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Henryk Ugowski

ON THE FIRST FOURIER PROBLEM FOR RANDOM PARABOLIC EQUATIONS OF THE SECOND ORDER

In this paper we consider the first Fourier problem

(0.1) $$Mu \equiv \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x,t)u_{x_i} + c(x,t)u - u_t = f(x,t), \quad \varphi(x,t) \in \bar{G} \setminus \Gamma,$$

(0.2) $$u(x,t) = \varphi(x,t), \qquad (x,t) \in \Gamma,$$

where $G \subset R^{n+1} = \{(x,t): x \in R^n, \ t \in R\}$ is a bounded domain, $\bar{G}$ denotes the closure of G and Γ is the parabolic boundary of G. Here M is a parabolic operator in $\bar{G}$ with real-valued coefficients defined in $\bar{G}$, whereas f and φ are random functions defined in $\bar{G}$ and Γ [1], respectively. At first we derive some a priori estimates of Friedman's type for a solution of the problem (0.1), (0.2). These estimates and the existence of a solution of the problem (0.1), (0.2) in the scalar case (i.e. f and φ are real-valued functions) enable us to prove an existence and uniqueness theorem for the problem in question.

[1] Notation and definitions will be stated in Section 1.

The last section of the paper deals with the first Fourier problem for the infinite system of random semilinear parabolic equations

$$(0.3)\quad M^k u^k \equiv \sum_{i,j=1}^{n} a_{ij}^k(x,t)u_{x_i x_j}^k + \sum_{i=1}^{n} b_i^k(x,t)u_x^k + $$

$$+ c^k(x,t)u^k - u_t^k = f^k(x,t,u,u_x), \quad (x,t)\in \bar{G}\setminus\Gamma,$$

$$(0.4)\quad u^k(x,t) = \varphi^k(x,t), \quad (x,t)\in\Gamma, \quad k=1,2,\dots,$$

where M^k $(k=1,2,\dots)$ satisfy the same conditions as M, $u = (u^1,u^2,\dots)$, $u_x = (u_{x_1},\dots,u_{x_n})$ with $u_{x_j} = (u_{x_j}^1,u_{x_j}^2,\dots)$ and f^k, φ^k are certain given random functions. Using the results obtained for the problem (0.1), (0.2) and the Banach fixed point theorem we prove the existence and uniqueness of a solution of the problem (0.3), (0.4).

In the paper [6] there was proved the existence of a solution of the Cauchy problem for the random equation (0.1) with random coefficients (as well as with random functions f and φ). Next, in the paper [7] there has been considered the Cauchy problem for random evolution equations which involves some particular cases of the problems (0.1), (0.2) and (0.3), (0.4) under different assumptions from those of the present paper. Our results obtained for the problem (0.1), (0.2) constitute an extension of appropriate ones concerning the scalar case (Theorems III.6, III.7 and VI.4 of [3][2)]) to the random case.

1. Notation and definitions

Let G be a bounded domain of the Euclidean space R^{n+1} of the variables $(x,t) = (x_1,\dots,x_n,t)$ whose boundary con-

[2)] Throughout this paper when referring to the monographs [1] - [4] we shall denote by a Roman numeral the chapter number.

sists of domains E_0 and E_T lying on the planes $t = 0$ and $t = T = const > 0$, respectively, and of a manifold S situated in the strip $\{(x,t): 0 \leq t \leq T\}$. The set $\Gamma = E_0 \cup S$ is called a parabolic boundary of G. The parabolic distance of points $Q(x,t)$, $Q'(x',t') \in R^{n+1}$ is defined as

$$d(Q,Q') = (|x-x'|^2 + |t-t'|)^{1/2}, \text{ where } |x-x'| = \left[\sum_{i=1}^{n} (x_i - x_i')^2\right]^{1/2}.$$

As in [3] (Sec. III.2), we introduce the following notation for a function $u: G \to R$:

$$\|u\|_G = \sup_{Q \in G} |u(Q)|, \quad H_G^{(\alpha)}(u) = \sup_{Q,Q' \in G} \frac{|u(Q)-u(Q')|}{[d(Q,Q')]^\alpha}\,^{3)},$$

$$\|u\|_G^{(\alpha)} = \|u\|_G + H_G^{(\alpha)}(u), \quad \|u\|_G^{(1+\alpha)} = \|u\|_G^{(\alpha)} + \sum_{i=1}^{n} \|u_{x_i}\|_G^{(\alpha)},$$

$$\|u\|_G^{(2+\alpha)} = \|u\|_G^{(\alpha)} + \sum_{i=1}^{n} \|u_{x_i}\|_G^{(\alpha)} + \sum_{i,j=1}^{n} \|u_{x_i x_j}\|_G^{(\alpha)} + \|u_t\|_G^{(\alpha)}.$$

The set $C^{(k+\alpha)}(G)$ $(k=0,1,2)$ of all functions u with the finite norm $\|u\|_G^{(k+\alpha)}$ is a Banach space.

The following norms will also be needed (see Sec.VII.2 of [5]):

$$\|u\|_G^{(1-0)} = \|u\|_G + \sup_{Q,Q' \in G} \left\{|u(Q)-u(Q')| \cdot \left[|x-x'| + |t-t'|\right]^{-1}\right\},$$

$$\|u\|_G^{(2-0)} = \|u\|_G^{(1-0)} + \sum_{i=1}^{n} \|u_{x_i}\|_G^{(1-0)}.$$

3) From now $\alpha \in (0,1)$ is an arbitrarily fixed number.

The set of all functions u with the finite norm $\|u\|_G^{(k-0)}$ $(k = 1,2)$ will be denoted by $C^{(k-0)}(G)$.

We state the following definitions concerning the manifold S (see [3], Sec. III.2 and VII.2). Suppose that for every point $Q \in S$ there exists an $(n+1)$-dimensional neighbourhood V such that:

1^0 $V \cap G$ is situated on only one side of the surface $V \cap S$;

2^0 $V \cap S$ can be represented for some i $(1 \le i \le n)$ by an equation of the form

$$x_i = h(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n,t).$$

If the functions h belong to $C^{(\gamma)}$ $(\gamma = \alpha, 1+\alpha, 2+\alpha, 1-0, 2-0)$, then we say that S is of class $C^{(\gamma)}$. If $S \in C^{(2+\alpha)}$ and the derivatives $h_{x_j t}$ $(j \ne i)$ exist and are continuous, then we say that S is of class $\bar{C}^{(2+\alpha)}$; if, moreover, h_{t^2} exist and are continuous, then S is said to belong to class $\bar{\bar{C}}^{(2+\alpha)}$. The manifold S of class $\bar{\bar{C}}^{(\gamma)}$ can be covered by a finite number of balls V_k such that $S_k = S \cap V_k$ is defined by the equation

$$(1.1) \qquad x_{i_k} = h_k(x_1,\dots,x_{i_k-1},x_{i_k+1},\dots,x_n,t),$$

where $h_k \in C^{(\gamma)}$.

Let $v: S \to R$, where $S \in C^{(\gamma)}$. Using (1.1) we can write the function $v(x,t)$ on S_k as a function of the variables $x_1,\dots,x_{i_k-1},x_{i_k+1},\dots,x_n,t$ in a certain region D_k. We then define

$$\|v\|_S^{(\gamma)} = \max_k \|v\|_{D_k}^{(\gamma)}$$

and we say that $v \in C^{(\gamma)}(S)$ if $\|v\|_S^{(\gamma)} < \infty$.

Now we introduce definitions concerning random functions. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. By $L_p = L_p(\Omega)$ $(1 \le p < \infty$[4]$)$ we denote the Banach space of all random variables $\xi : \Omega \to R$ with the finite norm

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4) Throughout the paper $p \in \langle 1, \infty \rangle$ is arbitrarily fixed.

$$\|\xi\|_p=\left[\int_\Omega |\xi(\omega)|^p \, P(d\omega)\right]^{1/p} \text{ if } 1\le p<\infty , \quad \|\xi\|_\infty = \operatorname*{ess\,sup}_{\omega\in\Omega}|\xi(\omega)| .$$

The limit, continuity and partial derivatives of a random function $u: G \to L_p$ are understood in the strong sense and they are called respectively the L_p-limit, L_p-continuity and L_p-derivatives of u.

