DEMONSTRATIO MATHEMATICA
Vol. XIV No 4 1981

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CLASSIFICATION OF THE SPACES WITH AFFINE CONNECTION OF COSSU TYPE

1. Introduction

We call space with affine connection L_n of Cossu type a space L_n such that the curvature tensor $R_{ijk}{}^{l}$ satisfies ([4], [7]) the condition

$$(1) \qquad R_{ijk}{}^{l} = \frac{2}{n-1} A^{l}_{[i}R_{jk]}, \qquad (n > 1).$$

In paper [2] I have proved that the abstract curvature tensor of type (1) is strictly equivalent to the abstract Ricci tensor R_{jk}. The n^2 coordinates of the Ricci tensor R_{jk} can be expressed in terms of the essential coordinates of the curvature tensor $R_{ijk}{}^{l}$ as follows

$$(2) \qquad R_{jk} = (n-1)R_{ijk}{}^{i}, \quad i \neq j,\ i,j,k = 1,\dots,n,$$

(where i is a fixed index) and besides that the following condition is satisfied

$$(3) \qquad R_{ijk}{}^{l} = 0, \quad \text{for} \quad l \neq i,\ l = 1,\dots,n.$$

To fix the ideas, when defining R_{jk} by means of formula (2), we take the values of the index i possibly least; by means of the remaining values of the index i we then get corresponding equalities for $R_{ijk}{}^{i}$.

2. Canonical forms of the curvature tensor

We call first canonical form of the curvature tensor in the space L_n of Cossu type the form of $R_{ijk}{}^{l}$ corresponding by formulas (2)-(3) to the canonical form of the Ricci tensor R_{jk}, [9].

Defining the following blocks

$$C_{ij} = \left[R_{ijk}{}^{l}\right], \quad i,j,k,l = 1,\dots,n, \tag{4}$$

where i,j are fixed and l is a row index we can write the coordinate matrix C of the curvature tensor $R_{ijk}{}^{l}$ in the form of a block-antisymmetric matrix

$$C = \left[C_{ij}\right] = \begin{bmatrix} 0 & C_{12} & \dots & & C_{1n} \\ -C_{12} & 0 & \dots & & C_{2n} \\ \vdots & & & & \vdots \\ & & & & C_{n-1,n} \\ -C_{1n} & -C_{2n} & \dots & -C_{n-1,n} & 0 \end{bmatrix}. \tag{5}$$

Theorem 1. The coordinate matrix C of the curvature tensor $R_{ijk}{}^{l}$ of the space L_n of Cossu type is elementarily equivalent to the following block-diagonal matrix

$$C \sim \begin{bmatrix} \left[R_{jk}\right] & 0 & \dots & 0 \\ & \left[R_{jk}\right] & \dots & \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & & \dots & \left[R_{jk}\right] \end{bmatrix}. \tag{6}$$

Proof. Taking into account formulas (2) and (4) and performing elementary operations on the lines of the matrix (5) we get the right-hand side of formula (6).

Corollary 1. By means of (5) and (6) we get the following formule for the rank of the matrix C

(7) $$r(C) = n \cdot r([R_{jk}]).$$

We call second canonical form of the curvature tensor $R_{ijk}{}^{l}$ of the space L_n of Cossu type the right-hand side of formula (6) written for the canonical form [9] of the Ricci tensor R_{jk}.

Formula (7) enables us to divide the curvature tensor $R_{ijk}{}^{l}$ of the space L_n of Cossu type into types (and the space L_n itself into classes) according to the rank of the matrix C.

3. Classification of the two-dimensional space

For n = 2 we have by (5) and (6)

(7) $$C = \begin{bmatrix} 0 & C_{12} \\ -C_{12} & 0 \end{bmatrix}, \quad C_{12} = \begin{bmatrix} R_{21} & R_{22} \\ -R_{11} & -R_{12} \end{bmatrix}$$

or

(8) $$C \sim \begin{bmatrix} [R_{jk}] & 0 \\ 0 & [R_{jk}] \end{bmatrix}, \quad j,k=1,2.$$

From [3] we infer that the Ricci tensor R_{jk} has twelve canonical forms which we can write in the following condensed form

