DEMONSTRATIO MATHEMATICA
Vol. XIV No 3 1981

Wojciech Żakowski

ON SOME CLASS OF NONLINEAR PROCESSES WITH A MEMORY IN CONTINUOUS TIME

Introduction

In [1] we have introduced the notion of a (α,k)-computation as a continuous function $x : \langle 0;+\infty) \to \mathbb{R}$ satisfying the condition

$$(1) \qquad x(t) = \int_0^k \alpha(s)x(t-k+s)ds \quad \text{for all} \quad t \geq k,$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is a non-zero polynomial and k denotes a positive number. (α,k)-computations seem to be useful in describing some linear processes investigated in continuous time and which are characterized by a "memory" whose length is k time units. Such processes occur in many technical, economical and biological problems, e.g. in control theory, renewal theory, in the description of cells reproduction [2] etc. Basic properties of (α,k)-computations were investigated by Żakowski in [3], [4].

In this paper we introduce the notion of a CGFk-process as a continuous function $x : \langle 0;+\infty) \to \mathbb{R}$ satisfying some integral, generally nonlinear condition (6). CGFk-processes seem to be useful in the description of some class of continuous processes, generally nonlinear, with "memory" whose length is k. We have also considered some qualitative properties of CGFk-processes.

1. Basic notations and definitions

Let $\mathbb{R}$ denote the set of all real numbers and k denote an arbitrary positive number. By Δ and Δ_1 we denote the following sets

(2) $$\Delta = \left\{(s,t,u) \in \mathbb{R}^3 : 0 \leqslant s \leqslant k \wedge t \geqslant k \wedge u \in \mathbb{R}\right\}$$

and

(3) $$\Delta_1 = \left\{(t,v) \in \mathbb{R}^2 : t \geqslant k \wedge v \in \mathbb{R}\right\}.$$

Let $F : \Delta \to \mathbb{R}$ and $G : \Delta_1 \to \mathbb{R}$ be continuous functions. We assume that there exist positive numbers L_F and L_G such that for every $s \in \langle 0;k \rangle$, $t \geqslant k$, $\tilde{u},\tilde{\tilde{u}} \in \mathbb{R}$ and $\tilde{v},\tilde{\tilde{v}} \in \mathbb{R}$ the inequalities

(4) $$\left|F(s,t,\tilde{u}) - F(s,t,\tilde{\tilde{u}})\right| \leqslant L_F \cdot \left|\tilde{u}-\tilde{\tilde{u}}\right|$$

and

(5) $$\left|G(t,\tilde{v}) - G(t,\tilde{\tilde{v}})\right| \leqslant L_G \cdot \left|\tilde{v}-\tilde{\tilde{v}}\right|$$

hold.

D e f i n i t i o n : The continuous function $x : \langle 0;+\infty) \to \mathbb{R}$ fulfilling for all $t \geqslant k$ the condition

(6) $$x(t) = G\left(t, \int_0^k F(s,t,x(t-k+s))ds\right)$$

is said to be CGFK-process (continuous GFk-process).

If $G(t,v) = v$ and $F(s,t,u) = \alpha(s)\cdot u$, where $\alpha : \mathbb{R} \to \mathbb{R}$ is a non-zero polynomial, then CGFk-process is a (α,k)-computation introduced in [1]. In this case the condition (6) is identical with the condition (1). Consequently, the notion of a CGFk-process is a generalization of the notion of (α,k)-computation. Any CGFk-process describes some real, continuous process which generally is nonlinear and has a "memory" of length k.

R e m a r k 1. Let $f : \langle 0;+\infty) \to \mathbb{R}$ be an arbitrary continuous function. For every $k > 0$ this function is a CGFk-process when for example $F \equiv 0$ and $G(t,v) = f(t)$ for all $t \geqslant k$ and $v \in \mathbb{R}$.

If $x : \langle 0;+\infty) \to \mathbb{R}$, then the restriction of x to the set $U \subseteq \langle 0;+\infty)$ is denoted by $x|U$. In particular, if x is a CGFk-process, then $x|\langle 0;k\rangle$ is called the initial state of x.

If $x : \langle 0;+\infty) \to \mathbb{R}$ and $T \geqslant 0$ then by x_T we denote the function $\langle 0;+\infty) \to \mathbb{R}$ such that

$$x_T(t) = x(t+T) \quad \text{for all} \quad t \geqslant 0. \tag{7}$$

The set of all real and continuous functions on the interval $\langle 0;a\rangle$ we denote by $C_{\langle 0;a\rangle}$.