Like in the scalar case we introduce the following notation for a random function $u:G \to L_p$:

$$\|u\|_{p,G} = \sup_{Q\in G} \|u(Q)\|_p, \quad H^{(\alpha)}_{p,G}(u) = \sup_{Q,Q'\in G} \frac{\|u(Q)-u(Q')\|_p}{[d(Q,Q')]^\alpha},$$

$$\|u\|^{(\alpha)}_{p,G} = \|u\|_{p,G} + H^{(\alpha)}_{p,G}(u), \quad \|u\|^{(1+\alpha)}_{p,G} = \|u\|^{(\alpha)}_{p,G} + \sum_{i=1}^{n} \|u_{x_i}\|^{(\alpha)}_{p,G},$$

$$\|u\|^{(2+\alpha)}_{p,G} = \|u\|^{(\alpha)}_{p,G} + \sum_{i=1}^{n} \|u_{x_i}\|^{(\alpha)}_{p,G} + \sum_{i,j=1}^{n} \|u_{x_ix_j}\|^{(\alpha)}_{p,G} + \|u_t\|^{(\alpha)}_{p,G}.$$

The set $C^{(k+\alpha)}(G;L_p)$ $(k = 0,1,2)$ of all functions u with the finite norm $\|u\|^{(k+\alpha)}_{p,G}$ is a Banach space[5].

D e f i n i t i o n 1.1. A function $\varphi : \Gamma \to L_p$ is said to be of class $C^{(2+\alpha)}(G;L_p)$ if there exists a continuous function $\Phi:\bar{G} \to L_p$ such that $\Phi \in C^{(2+\alpha)}(G;L_p)$ and $\Phi = \varphi$ on Γ. We then define

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5) The completness of $C^{(k+\alpha)}(G;L_p)$ can be proved in a standard manner by using the completeness of L_p and the classical theorem on termwise differentation of a sequence of random functions.

$$\|\varphi\|^{(2+\alpha)}_{p,G} = \inf_{\Phi} \|\Phi\|^{(2+\alpha)}_{p,G}.$$

R e m a r k 1.1. (see Sec. III.3 of [3]). If S is of class $C^{(2+\alpha)}$ and a function $\varphi: \Gamma \to L_p$ belongs to $C^{(2+\alpha)}(G;L_p)$, then for any extension $\Phi \in C^{(2+\alpha)}(G;L_p)$ of φ, the derivatives Φ_{x_i}, $\Phi_{x_i x_j}$ and Φ_t are uniquely defined (by continuity) on the boundary ∂E_0 of the domain E_0, and the definition is independent of Φ. We denote these derivatives (on ∂E_0) by φ_{x_i}, $\varphi_{x_i x_j}$ and φ_t, respectively.

By a solution u of the problem (0.1), (0.2) we shall always understand a regular L_p-solution, i.e. the function $u: \bar{G} \to L_p$ is L_p-continuous in $\bar{G}$, possesses L_p-derivatives appearing in Mu which are L_p-continuous in $\bar{G} \setminus \Gamma$ and u satisfies (0.1), (0.2) in the L_p-sense.

2. The uniqueness of a solution and a priori estimates of a solution of the problem (0.1), (0.2)

At first we derive some a priori estimate of the norm $\|u\|_{p,G}$ for a solution u of the problem (0.1), (0.2) which is a counterpart of the appropriate estimate in the scalar case (see Theorem 5 in Sec. 1 of [5]). For this purpose we need the following lemma.

L e m m a 2.1. Let the following assumptions be satisfied:

(2.I) The coefficients a_{ij}, b_i $(i,j = 1,\dots,n)$ and c are real-valued functions defined in $\bar{G} \setminus \Gamma$ and $a_{ij} = a_{ji}$.

(2.II) The operator M is parabolic in $\bar{G} \setminus \Gamma$, i.e. for any $(x,t) \in \bar{G} \setminus \Gamma$ and $\varrho = (\varrho_1,\dots,\varrho_n) \in R^n$, $\varrho \neq 0$ we have

$$a_{ij}(x,t) = a_{ji}(x,t), \quad i,j = 1,\dots,n, \quad \sum_{i,j=1}^{n} a_{ij}(x,t)\varrho_i\varrho_j > 0.$$

(2.III) $f: \bar{G} \setminus \Gamma \to L_p$ and $\varphi: \Gamma \to L_p$.

Supose that u is a regular L_p-solution of the problem (0.1), (0.2). Then for any functional $l \in L_p^*$ [6] the function $U = lu$ defined by $U(x,t) = l(u(x,t))$ is a regular solution of the scalar problem

$$MU = F(x,t), \quad (x,t) \in \bar{G} \setminus \Gamma, \tag{2.1}$$

$$U(x,t) = \Phi(x,t), \quad (x,t) \in \Gamma, \tag{2.2}$$

where $F = lf$, $\Phi = l\varphi$.

P r o o f . The L_p-continuity of the function u in $\bar{G}$ implies the continuity of U in $\bar{G}$ (see Sec. V.2 of [1] or Sec. III.2 of [4]). In view of Sec. V.3 of [1] there exist in $\bar{G} \setminus \Gamma$ the derivatives

$$U_{x_i}, \; U_{x_i x_j}, \; U_t \tag{2.3}$$

and moreover

$$U_{x_i} = lu_{x_i}, \quad U_{x_i x_j} = lu_{x_i x_j}, \quad U_t = lu_t. \tag{2.4}$$

Thus the derivatives (2.3) are continuous in $\bar{G} \setminus \Gamma$. Now, using (0.1), (0.2) and (2.4) one can easily find that U is a solution of the problem (2.1), (2.2).

In order to formulate a theorem concering the above-mentioned estimate we additionally introduce the following assumptions.

(2.IV) The coefficient c in (0.1) is bounded from above in $\bar{G} \setminus \Gamma$, i.e. $c(x,t) \leq N_1$, $(x,t) \in \bar{G} \setminus \Gamma$ for some positive constant N_1.

(2.V) The functions f and φ are L_p-bounded, i.e.

$$\| f \|_{p, \bar{G} \setminus \Gamma} < \infty \quad \text{and} \quad \| \varphi \|_{p, \Gamma} < \infty .$$

6) L_p^* denotes the adjoint space of L_p.

T h e o r e m 2.1. Let assumptions (2.I)-(2.V) be satisfied. Then for any solution u of the problem (0.1), (0.2) we have the estimate

$$(2.5) \qquad \|u\|_{p,\bar{G}} \leqslant e^{N_1 T}(\|\varphi\|_{p,\Gamma} + T\|f\|_{p,\bar{G}\setminus\Gamma}).$$

P r o o f . Take an arbitrary point $\bar{Q}(\bar{x},\bar{t}) \in \bar{G}$ and consider the case $u(\bar{Q}) \neq 0$. Then, by Theorem III.8.3 of [1], there exists a functional $l \in L_p^*$ with the norm $\|l\|_p = 1$ such that $lu(\bar{Q}) = \|u(\bar{Q})\|_p$. In view of Lemma 2.1 the function $U = lu$ is a regular solution of the problem (2.1), (2.2). Hence, using the above-mentioned Theorem 5 of [5], we obtain the estimate

$$(2.6) \qquad |U(Q)| \leqslant e^{N_1 T}(\|\Phi\|_{\Gamma} + T\|F\|_{\bar{G}\setminus\Gamma}), \qquad Q \in \bar{G}.$$

One easily find that

$$(2.7) \qquad \|\Phi\|_{\Gamma} \leqslant \|\varphi\|_{p,\Gamma}, \quad \|F\|_{\bar{G}\setminus\Gamma} \leqslant \|f\|_{p,\bar{G}\setminus\Gamma}.$$

Since $|U(\bar{Q})| = |lu(\bar{Q})| = \|u(\bar{Q})\|_p$, therefore inequalities (2.6), (2.7) immediately imply the estimate

$$(2.8) \qquad \|u(\bar{Q})\|_p \leqslant e^{N_1 T}(\|\varphi\|_{p,} + T\|f\|_{p,\bar{G}\setminus\Gamma}.$$

In the case $u(\bar{Q}) = 0$ the inequality (2.8) holds also true. Consequently (2.8) implies (2.5).

