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ON SOME NON-LOCAL MOVING BOUNDARY VALUE PROBLEM

1. Introduction

We shall deal with the following boundary value problem. Determine functions $u_1 = u_1(x,t)$, $u_2 = u_2(x,t)$, $s = s(t)$, satisfying in

$$\Omega = \left\{(x,t) : 0<x<s(t), \quad 0 < t < T\right\} \tag{1}$$

the system of two differential equations

$$\sum_{j=1}^{2} d_{\alpha_j} \frac{\partial^2 u_j}{\partial x^2} - \frac{\partial u_\alpha}{\partial t} = 0, \qquad (\alpha = 1,2) \tag{2}$$

and the boundary conditions

$$u_\alpha(x,0) = \varphi_\alpha(x), \quad 0 < x \leqslant s(0) \equiv s_0, \tag{3}$$

$$u_\alpha(s(t),t) = 0, \quad 0 < t < T, \tag{4}$$

$$\left.\frac{\partial u_1}{\partial x}\right|_{x=0} = f(t), \qquad 0 < t < T, \tag{5}$$

$$\left.\frac{\partial u_2}{\partial x}\right|_{x=0} - \lambda \left.\frac{\partial u_2}{\partial x}\right|_{x=s(t)} = g(t), \quad 0 < t < T, \tag{6}$$

$$(7)\qquad \sum_{j=1}^{2} d_{1j} \frac{\partial u_j}{\partial x}\Big|_{x=s(t)} = -s'(t), \quad 0 < t < T,$$

where the functions φ_α $(\alpha = 1,2)$, f, g, and the constants s_0, $d_{\alpha j}$, λ are given.

The problem (2) - (7) may be treated as a generalization of the moving boundary value problem for the heat equation, the so-called Stefan problem. Problems of Stefan type have been considered for over a century. In particular, the one-dimensional Stefan problem concerning the melting of ice was discussed by many authors, chiefly by A.Dacev [1], G.W.Evans [2], A.Friedman [4], A.Fasano and M.Primicerio [3], L.Rubinstein [6] and the literature quoted here .

In our paper we shall chiefly base on the results of G.W.Evans [loc.cit.] , respectively modified, in view of the non-local boundary condition (6).

The problem (2) - (7) arises in the theory of diffusion in three component systems (for instance metal alloys [5]). Concentrations of two components are to be determined from the boundary data. Condition (7) results from the law of conservation of mass on the free boundary.

Problem of the similar type was recently considered in [7], [8].

2. Definition of solution and list of assumptions

Let us denote

$$(8)\qquad \frac{\partial u_2}{\partial x}\Big|_{x=s(t)} \equiv c'(t), \quad c(0) \equiv c_0.$$

D e f i n i t i o n . The set of functions $\{u_1, u_2, s\}$ will be called a solution of the problem (2) - (7), if:

1° s is non-negative and continuously differentiable in $\langle 0,T)$,

2^o u_α $(\alpha = 1,2)$ are non-negative and continuous in $\bar{\Omega}$, and $\frac{\partial^2 u_\alpha}{\partial x^2}$, $\frac{\partial u}{\partial t}$ are continuous in Ω, besides $\frac{\partial u_\alpha}{\partial x}$ are continuous for $0 < x \leq s(t)$, $0 < t < T$,

3^o $\lambda c'(t) + g(t) \leq 0$ in $\langle 0,T)$,

4^o the equations (2) - (7) are satisfied.

In order to solve the problem (2) - (7) we admit the following assumptions:

I. The given square matrix $D = \left[d_{\alpha j}\right]_{2\times 2}$ is constant,

$$\det D > 0,\ d_{\alpha j} > 0,\quad (\alpha, j = 1,2),$$

II. The functions φ_α $(\alpha = 1,2)$ are continuously differentiable in $\langle 0,s_0\rangle$, $\varphi_\alpha \geq 0$ in $\langle 0,s_0\rangle$, and $\varphi_\alpha(s(t)) = 0$,

III. the function f is continuous in $\langle 0,T)$ and $f \leq 0$ in $\langle 0,T)$,

IV. the function g is continuous in $\langle 0,T)$ and $g \leq 0$ in $\langle 0,T)$,

V. the constant λ satisfies the condition $\lambda < 1$.

3. Integral equations equivalent to the problem (2)-(7)

To establish the existence of the solution of the problem (2)-(7) let us integrate the equations (2) over the domain Ω taking into account the boundary conditions (3)-(7) and admitting that $T = t$ is variable.