(9) $$\begin{cases} R_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad R_{2,3} = \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon \end{bmatrix}, & \begin{matrix} \text{a) } \varepsilon = 1 \\ \text{b) } \varepsilon = -1; \end{matrix} \\ R_{4,5,6} = \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix}, \quad \begin{matrix} \text{a) } \varepsilon_1 = \varepsilon_2 = 1, \\ \text{b) } \varepsilon_1 = -\varepsilon_2 = 1, \\ \text{c) } \varepsilon_1 = \varepsilon_2 = -1; \end{matrix} & \\ R_7 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad R_{8,9} = \begin{bmatrix} 1 & \varepsilon \\ -\varepsilon & \varepsilon \end{bmatrix}, & \begin{matrix} \text{a) } \varepsilon = 1, \\ \text{b) } \varepsilon = -1; \end{matrix} \\ R_{10} = \begin{bmatrix} 0 & 1+\alpha \\ 1-\alpha & 0 \end{bmatrix}, \quad \alpha \neq 0; & \\ R_{11;12} = \begin{bmatrix} \varepsilon & \varepsilon\beta \\ -\varepsilon\beta & \varepsilon \end{bmatrix}, \quad \beta \neq 0, & \begin{matrix} \text{a) } \varepsilon = 1, \\ \text{b) } \varepsilon = -1. \end{matrix} \end{cases}$$

The orbits of the tensor R_{jk} (or the tensor $R_{ijk}{}^{1}$) corresponding to the canonical forms (9) we denote by $\mathfrak{M}_{R_1}$, ..., $\mathfrak{M}_{R_{12}}$ (or by $\overline{\mathfrak{M}}_{R_1},\dots,\overline{\mathfrak{M}}_{R_{12}}$ for the tensor $R_{ijk}{}^{1}$), respectively. The exact characteristic of the orbits has been given in paper [3].

T h e o r e m 2. There exist twelve types of the space L_2 of Cossu type corresponding to the canonical forms (7) and (9) of the curvature tensor $R_{ijk}{}^{1}$. Moreover, for R_1 we have $r(C) = 0$; for $R_{2,3}$, $r(C) = 2$; whereas for $R_4,\dots,R_{12}$ $(\alpha \neq 0, \pm 1, \beta \neq 0)$, $r(C) = 4$.

4. Classification of the three-dimensional space

For $n = 3$ we have by (5) and (6)

$$C = \begin{bmatrix} 0 & C_{12} & C_{13} \\ -C_{12} & 0 & C_{23} \\ -C_{13} & -C_{23} & 0 \end{bmatrix}, \tag{10}$$

where the essentiál blocks have the form

$$\begin{cases} C_{12} = \frac{1}{2}\begin{bmatrix} R_{21} & R_{22} & R_{23} \\ -R_{11} & -R_{12} & -R_{13} \\ 0 & 0 & 0 \end{bmatrix}, \\ C_{13} = \frac{1}{2}\begin{bmatrix} R_{31} & R_{32} & R_{33} \\ 0 & 0 & 0 \\ -R_{11} & -R_{12} & -R_{13} \end{bmatrix}, \\ C_{23} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ R_{31} & R_{32} & R_{33} \\ -R_{21} & -R_{22} & -R_{23} \end{bmatrix}. \end{cases} \tag{11}$$

By Theorem 1 we have

$$(12)\qquad C \sim \begin{bmatrix} [R_{jk}] & 0 & 0 \\ 0 & [R_{jk}] & 0 \\ 0 & 0 & [R_{jk}] \end{bmatrix} \qquad j,k = 1,2,3.$$

By [8] the Ricci tensor R_{jk} has thirty one canonical forms which can be written in the following condensed form

(13)

$$R_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R_{2,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{matrix} a) & \varepsilon = 1, \\ b) & \varepsilon = -1; \end{matrix}$$

$$R_{4,5,6} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_2 \end{bmatrix}, \quad \begin{matrix} a) & \varepsilon_1 = \varepsilon_2 = 1, \\ b) & \varepsilon_1 = -\varepsilon_2 = 1, \\ c) & \varepsilon_1 = \varepsilon_2 = -1; \end{matrix}$$

$$R_{7,8,9,10} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}, \quad \begin{matrix} a) & \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1, \\ b) & \varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = 1, \\ c) & \varepsilon_1 = -\varepsilon_2 = -\varepsilon_3 = 1, \\ d) & \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1; \end{matrix}$$

$$R_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad R_{12,13} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \varepsilon \\ 0 & -\varepsilon & \varepsilon \end{bmatrix}, \quad \begin{matrix} a) & \varepsilon = 1, \\ b) & \varepsilon = -1; \end{matrix}$$

$$R_{14} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1-\alpha \\ 0 & 1-\alpha & 0 \end{bmatrix}, \ \alpha \neq 0; \quad R_{15,16} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \varepsilon & \varepsilon\beta \\ 0 & -\varepsilon\beta & \varepsilon \end{bmatrix}, \ \beta \neq 0, \quad \begin{matrix} a) & \varepsilon = 1, \\ b) & \varepsilon \neq -1 \end{matrix}$$