As a corollary from Theorem 2.1 we obtain in a standard manner the following uniqueness theorem for the problem (0.1), (0.2).

T h e o r e m 2.2. If assumptions (2.I)-(2.IV) are satisfied, then the problem (0.1), (0.2) possesses no more than one solution.

Now we shall derive a priori estimate of the norm $\|u\|_{p,G}^{(2+\alpha)}$ of a solution u of the problem (0.1), (0.2). For this purpose we introduce the following assumptions:

(2.VI) The coefficients of M are uniformly Hölder continuous (exponent α) in $\bar{G}$.

(2.VII) For any $(x,t) \in \bar{G}$ and $\varrho \in R^n$ we have

$$a_{ij}(x,t)=a_{ji}(x,t),\ i,j=1,\dots,n,\quad \sum_{i,j=1}^{n} a_{ij}(x,t)\varrho_i\varrho_j \geq N_2\|\varrho\|^2,$$

where N_2 is a positive constant.

Assumption (2.VI) implies that

$$\|a_{ij}\|_G^{(\alpha)},\ \|b_i\|_G^{(\alpha)},\ \|c\|_G^{(\alpha)} \leq N_3 \quad (i,j=1,\dots,n),$$

N_3 being a positive constant.

The following lemma will be also needed.

L e m m a 2.2. If $u \in C^{(\alpha)}(G;L_p)$, then for any functional $l \in L_p^*$ the function lu belongs to $C^{(\alpha)}(G)$ and

$$\|(lu)\|_G \leq \|l\|_p \|u\|_{p,G}, \tag{2.9}$$

$$H_G^{(\alpha)}(lu) \leq \|l\|_p H_{p,G}^{(\alpha)}(u) \tag{2.10}$$

and consequently

$$\|lu\|_G^{(\alpha)} \leq \|l\|_p \|u\|_{p,G}^{(\alpha)}. \tag{2.11}$$

P r o o f . The inequalities (2.9) and (2.10) result respectively from the following ones

$$|lu(Q)| \leq \|l\|_p \|u(Q)\|_p, \qquad Q \in G$$

and

$$\frac{|(lu)(Q)-(lu)(Q')|}{[d(Q,Q')]^\alpha} \leq \|l\|_p \frac{\|u(Q)-u(Q')\|_p}{[d(Q,Q')]^\alpha}, \quad Q,Q' \in G, \quad Q \neq Q'.$$

As a generalization of Lemma 2.2 we obtain the following lemma.

L e m m a 2.3. If u $C^{(k+\alpha)}(G;L_p)$ (k = 0,1,2), then for any functional $l \in L_p^*$ the function lu belongs to $C^{(k+\alpha)}(G)$ and

$$\| lu\|_G^{(k+\alpha)} \leq \|l\|_p \ \|u\|_{p,G}^{(k+\alpha)} . \tag{2.12}$$

P r o o f . For k = 0 the lemma is proved. If k = 1, then, using (2.4) and (2.11) for u_{x_i} (i=1,...,n), we conclude that

$$\|(lu)_{x_i}\|_G^{(\alpha)} = \|lu_{x_i}\|_G^{(\alpha)} \leq \|l\|_p \ \|u_{x_i}\|_{p,G}^{(\alpha)} . \tag{2,13}$$

According to (2.11) and (2.13) the inequality (2.12) holds for k = 1. Similarly for k = 2 we have

$$\|(lu)_{x_ix_j}\|_G^{(\alpha)} \leq \|l\|_p \ \|u_{x_ix_j}\|_{p,G}^{(\alpha)} , \quad i,j=1,\dots,n,$$

$$\|(lu)_t\|_G^{(\alpha)} \leq \|l\|_p \ \|u_t\|_{p,G}.$$

Hence and by (2.11), (2.13) the inequality (2.12) holds for k = 2.

T h e o r e m 2.3 (cf. Theorem III.6 of [3]). Suppose that:

1^0 assumptions (2.VI) and (2.VII) are satisfied;

2^0 $S \in C^{(2+\alpha)}$, $f \in C^{(\alpha)}(G;L_p)$ and $\varphi \in C^{(2+\alpha)}(G;L_p)$ (see Definition 1.1);

3^0 the function $u \in C^{(2+\alpha)}(G;L_p)$ is a solution of the problem (0.1), (0.2).

Then there exists a constant $K_1 > 0$ depending only on N_2, N_3, α and G such that

$$\|u\|_{p,G}^{(2+\alpha)} \leq K_1 \left[\|\varphi\|_{p,G}^{(2+\alpha)} + \|f\|_{p,G}^{(\alpha)} \right]. \tag{2.14}$$

P r o o f . Lemmas 2.1 and 2.3 and formulas (2.4) imply that for any functional $l \in L_p^*$ the function $U = lu$ is a solution of the problem (2.1), (2.2) and belongs to $C^{(2+\alpha)}(G)$. This enables us to apply to the function U the estimate (III.2.21) of [3].

Let us take arbitrary points $Q, Q' \in G$, $Q \neq Q'$. If $u(Q) \neq u(Q')$, then there exists a functional $l \in L_p^*$ such that $\|l\|_p = 1$ and and $l[u(Q)-u(Q')] = \|u(Q)-u(Q')\|_p$. Hence, according to the estimate (III.2.21) of [3] and Lemma 2.3 we have

$$\frac{\|u(Q)-u(Q')\|_p}{[d(Q,Q')]^\alpha} = \frac{|U(Q)-U(Q')|}{[d(Q,Q')]^\alpha} \leq$$

$$\leq K\left[\|l\varphi\|_G^{(2+\alpha)} + \|lf\|_G^{(\alpha)}\right] \leq K\left[\|\varphi\|_{p,G}^{(2+\alpha)} + \|f\|_{p,G}^{(\alpha)}\right] .$$

Obviously this inequality holds true in the case $u(Q) = u(Q')$ too. So we get the estimate

$$H_{p,G}^{(\alpha)}(u) \leq K\left[\|\varphi\|_{p,G}^{(2+\alpha)} + \|f\|_{p,G}^{(\alpha)}\right] . \tag{2.15}$$

Now let us consider u_{x_i} for some i $(1 \leq i \leq n)$. For any point $Q \in G$ such that $u_{x_i}(Q) \neq 0$ there exists a functional $l \in L_p^*$ such that $\|l\|_p = 1$ and $lu_{x_i}(Q) = \|u_{x_i}(Q)\|_p$. Like as above we obtain

$$\|u_{x_i}(Q)\|_p = |U_{x_i}(Q)| \leq K\left[\|\varphi\|_{p,G}^{(2+\alpha)} + \|f\|_{p,G}^{(\alpha)}\right],$$

whence

$$\|u_{x_i}\|_{p,G} \leq K\left[\|\varphi\|_{p,G}^{(2+\alpha)} + \|f\|_{p,G}^{(\alpha)}\right] . \tag{2.16}$$

Further, taking a functional $l \in L_p^*$ such that $\|l\|_p = 1$ and $l\left[u_{x_i}(Q)-u_{x_i}(Q')\right] = \|u_{x_i}(Q)-u_{x_i}(Q')\|_p$ (under the condition $u_{x_i}(Q) \neq u_{x_i}(Q')$) and arguing as in the proof of (2.15) we get the inequality

$$(2.17) \qquad H_{p,G}^{(\alpha)}(u_{x_i}) \leq K\left[\|\varphi\|_{p,G}^{(2+\alpha)} + \|f\|_{p,G}^{(\alpha)}\right], \qquad i=1,\dots,n.$$

For the derivatives $u_{x_i x_j}$, u_t one can also derive the estimates of the form (2.16), (2.17). Hence in view of the estimates (2.5), (2.15)-(2.17) the proof of Theorem 2.3 is completed.