Namely let us evaluate

$$(9)\qquad \int_0^t \int_0^{s(\tau)} \left(d_{11}\frac{\partial^2 u_1}{\partial x^2} + d_{12}\frac{\partial^2 u_2}{\partial x^2} - \frac{\partial u_1}{\partial t}\right) dx\, d\tau = 0,$$

$$(10)\qquad \int_0^t \int_0^{s(\tau)} \left(d_{21}\frac{\partial^2 u_1}{\partial x^2} + d_{22}\frac{\partial^2 u_2}{\partial x^2} - \frac{\partial u_2}{\partial t}\right) dx\, d\tau = 0$$

with boundary conditions (3)-(7).

After changing the order of integration and remarking that from (7) and (8)

$$\frac{\partial u_1}{\partial x}\Big|_{x=s(t)} = \frac{-s'(t) - d_{12}\, c'(t)}{d_{11}} ,$$

$$\frac{\partial u_2}{\partial x}\Big|_{x=0} = g(t) + \lambda c'(t),$$

we get the system

$$(11)\qquad s(t) + d_{12}\,\lambda\, c(t) = s_0 + d_{12}\, c_0 \lambda + \int_0^{s(t)} \varphi_1(x)dx - $$
$$- d_{11}\int_0^t f(t)dt - d_{12}\int_0^t g(t)dt - \int_0^{s(t)} u_1(x,t)dx,$$

$$(12)\qquad \frac{d_{21}}{d_{11}}\, s(t) + \left[d_{22}(\lambda-1) + \frac{d_{12}d_{21}}{d_{11}}\right] c(t) = \frac{d_{21}}{d_{11}}\, s_0 +$$
$$+ \left[d_{22}(\lambda-1) + \frac{d_{12}d_{21}}{d_{11}}\right] c_0 + \int_0^{s(t)} \varphi_2(x)dx - d_{21}\int_0^t f(t)dt -$$
$$- d_{22}\int_0^t g(t)dt - \int_0^{s(t)} u_2(x,t)dx.$$

According to the suppositions I, V, the determinant of the system (11), (12) is not equal to zero, thus we get from (11), (12)

$$(13)\qquad s(t) = \alpha_1 \int_0^{s(t)} u_1(x,t)dx + \beta_1 \int_0^{s(t)} u_2(x,t)dx + F_1(t),$$

$$(14)\qquad c(t) = \alpha_2 \int_0^{s(t)} u_1(x,t)dx + \beta_2 \int_0^{s(t)} u_2(x,t)dx + F_2(t),$$

where we have denoted:

$$\alpha_1 = \frac{-d_{12}d_{21} - d_{11}d_{22}(\lambda-1)}{(\lambda-1)\det D}, \quad \beta_1 = \frac{d_{11}d_{12}}{(\lambda-1)\det D}$$

(15)

$$\alpha_2 = \frac{d_{21}}{(\lambda-1)\det D}, \qquad \beta_2 = \frac{-d_{11}}{(\lambda-1)\det D}$$

and F_1, F_2 are given continuous functions defined by the boundary data

$$\varphi_\alpha,\ f,\ g,\ s_0,\ \lambda,\ d_{\alpha j},\ c_0 \quad (\alpha, j = 1,2).$$

4. Approximating solutions to the problem (2) - (7)

Let us define

(16) $$s^{(1)}(t) = s_0, \quad c^{(1)}(t) = c_0,$$

and by induction

(17) $$s^{(n+1)}(t) = \alpha_1 \int_0^{s^{(n)}(t)} u_1^{(n)}(x,t)dx + \beta_1 \int_0^{s^{(n)}(t)} u_2^{(n)}(x,t)dx + F_1(t),$$

(18) $$c^{(n+1)}(t) = \alpha_2 \int_0^{s^{(n)}(t)} u_1^{(n)}(x,t)dx + \beta_2 \int_0^{s^{(n)}(t)} u_2^{(n)}(x,t)dx + F_2(t),$$

$(n = 1,2,\ldots)$, where $u_1^{(n)}$, $u_2^{(n)}$ denote the solution of the problem (2) - (6) in Ω_n

(19) $$\Omega_n = \left\{(x,t) : 0 < x < s^{(n)}(t),\ 0 < t < T\right\},$$

with $\left[c^{(n)}(t)\right]'$ in the boundary condition (6) i.e.