2. Some properties of the CGFk-processes

It follows from the condition (6) that if f is the initial state of any CGFk-process then

$$f(k) = G\left(k, \int_0^k F(s,k,f(s))\,ds\right). \tag{8}$$

T h e o r e m 1. If the function $f \in C_{\langle 0;k\rangle}$ satisfies the condition (8), then there exists exactly one CGFk-process x such that $x|\langle 0;k\rangle = f$. This CGFk-process is a limit of the sequence (x_n) of successive approximations, defined as follows:

$$x_0(t) = \begin{cases} f(t) & \text{for} \quad 0 \leqslant t \leqslant k \\ f(k) & \text{for} \quad t > k \end{cases} \tag{9}$$

and

$$x_n(t) = \begin{cases} f(t) & \text{for} \quad 0 \leqslant t \leqslant k \\ G\left(t, \int_0^k F(s,t,x_{n-1}(t-k+s))\,ds\right) & \text{for} \quad t > k \end{cases} \tag{10}$$

$n = 1,2,\ldots$. The sequence (x_n) is almost uniformly convergent on the interval $\langle 0;+\infty)$.

P r o o f . Let δ denote an arbitrary positive number greater then k. We define the metric space:

$$\mathbf{C}_f^{(k,\delta)} = \left\{ x \in \mathbf{C}_{\langle 0;\delta\rangle} : x \,|\langle 0;k\rangle = f \right\} \tag{11}$$

with the metric

$$\varrho(\tilde{x},\tilde{\tilde{x}}) = \sup_{\langle 0;\delta\rangle} \left(e^{\lambda t} |\tilde{x}(t) - \tilde{\tilde{x}}(t)|\right), \tag{12}$$

where λ is a negative number such that

$$L_G \cdot L_F \cdot \frac{e^{\lambda k} - 1}{\lambda} < 1. \tag{13}$$

The space (11) is complete. On this space we define an operator A as follows

$$A[x(t)] = \begin{cases} f(t) & \text{for} \quad 0 \leqslant t \leqslant k \\ G(t, \int_0^k F(s,t,x(t-k+s))ds) & \text{for} \quad k \leqslant t \leqslant \delta. \end{cases} \tag{14}$$

On the basis of (8) we observe that the operator A transforms the space (11) into itself.

In view of (14), (4), (5) and (12) we have for every $t \in (k;\delta\rangle$ and for every $\tilde{x},\tilde{\tilde{x}} \in \mathbf{C}_f^{(k,\delta)}$:

$$e^{\lambda t} \left| A[\tilde{x}(t)] - A[\tilde{\tilde{x}}(t)] \right| \leqslant$$

$$\leqslant L_G \cdot L_F \int_0^k e^{\lambda(k-s)} e^{\lambda(t-k+s)} |\tilde{x}(t-k+s) - \tilde{\tilde{x}}(t-k+s)| ds \leqslant$$

$$\leqslant L_G \cdot L_F \cdot \varrho(\tilde{x},\tilde{\tilde{x}}) \int_0^k e^{\lambda(k-s)} ds = L_G \cdot L_F \cdot \varrho(\tilde{x},\tilde{\tilde{x}}) \cdot \frac{e^{\lambda k} - 1}{\lambda}.$$

This implies that for every $\tilde{x},\tilde{\tilde{x}} \in C_f^{(k,\delta)}$

$$\varrho\left(A[\tilde{x}],A[\tilde{\tilde{x}}]\right) \leq L_G \cdot L_F \frac{e^{\lambda k}-1}{\lambda} \varrho(\tilde{x},\tilde{\tilde{x}}).$$

From this and from (13), applying Banach's fixed point theorem, it follows that there exists exactly one function $x_* \in C_f^{(k,\delta)}$ such that $x_* = A[x_*]$. Moreover, $x_* = \lim_{n\to+\infty} x_n$, where x_n, $n = 0,1,2,\ldots$ is defined by (9) and (10) for $t \in \langle 0;\delta \rangle$. From (12) we note that the sequence (x_n) is uniformly convergent on the interval $\langle 0;\delta\rangle$. Because δ denotes an arbitrary positive number greater than k, the proof is complete.

R e m a r k 2. The application of the metric (9) causes that there is no contraints imposed on the numbers L_G and L_F in Lipschitz's conditione (4) and (5).

T h e o r e m 2. If x is an CGFk-process, and, moreover, if:

1° there exists a continuous function

$$M_F : \left\{(s,t) \in R^2 : 0 \leq s \leq k \wedge t > k\right\} \longrightarrow \langle 0;+\infty)$$

such that for every $(s,t,u) \in \Delta$ (see (2)) the condition

$$(15) \qquad \left|F(s,t,u)\right| \leq M_F(s,t)\cdot|u|$$

holds,

2° there exists a function

$$M_G : \langle k;+\infty) \longrightarrow \langle 0;+\infty)$$

such that for every $(t,v) \in \Delta_1$ the condition

$$(16) \qquad \left|G(t,v)\right| \leq M_G(t)\cdot|v|$$

holds,

3^0 there exists a number $t_0 \geqslant 0$ for which the function $x\,|\langle t_0;t_0+k\rangle$ is nonnegative (nonpositive) and nonzero and, moreover

$$(17)\qquad M_G(t_0+k)\cdot\int_0^k M_F(s,t_0+k)ds < 1,$$

then the function $x\,|\langle t_0;t_0+k\rangle$ is not nondecreasing (or not nonincreasing, respectivelly).