Before formulating the next theorem we introduce the following assumption and notation.

(2.VIII) The coefficients a_{ij} are uniformly Hölder continuous (exponent α) in $\bar{G}$ and they belong to $C^{(1-0)}(S)$; b_i and c are continuous in $\bar{G}$.

Thus for some constants N_4, $N_5 > 0$ we have

$$\sum_{i,j=1}^{n} \|a_{ij}\|_G^{(\alpha)} + \sum_{i=1}^{n} \|b_i\|_G + \|c\|_G \leq N_4, \quad \sum_{i,j=1}^{n} \|a_{ij}\|_S^{(1-0)} \leq N_5.$$

For any $\tau_1, \tau_2 \in \langle 0,T\rangle$ with $\tau_1 < \tau_2$ let us denote

$$G_{\tau_1,\tau_2} = G \cap \left\{(x,t): \tau_1 < t < \tau_2\right\}, \quad S_{\tau_1,\tau_2} = S \cap \left\{(x,t): \tau_1 \leq t \leq \tau_2\right\},$$

$$E_{\tau_1} = \left\{(x,\tau_1) \in \bar{G} \setminus S\right\}, \quad \Gamma_{\tau_1,\tau_2} = E_{\tau_1} \cup S_{\tau_1,\tau_2}.$$

T h e o r e m 2.4 (cf. Theorem VII.4 of [3]). Suppose that:

1° $S \in C^{(2+\alpha)} \cap C^{(2-0)}$ and $f \in C(\bar{G};L_p)$ [7];

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[7] $C(\bar{G};L_p)$ denotes the set of all functions $u:\bar{G} \to L_p$ which are L_p-continuous in $\bar{G}$.

2° assumptions (2.VII) and (2.VIII) are satisfied;
3° u is a solution of the problem

$$Mu = f(x,t),\ (x,t)\in \bar{G}_{\tau_1,\tau_2}\setminus \Gamma_{\tau_1,\tau_2},\ u(x,t) = 0,\ (x,t)\in\Gamma_{\tau_1,\tau_2}. \tag{2.18}$$

Then for any $\beta\in(0,1)$ there exists a constant $K_2>0$ depending only on β, N_2, N_4, N_5 and G such that

$$\|u\|^{(1+\beta)}_{p,G_{\tau_1,\tau_2}} \leqslant K_2(\tau_2 - \tau_1)^{(1-\beta)/2}\,\|f\|_{p,G_{\tau_1,\tau_2}}. \tag{2.19}$$

P r o o f . It follows from the proof of Theorem VII.4 of [3] that if u is a solution of the problem (2.18) in the scalar case, then there holds the estimate

$$\|u\|^{(1+\beta)}_{G_{\tau_1,\tau_2}} \leqslant K_2(\tau_2 - \tau_1)^{(1-\beta)/2}\,\|f\|_{G_{\tau_1,\tau_2}}. \tag{2.20}$$

Moreover, the condition $f(x,\tau_1) = 0$ on ∂E_{τ_1} (the boundary of E_{τ_1}) is superfluous. Using (2.20) and applying the same method as in the proof of Theorem 2.3, we get (2.19).

Now let u be a solution of the problem

$$Mu=f(x,t),\ (x,t)\in\bar{G}_{\tau_1,\tau_2}\setminus\Gamma_{\tau_1,\tau_2},\ u(x,t)=\varphi(x,t),\ (x,t)\in\Gamma_{\tau_1,\tau_2},$$

where the function $\varphi:\Gamma_{\tau_1,\tau_2}\to L_p$ belongs both to

$$C^{(2+\alpha)}(G_{\tau_1,\tau_2};L_p) \quad\text{and}\quad C^{(1+\beta)}(G_{\tau_1,\tau_2};L_p).$$

Then, under assumptions 1° and 2° of Theorem 2.4, we obtain from (2.19) (in a standard manner) the following estimate

$$(2.21)\qquad \|u\|^{(1+\beta)}_{p,G_{\tau_1,\tau_2}} \leq K_2(\tau_2-\tau_1)^{(1-\beta)/2}\Big[\|f\|_{p,G_{\tau_1,\tau_2}} + K_3\|\varphi\|^{(2+\alpha)}_{p,G_{\tau_1,\tau_2}}\Big] + \|\varphi\|^{(1+\beta)}_{p,G_{\tau_1,\tau_2}},$$

$K_3 > 0$ being a constant depending only on N_4.

3. <u>The existence of a solution of the problem (0.1), (0.2)</u>

At first we introduce some notation. Let us denote

$$\lambda = \{\Omega_1, \ldots, \Omega_m\},$$

where $\Omega_i \in \mathcal{F}$ $(i=1,\ldots,m)$, $\Omega_1 \cup \ldots \cup \Omega_m = \Omega$ and $\Omega_i \cap \Omega_j = \phi$ for $i \neq j$. The set λ will be called a finite partition (or shortly a partition) of the space Ω. By Λ we denote the set of all partitions of Ω. It can be partially ordered as follows: $\lambda \leq \lambda'$ if every set belonging to λ is a sum of those belonging to λ'. So $(\Lambda, \leq)$ is a directed set (see Definition I.7.1 of [2]).

For any $\lambda \in \Lambda$ define an operator A_λ setting

$$(3.1)\qquad A_\lambda \xi = \sum_{i=1}^{m} A_{\lambda i}(\xi)\chi_{\Omega_i}, \qquad \xi \in L_p,$$

where χ_{Ω_i} is the characteristic function of Ω_i and

$$(3.2)\qquad A_{\lambda i}(\xi) = \begin{cases} [P(\Omega_i)]^{-1}\int_{\Omega_i} \xi(\omega)P(d\omega) & \text{if } P(\Omega_i) > 0, \\ 0 \text{ if } P(\Omega_i) = 0. \end{cases}$$

Obviously $A_\lambda : L_p \longrightarrow L_p$ and it results from the proof of Theorem IV.8.18 of [2] that

$$(3.3)\qquad \|A_\lambda\|_p \leq 1, \quad \lambda \in \Lambda$$

and

$$\lim_{\lambda\in\Lambda} \|A_\lambda \xi - \xi\|_p = 0, \qquad \xi \in L_p \tag{3.4}$$

(see also Sec. I.7.1 of [2]).

The following lemmas concerning generalized sequences of random functions will be needed.

Lemma 3.1. If $u \in C(\bar{G};L_p)$, then for any $\lambda \in \Lambda$ the function $A_\lambda u$, defined by $(A_\lambda u)(Q) = A_\lambda[u(Q)]$, belongs to $C(\bar{G};L_p)$ and

$$\lim_{\lambda\in\Lambda} \|A_\lambda u-u\|_{p,G} = 0. \tag{3.5}$$

Proof. In virtue of (3.2) we have $A_{\lambda i} \in L_p^*$, whence $A_{\lambda i} u \in C(\bar{G})$. This implies, by (3.1), the relation

$$A_\lambda u \in C(\bar{G};L_\infty) \subset C(\bar{G};L_p).$$

According to the uniform L_p-continuity of u in $\bar{G}$ given any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\|u(Q)-u(Q')\|_p < \frac{\varepsilon}{3} \quad \text{if} \quad d(Q,Q') < \delta. \tag{3.6}$$

Let $\{Q_1,\dots,Q_k\} \subset \bar{G}$ be a δ-net of $\bar{G}$[8]. By (3.4)

$$\lim_{\lambda\in\Lambda} \|A_\lambda u(Q_i)-u(Q_i)\|_p = 0, \quad i = 1,\dots,k$$

and consequently there is a $\lambda_0 \in \Lambda$ such that

$$\|A_\lambda u(Q_i)-u(Q_i)\|_p < \frac{\varepsilon}{3}, \quad i = 1,\dots,k, \quad \lambda \geq \lambda_0. \tag{3.7}$$

8) I.e. for any $Q \in \bar{G}$ there is Q_i such that $d(Q,Q_i) < \delta$.

