l \end{Bmatrix} \begin{Bmatrix} l \\ j\ k \end{Bmatrix} - \begin{Bmatrix} h \\ j\ l \end{Bmatrix} \begin{Bmatrix} l \\ i\ k \end{Bmatrix}, \tag{1.4}$$

$$R_{jk} = R^i_{ijk} \quad \text{and} \quad R = R_{jk}\, g^{jk}$$

be the Riemannian curvature tensor, the Ricci tensor and the scalar curvature, respectively.

Recently, Tachibana [3] has defined the Bochner curvature tensor (with respect to a real coordinate system) by

$$(1.5)\qquad K^h_{ijk} = R^h_{ijk} + \frac{1}{n+4}(R_{ik}S^h_j - R_{jk}S^h_i + g_{ik}R^h_j - g_{jk}R^h_i + S_{ik}F^h_j - S_{jk}F^h_i + F_{ik}S^h_j - F_{jk}F^h_i + 2S_{ij}F^h_k + 2F_{ij}S^h_k) - \frac{R}{(n+2)(n+4)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k),$$

where

$$S_{ij} = F^\alpha_i R_{\alpha j}.$$

The Kählerian conharmonic curvature tensor [2] is given by

$$(1.6)\qquad T^h_{ijk} = R^h_{ijk} + \frac{1}{n+4}(R_{ik}\delta^h_j - R_{jk}\delta^h_i + g_{ik}R^h_j - g_{jk}R^h_i + S_{ik}F^h_j - S_{jk}F^h_i + F_{ik}S^h_j - F_{jk}S^h_i + 2S_{ij}F^h_k + 2F_{ij}S^h_k).$$

Let us suppose that a Kählerian space is an Einstein one. Then the Ricci tensor satisfies

$$(1.7)\qquad R_{ij} = \frac{R}{n}g_{ij}, \quad R_{,\alpha} = 0$$

from which we obtain

(1.8) $$R_{ij,\alpha} = 0,\ S_{ij,\alpha} = 0 \text{ and } S_{ij} = \frac{R}{n} F_{ij}.$$

If a Kählerian space is an Einstein one, then the Bochner curvature tensor and the Kählerian conharmonic curvature tensor reduce to the forms:

(1.9) $$U^h_{ijk} = R^h_{ijk} + \frac{R}{n(n+2)} (g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k),$$

and

(1.10) $$E^h_{ijk} = R^h_{ijk} + \frac{2R}{n(n+4)} (g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k).$$

In view of equations (1.9) and (1.10), we have

(1.11) $$U^h_{ijk} = E^h_{ijk} - \frac{R}{(n+2)(n+4)} (g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k).$$

We shall use the following

D e f i n i t i o n [1]. A Kähler space is said to be recurrent if we have

(1.12) $$R^h_{ijk,\alpha} - \lambda_\alpha R^h_{ijk} = 0$$

for some non-zero recurrence vector $\lambda_\alpha$ and is called Ricci--recurrent if it satisfies the relation

(1.13) $$R_{ij,\alpha} - \lambda_\alpha R_{ij} = 0.$$

Multiplying the above equation by $g^{ij}$, we get

(1.14) $$R_{,\alpha} - \lambda_\alpha R = 0.$$

## 2. Einstein-Kählerian conharmonic recurrent space

D e f i n i t i o n 2.1. An Einstein-Kähler space satisfying the relation

$$E^h_{ijk,\alpha} - \lambda_\alpha E^h_{ijk} = 0, \tag{2.1}$$

where $\lambda_\alpha$ is a non-zero recurrence vector, will be called an Einstein-Kählerian conharmonic recurrent space or briefly an $E - K^*$ space.

D e f i n i t i o n 2.2. An Einstein-Kähler space satisfying the relation

$$U^h_{ijk,\alpha} - \lambda_\alpha U^h_{ijk} = 0 \tag{2.2}$$

where $\lambda_\alpha$ is a non-zero recurrence vector, will be called an Einstein-Kählerian space with recurrent Bochner curvature tensor.

We have the following

T h e o r e m 2.1. A necessary and sufficient condition for an $E\text{-}K^*$ space to be H Kählerian recurrent is that the scalar curvature be equal to zero.

P r o o f . Suppose that an $E\text{-}K^*$ space is Kählerian recurrent. Making use of equations (1.7), (1.8) and (1.10) in (2.1) we obtain

$$R^h_{ijk,\alpha} = \lambda_\alpha\Big[R^h_{ijk} + \frac{2R}{n(n+4)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k)\Big]. \tag{2.3}$$

Since, an $E\text{-}K^*$ space is Kählerian recurrent, equations (2.3) reduces to

$$\frac{2R}{n(n+4)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k) = 0, \tag{2.4}$$

which gives $R = 0$.

Conversely, if an E-K* space satisfies R = 0, then equation (2.3) reduces to

$$R^h_{ijk,\alpha} - \lambda_\alpha R^h_{ijk} = 0,$$

which shows that the space is Kählerian recurrent. This completes the proof.

Similarly in view of Theorem 2.1 and equations (1.7), (1.8) and (1.11) we can prove the following theorem:

T h e o r e m 2.2. A necessary and sufficient condition for an Einstein-Kählerian space with recurrent Bochner curvature tensor to be Kählerian recurrent is that the scalar curvature be equal to zero.

We have the following (See. [4] and [5]).

L e m m a 2.1. The curvature tensor $R_{hijk}$ satisfies the identity

$$(2.5) \qquad R_{hijk,lm}-R_{hijk,ml}+R_{jklm,hi}-R_{jklm,ih}+R_{lmhi,jk}-R_{lmhi,kj}=0,$$

where

$$R_{hijk,l,m} \overset{\text{def}}{=} R_{hijk,lm}.$$

L e m m a 2.2. If $a_{\alpha\beta}$, $b_\gamma$ are quantities satisfying

$$(2.6) \qquad a_{\alpha\beta} = a_{\beta\alpha} \quad \text{and} \quad a_{\alpha\beta}b_\gamma + a_{\beta\gamma}b_\alpha + a_{\gamma\alpha}b_\beta = 0$$

for $\alpha, \beta, \gamma = 1,2,\dots,N$, then either all the $a_{\alpha\beta}$ are zero or all the $b_\gamma$ are zero.

With the aid of above Lemmas, we shall prove the following

T h e o r e m 2.3. In an E-K* space, either recurrence vector is gradient or the space is of constant holomorphic sectional curvature.

P r o o f . Differentiating (2.3) covariantly and using equations (1.3), (1.7), (1.8) and (2.3), we obtain

(2.7) $$R_{ijkh,ab} = (\lambda_{a,b} + \lambda_a\lambda_b)E_{ijkh},$$

where

$$E_{ijkh} = R_{ijkh} + \frac{2R}{n(n+4)}(g_{ik}g_{hj} - g_{ik}g_{hi} + F_{ik}F_{jh} - F_{jk}F_{ih} + 2F_{ij}F_{kh}).$$

From (2.7) and the identity (2.5), we get

(2.8) $$\lambda_{ab}E_{ijkh} + \lambda_{ij}E_{khab} + \lambda_{kh}E_{abij} = 0,$$

where

$$\lambda_{ab} \stackrel{\text{def}}{=\!=} \lambda_{b,a} - \lambda_{a,b}.$$

Equation (2.8) is of the form (2.6) since $E_{ijkh} = E_{khij}$. Thus, from Lemma 2.2, we have Theorem 2.3. This completes the proof.

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