$$\sum_{j=1}^{2} d_{\alpha j} \frac{\partial^2 u_j^{(n)}}{\partial x^2} - \frac{\partial u^{(n)}}{\partial t} = 0 \quad \text{in} \quad \Omega_n, \tag{20}$$

$$u_\alpha^{(n)}(x,0) = \varphi_\alpha(x), \qquad 0 < x < s^{(n)}(0) \equiv s_0 \tag{21}$$

$$u_\alpha^{(n)}(s^{(n)}(t),t) = 0, \quad 0 < t < T, \tag{22}$$

$$\left.\frac{\partial u_1^{(n)}}{\partial x}\right|_{x=0} = f(t), \qquad 0 < t < T, \tag{23}$$

$$\left.\frac{\partial u_2^{(n)}}{\partial x}\right|_{x=0} = g(t) + \lambda \left[c^{(n)}(t)\right]', \quad 0 < t < T. \tag{24}$$

L e m m a 1. Under the assumptions I - V there exists the unique solution of the auxiliary problem (20) - (24), $s^{(n)}$, $\left[c^{(n)}\right]'$ being supposed to be given.

P r o o f . According to the supposition I the matrix D has eigenvalues λ_1, λ_2, which are positive and single. Thus, similarly as in [7] one can transform the system (20) into two separate equations of heat conduction type with boundary conditions being linear combinations of the conditions (21)-(24). Existence and uniqueness of such problems follows e.g. from the results of A.Fasano and M.Primicerio [loc.cit.]. Thus, there exists a solution $u_1^{(n)}$, $u_2^{(n)}$ of the auxiliary problem (20)-(24) and is unique.

L e m m a 2. If

$$u_\alpha(x,t) = \lim_{n\to\infty}\left[u_\alpha^{(n)}(x,t)\right], \quad (\alpha = 1,2) \tag{25}$$

form a solution of the problem (20)-(24) with

$$s(t) = \lim_{n\to\infty}\left[s^{(n)}(t)\right], \tag{26}$$

$$c'(t) = \lim_{n\to\infty} \left[c^{(n)}(t)\right]', \tag{27}$$

then the boundary condition (7) is satisfied.

P r o o f . Taking into account that u_1, u_2 satisfy equations (20) we may differentiate equations (13), (14) with respect to t and find that

$$\sum_{j=1}^{2} d_{1j} \frac{\partial u_j}{\partial x}\bigg|_{x=s(t)} = -s'(t),$$

with

$$\frac{\partial u_2}{\partial x}\bigg|_{x=0} = g(t) + \lambda c'(t).$$

L e m m a 3. If the functions u_α $(\alpha = 1,2)$ as the limits of iterations (25), with s and c' being the limits (26), (27), form a solution of the problem (20)-(24), then u_α form the solution of the problem (2)-(7) in the sense of definition.

P r o o f . It is evident that u_α $(\alpha = 1,2)$ satisfy the differential equations (2) and the boundary conditions (3) - (6) because they satisfy equations (20) - (24) in the limiting case $s(t) = \lim\limits_{n\to\infty} s^{(n)}(t)$, with $c(t) = \lim c^{(n)}(t)$. The fulfilment of the boundary condition (7) was proved in the Lemma 2.

It remains to show that the functions u_α and s are nonnegative and satisfy conditions 1^0, 2^0 of the definition, and the function c' satisfies the condition 3^0. Non-negativeness of the functions u_α results from the maximum principle applied to the system (2) transformed into two heat conduction equations [7]. Their other properties are easily proved.

Non-negativeness of the function s and condition 3^0 result from equations (13), (14) and

$$s' = \lim_{n \to \infty} \left[s^{(n)}\right]', \qquad c' = \lim_{n \to \infty} \left[c^{(n)}\right]'.$$

The continuity of the functions s' and c' is also evident.

The proof of existence of approximate solutions defined by (17), (18) and of their convergence does not differ essentially from that of G.W.Evans [2] and from the general principles of the method of successive approximations. This results from the fact that the type of equations (13), (14), is the same as the type of the equation obtained by G.W.Evans.

Thus we can formulate the following

T h e o r e m . If the assumptions I - V are satisfied, then there exists for sufficiently small t a solution of the problem (2) - (7) in the sense of definition formulated in part II.

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