In view of (3.6) for any $Q \in \bar{G}$ there exists Q_i such that

$$(3.8) \qquad \|u(Q)-u(Q_i)\|_p < \frac{\varepsilon}{3} .$$

Taking advantage of inequalities (3.3), (3.7) and (3.8) we obtain

$$\|A_\lambda u(Q)-u(Q)\| \leq \|A_\lambda[u(Q)-u(Q_i)]\|_p +$$

$$+\|A_\lambda u(Q_i)-u(Q_i)\|_p + \|u(Q_i)-u(Q)\|_p < \varepsilon$$

for any $Q \in \bar{G}$ and $\lambda \geq \lambda_0$. This gives

$$\|A_\lambda u-u\|_{p,G} < \varepsilon , \qquad \lambda \geq \lambda_0$$

which implies (3.5).

L e m m a 3.2. If $u \in C^{(k+\alpha)}(G;L_p)$ $(k=0,1,2)$, then for any $\lambda \in \Lambda$ we have $A_\lambda u \in C^{(k+\alpha)}(G;L_p)$ and

$$(3.9) \qquad \|A_\lambda u\|^{(k+\alpha)}_{p,G} \leq \|u\|^{(k+\alpha)}_{p,G} .$$

P r o o f . If $k = 0$, then it follows from (3.3) that

$$\|A_\lambda u(Q)\|_p \leq \|u(Q)\|_p, \quad Q \in G, \quad \lambda \in \Lambda$$

and

$$\frac{\|A_\lambda u(Q)-A_\lambda u(Q')\|_p}{[d(Q,Q')]^\alpha} \leq \frac{\|u(Q)-u(Q')\|_p}{[d(Q,Q')]^\alpha} , \quad Q,Q' \in G, \ Q \neq Q', \quad \lambda \in \Lambda.$$

So we have $A_\lambda u \in C^{(\alpha)}(G;L_p)$ and

$$\|A_\lambda u\|_{p,G} \leq \|u\|_{p,G}, \qquad H^{(\alpha)}_{p,G}(A_\lambda u) \leq H^{(\alpha)}_{p,G}(u).$$

Consequently

$$\|A_\lambda u\|_{p,G} \leqslant \|u\|_{p,G}, \tag{3.10}$$

which proves (3.9) for $k = 0$.

In the case $k = 1$ the relations (3.1) and (2.4) yield the formula

$$(A_\lambda u)_{x_j} = A_\lambda(u_{x_j}), \quad j = 1,\dots,n. \tag{3.11}$$

Hence, according to the previous considerations

$$(A_\lambda u)_{x_j} \in C^{(\alpha)}(G;L_p) \quad j = 1,\dots,n$$

and

$$\|(A_\lambda u)_{x_j}\|^{(\alpha)}_{p,G} \leqslant \|u_{x_j}\|_{p,G}, \quad j = 1,\dots,n. \tag{3.12}$$

Thus we have $A_\lambda u \in C^{(1+\alpha)}(G;L_p)$ and (3.10), (3.12) imply (3.9) for $k = 1$.

Similarly for $k = 2$ one can find that

$$(A_\lambda u)_{x_j x_i},\ (A_\lambda u)_t \in C^{(\alpha)}(G;L_p), \quad j,i = 1,\dots,n$$

and

$$\|(A_\lambda u)_{x_j x_i}\|^{(\alpha)}_{p,G} \leqslant \|u_{x_j x_i}\|^{(\alpha)}_{p,G}, \quad j,i = 1,\dots,n, \tag{3.13}$$

$$\|(A_\lambda u)_t\|^{(\alpha)}_{p,G} \leqslant \|u_t\|^{(\alpha)}_{p,G}. \tag{3.14}$$

Consequently $A_\lambda u \in C^{(2+\alpha)}(G;L_p)$ and in view of (3.10), (3.12)-(3.14) the inequality (3.9) holds true for $k = 2$.

L e m m a 3.3. If $u \in C^{(k+\alpha)}(G;L_p)$ $(k = 0,1,2)$, then

$$\lim_{\lambda\in\Lambda} \|A_\lambda u-u\|^{(k+\beta)}_{p,G} = 0 \quad \text{for any} \quad \beta\in(0,\alpha). \tag{3.15}$$

P r o o f . In the case $k = 0$ let us introduce the function

$$v(Q,Q') = \begin{cases} \dfrac{u(Q)-u(Q')}{[d(Q,Q')]^{\beta}}, & Q,Q'\in \bar{G}, \quad Q' \neq Q, \\ 0, & Q = Q'\in \bar{G}, \end{cases}$$

where $\beta\in(0,\alpha)$. Since v is L_p-continuous in $\bar{G}\times\bar{G}$, therefore, by Lemma 3.1,

$$\lim_{\lambda\in\Lambda} \|A_\lambda v-v\|_{p,G\times G} = 0.$$

Hence, taking into considerations the equality

$$\|A_\lambda v-v\|_{p,G\times G} = H^{(\beta)}_{p,G}(A_\lambda u-u)$$

and (3.5), it follows that

$$\lim_{\lambda\in\Lambda} \|A_\lambda u-u\|^{(\beta)}_{p,G} = 0. \tag{3.16}$$

In the case $k = 1$ the formula (3.11) enables us to apply (3.16) to u_{x_j} $(j = 1,\dots,n)$ and so we get

$$\lim_{\lambda\in\Lambda}\|(A_\lambda u-u)_{x_j}\|^{(\beta)}_{p,G} = \lim_{\lambda\in\Lambda} \|A_\lambda u_{x_j}-u_{x_j}\|^{(\beta)}_{p,G} = 0. \tag{3.17}$$

Relations (3.16), (3.17) imply (3.15) for $k = 1$.

If $k = 2$, then arguing like as above we find that

$$\lim_{\lambda\in\Lambda} \|(A_\lambda u-u)_{x_j x_i}\|^{(\beta)}_{p,G} = 0, \quad j,i = 1,\dots,n,$$

$$\lim_{\lambda\in\Lambda} \|(A_\lambda u-u)_t\|^{(\beta)}_{p,G} = 0.$$

Hence, by (3.16) and (3.17), the relation (3.15) holds true for $k = 2$.

Now we prove an existence theorem for the problem (0.1), (0.2) which is a counterpart of Theorem III.7 of [3].

Theorem 3.1. Let assumptions (2.VI) and (2.VII) be satisfied and suppose that $S \in \bar{\bar{C}}^{(2+\alpha)}$, $f \in C^{(\alpha)}(G;L_p)$, $\varphi \in C^{(2+\alpha)}(G;L_p)$ (see Definition 1.1) and

$$(3.18) \qquad M\varphi = f \quad \text{on } \partial E_0 \quad \text{(see Remark 1.1).}$$

Then the problem (0.1), (0.2) has a unique solution $u \in C^{(2+\alpha)}(G;L_p)$.

