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ON SOME PROPERTIES OF THE CHARACTERISTIC OF A PROJECTIVE TRANSFORMATION

We shall say that a transformation $\varphi: P_n \to P_n$ has a property l if every point of P_n lies in an l-dimensional subspace $H \subset P_n$ invariant under φ. Obviously, for every projective transformation φ of P_n onto itself there exists at least one integer l such that φ has the property l $(0 \leq l \leq n)$. Let φ have property m, and let φ do not have property k for all $k = 0,\dots,m-1$. Then m will be called the *characteristic* of φ -in symbols char φ (see [1]).

Certainly, the defined notion has a geometrical character. The question arises how to find the characteristic of a projective collineation which is algebraically determined.

Let $P_n(F)$ denote the n-dimensional projective space over the field F. Projective collineations in $P_n(F)$ will be written in the matrix form $y = Ax$ $(A = [a_{ij}],\ a_{ij} \in F,$ $\det A \neq 0)$. In this work we shall consider fields of characteristic 0.

L e m m a 1. Let F be a field, let F' be an extension of F, and let $A = [a_{ij}]$ be an $(n+1)\times(n+1)$ matrix such that $a_{ij} \in F$. If φ and φ' are nonsingular collineations in $P_n(F)$ and $P_n(F')$, respectively, determined by the same formula $y = Ax$, then char φ = char φ'.

P r o o f . First of all, notice the char $\varphi \leq$ char φ'. Let F_0 be the smallest subfield of F containing all numbers a_{ij}. Let next φ_0 denote the transformation $y = Ax$ of $P_n(F_0)$. If char $\varphi_0 = n$, then obviously char $\varphi = n$. Assume that char $\varphi_0 = m < n$. This means that for every point $x \in P_n(F_0)$ the points $x, Ax, \dots, A^{m+1}x$ are linearly dependent. It follows from this that the rank of the matrix X columns of which are $x, Ax, \dots, A^{m+1}x$ is not greater than $m+1$ for every $x \in P_n(F_0)$. Hence all minors of X the degree of which is greater than $m+1$ are equal to zero for every $x \in P_n(F_0)$. Since these minors are homogeneous polynomials of the variables $x_0, \dots, x_n$, all coefficients of them must be equal to zero. This ends the proof.

Thus we have shown that the characteristic of a projective transformation depends of its matrix, only.

Assume now that F is an algebraically closed field. Then every collineation φ in $P_n(F)$ has in an allowable coordinate system the matrix of the form [2]

$$(1)\quad A = \begin{bmatrix} A_{e_1}^{(\alpha_1)} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & A_{e_r}^{(\alpha_r)} \end{bmatrix}, \quad A_e(\alpha) = \begin{bmatrix} \alpha_1 & \dots & 0 \\ \vdots & \ddots & 1 \\ 0 & \dots & \alpha \end{bmatrix}, \quad 0 \neq \alpha_i \in F, \quad \sum_{i=1}^{r} e_i = n+1.$$

Certainly, we may assume here that if $\alpha_{i_1} = \alpha_{i_2} = \dots = \alpha_{i_k}$, then $i_k = i_{k-1} + 1 = \dots = i_1 + k - 1$ as well as $e_{i_1} \leqslant e_{i_2} \leqslant \dots \leqslant \dots \leqslant e_{i_k}$. Then using Segre's symbols we may characterize as follows [2]

$$(2)\quad \left[(e_{h_0+1}, \dots, e_{h_1})(e_{h_1+1}, \dots, e_{h_2}) \dots (e_{h_s+1}, \dots, e_{h_s})\right],$$

where $h_0 = 0$, $h_s = r$.

L e m m a 2. Let φ be a collineation with the matrix of the form (1). Then char $\varphi = n$ if and only if $\alpha_i \neq \alpha_j$ for $i \neq j$.

P r o o f . The necessity is evident. We shall prove the sufficiency. The condition $\alpha_i \neq \alpha_j$ implies the existence of exactly r fixed points of φ, and consequently r hyperplanes invariant under φ. If $n = 1$, then the lemma is obvious. Assume its validity for $n = k-1$, and consider $n = k$. If the characteristic of φ were equal to $k-1$, then there would exist an infinite number of hyperplanes invariant under φ, which is not possible. Suppose that char $\varphi \leq k-2$. Let us denote by H_1 the hyperplane $x_{e_1} = 0$. Obviously H_1 is invariant under φ. It is evident that the matrix of $\varphi_{/H_1}$ has the form (1), where $\alpha_i \neq \alpha_j$ for $i \neq j$. From the inductive assumption it follows that char $\varphi_{/H_1} = k-1$. On the other hand, char $\varphi_{/H_1} \leq k-2$, since char$\varphi \leq k-2$. In view of this contradiction we infer that char $\varphi = k$. q.e.d.