$$R_{17} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \quad R_{18,19} = \begin{bmatrix} 0 & 0 & \varepsilon \\ 0 & \varepsilon & \varepsilon \\ \varepsilon & -\varepsilon & 0 \end{bmatrix}, \quad \begin{matrix} a) & \varepsilon = 1, \\ b) & \varepsilon = -1; \end{matrix}$$

$$R_{20,21} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}, \quad \begin{matrix} a) & \varepsilon = 1, \\ b) & \varepsilon = -1; \end{matrix}$$

$$R_{22,23,24,25} = \begin{bmatrix} 0 & \varepsilon_1 & 0 \\ -\varepsilon_1 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_2 \end{bmatrix}, \quad \begin{array}{l} a)\ \varepsilon_1 = \varepsilon_2 = 1, \\ b)\ \varepsilon_1 = -\varepsilon_2 = 1, \\ c)\ \varepsilon_1 = -\varepsilon_2 = -1, \\ d)\ \varepsilon_1 = \varepsilon_2 = -1; \end{array}$$

$$R_{26,27} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & 0 & 1+\alpha \\ 0 & 1-\alpha & 0 \end{bmatrix}, \quad \alpha \neq 0 \quad \begin{array}{l} a)\ \varepsilon = 1, \\ b)\ \varepsilon = -1; \end{array}$$

$$R_{28,29,30,31} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & \varepsilon_2\beta \\ 0 & -\varepsilon_2\beta & \varepsilon_2 \end{bmatrix}, \quad \beta \neq 0 \quad \begin{array}{l} a)\ \varepsilon_1 = \varepsilon_2 = 1, \\ b)\ \varepsilon_1 = -\varepsilon_2 = 1, \\ c)\ \varepsilon_1 = -\varepsilon_2 = -1, \\ d)\ \varepsilon_1 = \varepsilon_2 = -1. \end{array}$$

The orbits (or families of orbits) of the Ricci tensor R_{jk} (or the curvature tensor $R_{ijk}{}^{1}$) corresponding to the canonical forms (13) we denote by $\mathfrak{M}_{R_1}, \ldots, \mathfrak{M}_{R_{31}}$ (or by $\overline{\mathfrak{M}}_{R_1}, \ldots$ $\ldots, \overline{\mathfrak{M}}_{R_{31}}$ for the tensor $R_{ijk}{}^{1}$), respectively. The exact characteristic of the orbits may be found in paper [8].

T h e o r e m 3. There exist thirty one types of space L_3 of Cossu type corresponding to the canonical forms (10)-(11) and (13) of the curvature tensor $R_{ijk}{}^{1}$. Moreover, for R_1 we have $r(C) = 0$; for $R_{2,3}$, $r(C) = 3$; for $R_{4,5,6}$, R_{11}, $R_{12,13}$, R_{14} $(\alpha \neq \pm 1)$, $R_{15,16}$ $(\beta \neq 0)$, R_{17}, $r(C) = 6$; for the remaining canonical forms of the tensor R_{jk} we have $r(C) = 9$.

BIBLIOGRAPHY

[1] S. G o ł ą b : Tensor calculus. Warszawa 1974.

[2] L. B i e s z k : O rzędzie przestrzeni o koneksji afinicznej L_n typu Cossu, (to appear in Zeszyty Nauk. Politech. Szczecin.), 1980.

[3] L. B i e s z k , D. S t y g a r : On the transitive fibres of tensors of second order in the two-dimensional space X^2, Demonstratio Math. 13 (1980) 147-163.

[4] A. C o s s u : Proprietà di curvatura di una particolare classe di varietà a connessione affine, Atti Accad. Naz. Lincei, 6 (1949) 702-707.

[5] M. K u c h a r z e w s k i : Elements of the theory of geometric objects. Katowice 1969.

[6] E. S i w e k : Sur les domaines de transitivité du groupe de transformations des composantes d'un tenseur covariant du second ordre, Ann. Polon. Math., 10 (1960) 217-224.

[7] J.A. S c h o u t e n : Ricci-Calculus. Berlin-Göttingen-Heidelberg 1954.

[8] D. S t y g a r : Włókna tranzytywne tensora dwukrotnie kowariantnego w przestrzeni trójwymiarowej. Doctoral Thesis, Technical University of Warsaw, Warsaw 1974.

[9] A. Z a j t z : Komitanten der Tensoren zweiter Ordnung, Zeszyty Nauk. Uniw. Jagiello. Prace Mat. 8 (1964) 1-53.

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Received April 2nd, 1979.