Proof. The unicity was proved in Sec. 2. In order to prove the existence of a solution we consider, for any $\lambda \in \Lambda$, the problem

$$(3.19) \qquad Mu = A_\lambda f(x,t), \ (x,t) \in \bar{G} \setminus \Gamma,$$

$$(3.20) \qquad u(x,t) = A_\lambda \varphi(x,t), \ (x,t) \in \Gamma .$$

In virtue of Lemma 3.2 we have $A_\lambda f \in C^{(\alpha)}(G;L_p)$, $A_\lambda \varphi \in C^{(2+\alpha)}(G;L_p)$ and moreover, by (3.1),

$$(3.21) \qquad A_\lambda f(x,t) = \sum_{i=1}^{m} A_{\lambda i}\left[f(x,t)\right] \chi_{\Omega_i},$$

$$(3.22) \qquad A_\lambda \varphi(x,t) = \sum_{i=1}^{m} A_{\lambda i}\left[\varphi(x,t)\right] \chi_{\Omega_i}.$$

Now for $i = 1,\dots,m$ consider the scalar problem

$$(3.23) \qquad Mu = A_{\lambda i}\left[f(x,t)\right], \ (x,t) \in \bar{G} \setminus \Gamma,$$

$$(3.24) \qquad u(x,t) = A_{\lambda i}\left[\varphi(x,t)\right], \quad (x,t) \in \Gamma .$$

According to Lemma 2.3 $A_{\lambda i}f \in C^{(\alpha)}(G)$ and $A_{\lambda i}\varphi \in C^{(2+\alpha)}(G)$. Moreover, relations (2.4) and (3.18) imply that $M(A_{\lambda i}\varphi) = A_{\lambda i}f$ on ∂E_0. Thus, by Theorem III.7 of [3], the problem (3.23), (3.24) has a unique solution $u_{\lambda i} \in C^{(2+\alpha)}(G)$ $(i = 1,\ldots,m)$. Consequently the function

$$u_\lambda = \sum_{i=1}^{m} u_{\lambda i}\chi_{\Omega_i}$$

belongs to $C^{(2+\alpha)}(G;L_p)$ and, in view of (3.21) and (3.22), it is a solution of the problem (3.19), (3.20).

Take an arbitrarily fixed $\beta \in \left(\frac{\alpha}{2},\alpha\right)$. By Lemma 3.3 we have

$$\lim_{\lambda\in\Lambda} \|A_\lambda f - f\|^{(\beta)}_{p,G} = 0, \tag{3.25}$$

$$\lim_{\lambda\in\Lambda} \|A_\lambda\varphi - \varphi\|^{(2+\beta)}_{p,G} = 0 \tag{3.26}$$

and therefore the generalized sequences $(A_\lambda f)$ and $(A_\lambda\varphi)$ satisfy the Cauchy condition in the spaces $C^{(\beta)}(G;L_p)$ and $C^{(2+\beta)}(G;L_p)$, respectively (see Sec. I.7.4 of [2]). Observe that for any $\lambda, \lambda' \in \Lambda$ hold the relations

$$M(u_\lambda - u_{\lambda'}) = A_\lambda f(x,t) - A_{\lambda'}f(x,t),\ (x,t)\in \overline{G}\setminus\Gamma,$$

$$(u_\lambda - u_{\lambda'})(x,t) = A_\lambda\varphi(x,t) - A_{\lambda'}\varphi(x,t),\quad (x,t)\in\Gamma,$$

$$u_\lambda - u_{\lambda'} \in C^{(2+\beta)}(G;L_p),\qquad A_\lambda f - A_{\lambda'}f \in C^{(\beta)}(G;L_p),$$

$$A_\lambda\varphi - A_{\lambda'}\varphi \in C^{(2+\beta)}(G;L_p).$$

So we have, by Theorem 2.3, the estimate

$$\|u_\lambda - u_{\lambda'}\|^{(2+\beta)}_{p,G} \le K_1(\beta)\left[\|A_\lambda\varphi - A_{\lambda'}\varphi\|^{(2+\beta)}_{p,G} + \|A_\lambda f - A_{\lambda'}f\|^{(\beta)}_{p,G}\right].$$

Hence, in view of the above-mentioned Cauchy condition, given any $\varepsilon > 0$, there is a $\lambda_0 \in \Lambda$ such that

$$\|u_\lambda - u_{\lambda'}\|_{p,G}^{(2+\beta)} < \varepsilon \quad \text{for any} \quad \lambda, \lambda' \geq \lambda_0.$$

This implies, by Lemma I.7.5 of [2], the existence of $u \in C^{(2+\beta)}(G;L_p)$ such that

$$\lim_{\lambda\in\Lambda} \|u_\lambda - u\|_{p,G}^{(2+\beta)} = 0. \tag{3.27}$$

In virtue of the relations (3.25)-(3.27) the function u is a solution of the problem (0.1), (0.2).

It remains to show that

$$u \in C^{(2+\alpha)}(G;L_p). \tag{3.28}$$

Indeed, by Theorem 2.3, we have

$$\|u\|_{p,G}^{(2+\beta)} \leq K_1(\beta)\left[\|\varphi\|_{p,G}^{(2+\beta)} + \|f\|_{p,G}^{(\beta)}\right] \tag{3.29}$$

for any $\beta \in \left(\frac{\alpha}{2}, \alpha\right)$. The proofs of Theorems III.6 of [3] and 2.3 imply that

$$K_1(\beta) < K', \quad \beta \in \left(\frac{\alpha}{2}, \alpha\right) \tag{3.30}$$

for some $K' > 0$. Moreover, we have

$$\|\varphi\|_{p,G}^{(2+\beta)} \leq K''\|\varphi\|_{p,G}^{(2+\alpha)}, \quad \|f\|_{p,G}^{(\beta)} \leq K''\|f\|_{p,G}^{(\alpha)}, \quad \beta \in \left(\frac{\alpha}{2}, \alpha\right) \tag{3.31}$$

for some $K'' > 0$. Relations (3.29)-(3.31) give the estimate

$$\|u\|_{p,G}^{(2+\beta)} \leq K'K''\left[\|\varphi\|_{p,G}^{(2+\alpha)} + \|f\|_{p,G}^{(\alpha)}\right] \quad \text{for any} \quad \beta \in \left(\frac{\alpha}{2}, \alpha\right)$$

from which (3.28) follows.

4. On the existence and uniqueness of a solution of the problem (0.3), (0.4)

We introduce the following assumptions concerning operators M^k $(k = 1,2,\dots)$ defined by (0.3).

(4.I) The coefficients of M^k are uniformly Hölder continnuous (exponent α) in $\bar{G}$ and moreover

$$\|a_{ij}^k\|_G^{(\alpha)}, \quad \|b_i^k\|_G^{(\alpha)}, \quad \|c^k\|_G^{(\alpha)} \leq B_1,$$

B_1 being a positive constant.

(4.II) For any $(x,t) \in \bar{G}$, $\varrho \in R^n$ and $k = 1,2,\dots$ we have

$$a_{ij}^k(x,t) = a_{ji}^k(x,t), \quad i,j=1,\dots,n, \quad \sum_{i,j=1}^{n} a_{ij}^k(x,t)\varrho_i\varrho_j \geq B_2|\varrho|^2,$$

where B_2 is a positive constant.

(4.III) The coefficients a_{ij}^k $(i,j=1,\dots,n;\ k=1,2,\dots)$ belong to $C^{(1-0)}(S)$ and

$$\sum_{i,j=1}^{n} \|a_{ij}^k\|_S^{(1-0)} \leq B_3 \quad (k=1,2,\dots),$$

B_3 being a positive constant.

In order to formulate assumptions concerning the functions f^k and φ^k $(k=1,2,\dots)$ we introduce the following notation. Let V denote the set of all functions $u = (u^1, u^2, \dots)$ such that $u^j \in L_p$, $j=1,2,\dots$ and

(4.1) $$|u|_p = \sum_{j=1}^{\infty} B_{4j} \|u^j\|_p < \infty,$$

where B_{4j} $(j=1,2,\dots($ are some positive constants such that

$$B_4 \equiv \sum_{j=1} B_{4j} < \infty .$$

As is easily seen the space V with the norm (4.1) is a Banach space. Moreover, by $V^{(i+\alpha)}(G)$ $(i=0,1,2)$ we denote the set of all functions $u=(u^1,u^2,\dots)$ such that $u^j \in C^{(i+\alpha)}(G;L_p)$ $(j=1,2,\dots)$ and

$$(4.2) \qquad |u|_{p,G}^{(i+\alpha)} = \sum_{j=1}^{\infty} B_{4j} \, \|u^j\|_{p,G}^{(i+\alpha)} < \infty .$$

The set $V^{(i+\alpha)}(G)$ $(i=0,1,2)$ with the norm (4.2) is a Banach space too.