L e m m a 3. Let F be an algebraically closed field. If φ is a nonsingular collineation in $P_n(F)$ and U is a hyperplane invariant under φ, then it can be found an allowable coordinate system in which φ has the matrix of the form (1) and U has one of the following equations

$$x_{e_1-1} = 0, \ldots, x_{e_1+e_2+\ldots+e_r-1} = 0.$$

P r o o f . Let us denote the hyperplanes mentioned in the thesis by the symbols $U_1, \ldots, U_r$, respectively. Obviously, there exists a coordinate system such that the matrix A of φ has the form (1), and its submatrices are ordered as in (2). The transformation $y = Ax$ induces the transformation $u = A^T v$ of hyperplane coordinates (if $x \in u$, then $Ax \in v$). U is associated with some of eigen-values of A, say α_1. Then there exist numbers $\lambda_i \in F$ $i = 0, \ldots, h_1-1$ not all zero such that $U = \lambda_0 U_1 + \ldots + \lambda_{h_1-1} U_{h_1}$. Assume that $\lambda_i = 0$ for $i < m-1$ and $\lambda_{m-1} \neq 0$. Take into account the formulae

$$u^*_{e_1+\dots+e_{i-1}+j} = \lambda_{m-1} u_{e_1+\dots+e_{i-1}+j} \qquad j=0,\dots,e_i-e_m-1$$

(3)

$$u^*_{e_1+\dots+e_i-e_m+j} = \lambda_{m-1} u_{e_1+\dots+e_i-e_m+j} - \lambda_{i-1} u_{e_1+\dots+e_{m-1}+j}$$

$$j=0,\dots,e_m-1,$$

$u^*_k = u_k$ for all remaining indices
where $i = m+1,\dots,h_1$.

One can check easily that the formulae (3), determining a new coordinate system, preserve the matrix A of φ and at the same time they transfer the equation of U onto $x^*_{e_1+\dots e_m-1} = 0$ (see analogous Theorem I, § 3, VIII, [2]).

L e m m a 4. Let F be an algebraically closed field, let φ be a nonsingular collineation in $P_n(F)$, and let k be an integer such that char $\varphi \leqslant k \leqslant n$. Then φ has property k.

P r o o f . If $n = 1$, then clearly the lemma is true. Assume that it is true for $n \leqslant s-1$, and consider $n = s$. Now we have two possibilities: m = char $\varphi = s$ or $m < s$. Since the first case is trivial, we shall deal with the second. The inequality $m < n$ implies that there exists a fundamental subspace of φ the dimension of which is not less than 1 (Lemma 2). Hence there exists a pencil of hyperplanes invariant under φ. Through each point of $P_n(F)$ passes a hyperplane belonging to the pencil. Let H be one of those hyperplanes. If char $\varphi_{/H}$ were greater than m, then there would exist a point $B \in H$ such that the points B, $\varphi_{/H}(B),\dots,\varphi^{m+1}_{/H}(B)$ would be linearly independent which contradicts the assumption char $\varphi = m$. Thence char $\varphi_{/H} \leqslant m$, and with respect to the inductive assumption $\varphi_{/H}$ has properties $m, m+1,\dots,s-1$. q.e.d.

T h e o r e m 1. Let F be an algebraically closed field of characteristic 0, and let φ be a collineation in $P_n(F)$ elementary divisors of which are described by the symbols (2). Then char $\varphi = \sum_{i=1}^{s} e_{h_i} - 1$.

P r o o f . Since the theorem is evident in the case $h_i - h_{i-1} = 1$ for $i = 1,\dots,s$ (Lemma 2), we may assume that e.g. $h_1 \geq 2$ (i.e. char $\varphi < n-1$). Then for $n = 1$ we obtain only one transformation characterized by $[1,1)]$, i.e. the identity. This means that the thesis holds in this case. Assume now that the theorem is true for $n = k-1$, and consider $n = k$. As previously, we denote hyperplanes $x_{e_1-1} = 0,\dots,$ $x_{e_1+\dots e_r-1} = 0$ by $U_1,\dots,U_r$, respectively. According to the inductive assumption, we have char $\varphi_{/U_1} = \dots = \text{char}\,\varphi_{/U_{h_1-1}} = \sum_{i=1}^{s} e_{h_i} - 1 = m$. Now there are two possibilities: $e_{h_1} = e_{h_1-1}$ or $e_{h_1} > e_{h_1-1}$. If $e_{h_1} = e_{h_1-1}$, then char $\varphi_{/U_{h_1}} =$ $= \text{char}\,\varphi_{/U_{h_1-1}}$. If however $e_{h_1} > e_{h_1-1}$, then char $\varphi_{/U_{h_1}} =$ $= \text{char}\,\varphi_{/U_{h_1-1}} - 1$. From this, with respect to Lemma 3, we infer that char $\varphi_{/H} \leq m$, where H is an arbitrary hyperplane invariant under φ. Hence, according to Lemma 4, char $\varphi = m$. q.e.d.

Now, let F be an arbitrary field of characteristic 0. Then every nonsingular collineation φ in $P_n(F)$ has in some allowable coordinate system the matrix of the form

$$(4) \qquad A = \begin{bmatrix} A_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & A_t \end{bmatrix}, \qquad A_i = \begin{bmatrix} 0\,1 & \dots & 0\,0 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 0 & & 1 \\ -a_{ir_i} & \dots & -a_{i1} \end{bmatrix},$$

where a_{ij} are coefficients of invariant factors of φ, $a_{ir_i} \neq 0$ for $i = 1,\dots,t$.

T h e o r e m 2. If F is a field of characteristic 0 and φ is a nonsingular collineation in $P_n(F)$ the matrix of which has the form (4), then char $\varphi = r_t - 1$.

P r o o f . By extending F to an algebraically closed field $\bar{F}$ we may write the matrix of φ in the form (1). One can easily see that the degree of the maximal invariant factor is equal to the sum of degrees of maximal distinct elementary divisors of φ, i.e. $r_t = \sum_{i=1}^{s} e_{h_i}$.

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