Now we formulate the further assumptions.

(4.IV) The functions f^k $(k=1,2,\dots)$ are defined for $(x,t) \in \bar{G}$, $u,v_1,\dots,v_n \in V$ with values in L_p and they satisfy the following Hölder condition: for any $a>0$ there is a constant $B_5(a)>0$ such that

$$\|f^k(Q,u,v_1,\dots,v_n) - f^k(Q',u,v_1,\dots,v_n)\|_p \leq B_5(a)\left[d(Q,Q')\right]^\alpha$$

for any $Q,Q' \in G$, $u,v_1,\dots,v_n \in V_a$, $k=1,2,\dots$, where

$$V_a = \left\{u \in V : |u|_p \leq a\right\}.$$

There is a constant $B_6>0$ such that

$$(4.3) \qquad \|f^k(Q,u,v_1,\dots,v_n) - f^k(Q,u',v_1',\dots,v_n')\|_p \leq$$
$$\leq B_6 \left(|u-u'|_p + \sum_{j=1}^{n} |v_j - v_j'|_p\right)$$

for any $Q\in\bar{G}$, $u,u',v_i,v_i'\in V$ $(i=1,\dots,n)$, $k=1,2,\dots$. Moreover, the sequence $(\|f^k(\cdot,0,\dots,0)\|_{p,G}$ is bounded.

(4.V) The function $\varphi=(\varphi^1,\varphi^2,\dots)$, where $\varphi^k:\Gamma\to L_p$ $(k=1,2,\dots)$, belongs to $V^{(2+\alpha)}(G)\cap V^{(1+\beta)}(G)$ with $\beta\in(0,1)$[9] and

$$(4.4)\qquad M^k\varphi^k = f^k(x,0,\varphi,\varphi_x) \quad \text{on } \partial E_0$$

(see Definition 1.1 and Remark 1.1).

Theorem 4.1. Suppose that $S\in\bar{\bar{C}}^{(2+\alpha)}\cap C^{(2-0)}$ and let assumptions (4.I)-(4.V) be satisfied. Then the problem (0.3), (0.4) has a unique solution $u\in V^{(2+\alpha)}(G)\cap V^{(1+\beta)}(G)$.

Proof. At first we prove the uniqueness of a solution of the problem (0.3), (0.4) in the space $V^{(1+\beta)}(G)$. We proceed in a standard manner step by step. Namely, let $u,\bar{u}\in V^{(1+\beta)}(G)$ be two solutions of the problem (0.3), (0.4). So we have, for any $\tau\in(0,T\rangle$ and $k=1,2,\dots$, the relations

$$(4.5)\qquad M^k(u^k-\bar{u}^k) = F^k(x,t)-\bar{F}^k(x,t), \quad (x,t)\in\bar{G}_\tau\setminus\Gamma_\tau ,$$

$$(4.6)\qquad (u^k-\bar{u}^k)(x,t) = 0, \quad (x,t)\in\Gamma_\tau ,$$

where $G_\tau=G_{0,\tau}$, $\Gamma_\tau=\Gamma_{0,\tau}$ (see the notation before Theorem 2.4) and

$$(4.7)\qquad F^k(x,t) = f^k(x,t,u,u_x), \quad \bar{F}^k(x,t) = f^k(x,t,\bar{u},\bar{u}_x).$$

In view of the assumption (4.IV) $F^k,\bar{F}^k\in C(\bar{G}_\tau;L_p)$ and therefore, applying to (4.5), (4.6) Theorem 2.4 and using (4.3) we obtain the inequalities

[9] Obviously if $\beta\in(0,\alpha\rangle$, then $V^{(1+\beta)}(G)\subset V^{(2+\alpha)}(G)$.

$$\|\dot{u}^k-\bar{u}^k\|_{p,G}^{(1+\beta)} \leqslant K_2B_6\tau^{(1-\beta)/2} \| u-\bar{u}\|_{p,G_\tau}^{(1+\beta)} \qquad (k=1,2,\dots).$$

Multiplying each of these inequalities by B_{4k}, then summing over k and using (4.2) we get

$$\| u-\bar{u}\|_{p,G}^{(1+\beta)} \leqslant K_2B_6B_4\tau^{(1-\beta)/2} \| u-\bar{u}\|_{p,G_\tau}^{(1+\beta)} . \tag{4.8}$$

Now let us fix $\tau\in(0,T>$ such that

$$B_7 \equiv K_2B_6B_4\tau^{(1-\beta)/2} < 1. \tag{4.9}$$

This implies, by (4.8), the equality $u = \bar{u}$ in $\bar{G}_\tau$. If

$$K_2B_6B_4T^{(1-\beta)/2} < 1, \tag{4.10}$$

then we can take $\tau = T$ and the proof of the uniqueness is completed.

In the case

$$K_2B_6B_4T^{(1-\beta)/2} \geqslant 1 \tag{4.11}$$

we have $\tau < T$. Let us put

$$\delta = \begin{cases} \tau, & \text{if } 2\tau < T, \\ T-\tau, & \text{if } 2\tau \geqslant T. \end{cases}$$

So we have $\delta\in(0,t>$ and

$$M^k(u^k-\bar{u}^k) = F^k(x,t)-\bar{F}^k(x,t), \quad (x,t)\in \bar{G}_{\delta,\delta+\tau}\setminus\Gamma_{\delta,\delta+\tau},$$

$$(u^k-\bar{u}^k)(x,t) = 0, \quad (x,t)\in\Gamma_{\delta,\delta+\tau}.$$

Further, arguing as above, one can find that $u = \bar{u}$ in $\bar{G}_{\delta+\tau}$. So after finite number of steps we get $u = \bar{u}$ in $\bar{G}$.

Now we shall prove the existence of a solution of the problem (0.3), (0.4). It will be applied a standard manner of extension of a solution.

Let us consider the problem

$$(4.12) \qquad M^k u^k = f^k(x,t,u,u_x), \quad (x,t) \in \bar{G}_\tau \setminus \Gamma_\tau ,$$

$$(4.13) \qquad u^k(x,t) = \varphi^k(x,t) \in \Gamma_\tau ,$$

where $\tau \in (0,T\rangle$ is fixed and satisfies the condition (4.9). We denote by W_τ the set of all functions $u \in V^{(1+\beta)}(G_\tau)$ such that

$$(4.14) \qquad u(x,t) = \varphi(x,t), \quad (x,t) \in \Gamma_\tau .$$

Obviously W_τ is a closed set of the space $V^{(1+\beta)}(G_\tau)$. For any $u \in W_\tau$, $k=1,2,\ldots$ consider the problem

$$(4.15) \qquad M^k v^k = F^k(x,t), \quad (x,t) \in \bar{G}_\tau \setminus \Gamma_\tau ,$$

$$(4.16) \qquad v^k(x,t) = \varphi^k(x,t), \quad (x,t) \in \Gamma_\tau ,$$

where F^k are defined by (4.7). It follows from assumption (4.IV) that

$$(4.17) \qquad F^k \in C^{(\alpha)}(G;L_p), \quad \|F^k\|^{(\alpha)}_{p,G} \leq B_8, \quad k=1,2,\ldots ,$$

B_8 being a positive constant. In view of (4.4) and (4.14) we have

$$M^k \varphi^k = F(x,0), \quad (x,0) \in \partial E_0, \quad k = 1,2,\ldots$$

So, by Theorem 3.1, the problem (4.15), (4.16) has a unique solution $v^k \in C^{(2+\alpha)}(G_\tau; L_p)$, $k=1,2,\dots$ Moreover, relations (2.14), (2.21) and (4.17) imply that

$$\|v^k\|^{(2+\alpha)}_{p,G_\tau} \leq B_9 \left[1 + \|\varphi^k\|^{(2+\alpha)}_{p,G_\tau}\right],$$

$$\|v^k\|^{(1+\beta)}_{p,G} \leq B_{10}\left[1 + \|\varphi^k\|^{(2+\alpha)}_{p,G_\tau}\right] + \|\varphi^k\|^{(1+\beta)}_{p,G_\tau} ,$$

where B_9 and B_{10} are some positive constants. Thus, by (4.V), we have

$$(4.18) \qquad v = (v^1, v^2, \dots) \in V^{(2+\alpha)}(G_\tau) \cap V^{(1+\beta)}(G_\tau).$$

At the same time we have proved that the operator Z defined by $Zu = v$ maps W_τ into itself.

For any $u, \bar{u} \in W_\tau$ hold the relations

$$M^k(v^k - \bar{v}^k) = F^k(x,t) - \bar{F}^k(x,t), \quad (x,t) \in \bar{G}_\tau \setminus \Gamma_\tau,$$

$$(v^k - \bar{v}^k)(x,t) = 0, \quad (x,t) \in \Gamma_\tau ,$$

where F^k and $\bar{F}^k$ are given by (4.7). Further, arguing as in the proof of (4.8) one can obtain the inequality

$$\| Zu - Z\bar{u}\|^{(1+\beta)}_{p,G_\tau} < B_7 \, \|u - \bar{u}\|^{(1+\beta)}_{p,G_\tau} ,$$

B_7 being defined by (4.9). According to the Banach fixed point theorem the operator Z has a unique fixed point u. This function u is a solution of the problem (4.12), (4.13) and, by (4.18), $u \in V^{(2+\alpha)}(G_\tau) \cap V^{(1+\beta)}(G_\tau)$.

In the case (4.10) we can set $\tau = T$ and so the proof of the existence of a solution of the problem (0.3), (0.4) is completed. If (4.11) holds, then we consider the problem

$$(4.19)\qquad M^k v^k = f^k(x,t,v,v_x), \quad (x,t)\in \overline{G}_{\delta,\delta+\tau}\setminus\Gamma_{\delta,\delta+\tau},$$

$$(4.20)\qquad v^k(x,t) = \overline{\varphi}^k(x,t), \quad (x,t)\in\Gamma_{\delta,\delta+\tau},$$

where $\delta\in(0,\tau)$ is selected such that $\delta+\tau\leqslant T$,

$$\overline{\varphi}^k(x,t) = \begin{cases} u^k(x,\delta), & (x,\delta)\in E_\delta, \\ \varphi^k(x,t), & (x,t)\in S_{\delta,\delta+\tau} \end{cases}$$

and $u=(u^1,u^2,\dots)$ is the above obtained solution of the problem (4.12), (4.13). For $k=1,2,\dots$ let us put

$$\overline{\phi}^k(x,t) = \begin{cases} \phi^k(x,t)\,\left[1-g(t)\right] + g(t)u^k(x,t), & (x,t)\in\overline{G}_{\delta,\tau}, \\ \phi^k(x,t), \quad (x,t)\in\overline{G}_{\tau,\delta+\tau}, \end{cases}$$

where $\phi=(\phi^1,\phi^2,\dots)\in V^{(2+\alpha)}(G;L_p)\cap V^{(1+\beta)}(G;L_p)$ is an extension of $\varphi=(\varphi^1,\varphi^2,\dots)$ and $g:<\delta,\delta+\tau>\to R$ is a smooth function such that $g(\delta)=1$ and $g(t)=0$ for $t\in<\tau,\delta+\tau>$. Thus $\overline{\phi}^k$ $(k=1,2,\dots)$ is an extension of $\overline{\varphi}^k$ and

$$\overline{\phi}\in V^{(2+\alpha)}(G_{\delta,\delta+\tau})\cap V^{(1+\beta)}(G_{\delta,\delta+\tau}),$$

i.e. $\overline{\varphi}=(\overline{\varphi}^1,\overline{\varphi}^2,\dots)\in V^{(2+\alpha)}(G_{\delta,\delta+\tau})\cap V^{(1+\beta)}(G_{\delta,\delta+\tau})$.
Moreover we have

$$M^k\overline{\varphi}^k = f^k(x,\delta,\overline{\varphi},\overline{\varphi}_x) \quad \text{on } \partial E_\delta.$$

Further, applying the same argumentation as that used for the problem (4.12), (4.13) one can prove the existence of a unique solution

$$v\in V^{(2+\alpha)}(G_{\delta,\delta+\tau})\cap V^{(1+\beta)}(G_{\delta,\delta+\tau})$$

of the problem (4.19), (4.20). Since

$$v^k(x,t) = u^k(x,t), \quad (x,t)\in\Gamma_{\delta,\tau}, \quad k=1,2,\dots ,$$

therefore in view of (4.12), (4.19) and by the uniqueness of a solution we have $v = u$ in $\bar{G}_{\delta,\tau}$. Consequently the function $w = (w^1,w^2,\dots)$ with w^k $(k=1,2,\dots)$ defined by formula

$$w^k(x,t) = \begin{cases} u^k(x,t), & (x,t)\in\bar{G}_\tau, \\ v^k(x,t), & (x,t)\in\bar{G}_{\tau,\delta+\tau} \end{cases}$$

belongs to

$$V^{(2+\alpha)}(G_{\tau+\delta})\cap V^{(1+\beta)}(G_{\tau+\delta})$$

and it is a solution of the problem

$$M^k w^k = f^k(x,t,w,w_x), \quad (x,t)\in\bar{G}_{\tau+\delta}\setminus\Gamma_{\tau+\delta},$$

$$w^k(x,t) = \varphi^k(x,t), \quad (x,t)\in\Gamma_{\tau+\delta}, \quad k=1,2,\dots$$

Proceeding in the above manner we obtain, after finite number of steps, a solution of the problem (0.3), (0.4). Thus the proof of Theorem 4.1 is completed.

Now let us consider the particular case

$$(4.21)\qquad f^k(Q,u,v_1,\dots,v_n) = g^k(Q) + \sum_{i=1}^{\infty} d_i^k(Q)u^i + \\ + \sum_{j=1}^{n}\sum_{i=1}^{\infty} e_{ji}^k(Q)v_j^i, \quad Q\in\bar{G}, \quad u,v_1,\dots,v_n\in V, \quad k=1,2,\dots$$

The following assumptions are introduced.

(4.VI) The functions $g^k:\bar{G} \to L_p$ (k=1,2,...) belong to $C^{(\alpha)}(G;L_p)$ and $\sup_k \|g^k\|^{(\alpha)}_{p,G} < \infty$.

(4.VII) The functions $d^k_i:\bar{G} \to L_\infty$, $e^k_{ji}:\bar{G} \to L_\infty$ (k,i= =1,2,..., j=1,...,n) belong to $C^{(\alpha)}(G;L_\infty)$ and moreover

$$\|d^k_i\|^{(\alpha)}_{\infty,G},\ \|e^k_{ji}\|^{(\alpha)}_{\infty,G} \leq B_{11}B_{4k},$$

where B_{11} is a positive constant.

It is easily verified that assumptions (4.VI), (4.VII) imply (4.IV) for the case (4.21). Thus Theorem 4.1 holds true in case (4.21).

REFERENCES

[1] A. Alexiewicz: Functional analysis, Warszawa 1969. Polish).

[2] N. Dunford, J.T. Schwartz: Linear operators, Part I, New York, London 1958.

[3] A. Friedman: Partial differential equations of parabolic type, Englewood Cliffs 1964.

[4] E. Hille, R.S. Phillips: Functional analysis and semi-groups, Providence, Rhode Island 1957.

[5] A.M. Il'in, A.S. Kalashnikov, O.A. Oleinik: Second order linear equations of parabolic type, Uspehi Mat. Nauk 17 (1962) 3-141 (in Russian).

[6] H. Ugowski: On a random linear parabolic equation, Demonstratio Math. 11 (1978) 465-498.

[7] H. Ugowski: On random evolution equations, Zeszyty Nauk. Politechn. Gdańsk. Matematyka (in press).

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY, GDAŃSK
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