P r o o f . If the function $x\,|\langle t_0;t_0+k\rangle$ is nonnegative and nonzero, then in view of (6), (15) and (16) we have

$$x(t_0+k) \leqslant M_G(t_0+k)\cdot\int_0^k M_F(s,t_0+k)x(t_0+s)ds.$$

Hence we get

$$(18)\qquad x(t_0+k) \leqslant \sup_{\langle t_0;t_0+k\rangle} x\cdot M_G(t_0+k)\cdot\int_0^k M_F(s,t_0+k)ds.$$

If

$$M_G(t_0+k) = 0 \qquad \text{or} \qquad \int_0^k M_F(s,t_0+k)ds = 0$$

then we have $x(t_0+k) = 0$. This implies that the function $x\,|\langle t_0;t_0+k\rangle$ is not nondecreasing. In the case if

$$M_G(t_0+k)\int_0^k M_F(s,t_0+k)ds > 0$$

the inequality

$$x(t_0+k) < \sup_{\langle t_0;t_0+k\rangle} x$$

holds on the basis of (17) and (18). On the other hand, there exists a number $c \in \langle t_0;t_0+k\rangle$ such that $x(c) = \sup_{\langle t_0;t_0+k\rangle} x$, whence it follows that $x(t_0+k) < x(c)$, which completes the proof for the case of nonnegative and nonzero function $x|\langle t_0;t_0+k\rangle$. In the case when this function is nonpositive and nonzero, the proof is analogous. Q.E.D.

T h e o r e m 3. If x is an CGFk-process and the hypotheses 1^0 and 2^0 of Theorem 2 hold and, moreover, for every $t > k$ the condition

$$(19) \qquad M_G(t)\cdot\int_0^k M_F(s,t)ds \leqslant 1$$

holds, then there exists a number $c \in \langle 0;k\rangle$ such that for every $t > 0$

$$(20) \qquad |x(t)| \leqslant |x(c)|.$$

P r o o f . The CGFk-process x is a limit of the sequence (x_n) of successive approximations, defined by equalities (9) and (10), where $f = x|\langle 0;k\rangle$. Let $M = \sup_{\langle 0;k\rangle}|x(t)|$. On the basis of (15) and (16) we have for every $t > k$ and $n = 1,2,\dots$

$$(21) \qquad |x_n(t)| \leqslant M_G(t)\cdot\int_0^k M_F(s,t)|x_{n-1}(t-k+s)|ds.$$

We observe (see (9)) that $|x_0(t)| \leqslant M$ for every $t > 0$. From this and from inequalities (19) and (21) applying mathematical induction we get $|x_n(t)| \leqslant M$ for every $t \geqslant 0$ and $n = 0,1,2,\dots$. Passing in this inequality to the limit with $n \to +\infty$ we get $|x(t)| \leqslant M$ for every $t \geqslant 0$. Obviously, there exists a number $c \in \langle 0;k\rangle$ such that $|x(c)| = M$, so we get inequality (20). Q.E.D.

C o r o l l a r y 1. If hypotheses 1^0 and 2^0 of Theorem 2 and the condition (19) hold, then any CGFk-process is bounded on the interval $\langle 0;+\infty)$.

Theorems 2 and 3 generalize some analogous theorems of the paper [3].

T h e o r e m 4. If x is an CGFk-process and if there exists a finite limit

(22) $$\lim_{t \to +\infty} x(t) = g$$

and, moreover, if $F(s,t,g)$ converges for $t \to +\infty$ to the finite limit $\tilde{F}(s)$, uniformly with $s \in \langle 0;k \rangle$, i.e. for every $\varepsilon > 0$ there is a $\delta \geqslant k$ such that for every $s \in \langle 0;k \rangle$

(23) $$t \geqslant \delta \Rightarrow \left| F(s,t,g) - \tilde{F}(s) \right| < \varepsilon,$$

then

(24) $$g = \lim_{t \to +\infty} G\left(t, \int_0^k \tilde{F}(s)ds\right).$$

P r o o f . From the condition (22) we have for every $t > k$

$$g + \varepsilon(t) = G\left(t, \int_0^k F(s,t,g + \varepsilon(t-k+s))ds\right),$$

where $\varepsilon(t) \to 0$ if $t \to +\infty$. Hence, in view of (4) and (5)

$$g + \varepsilon(t) = G\left(t, \int_0^k F(s,t,g)ds\right) + \mu(t),$$

where $\mu(t) \to 0$ if $t \to +\infty$. The assumptions imply that the function $\tilde{F}$ is continuous on the interval $\langle 0;k \rangle$. Consequently, by (5) and (23) we have

(25) $$g + \varepsilon(t) = G\left(t, \int_0^k \tilde{F}(s)ds\right) + \tilde{\mu}(t),$$

where $\tilde{\mu}(t) \to 0$ if $t \to +\infty$. Passing to the limit with $t \to +\infty$ in equality (25) we get condition (24). Q.E.D.

R e m a r k 3. In the case if $G(t,v) = v$ and $F(s,t,u) = \alpha(s)\cdot u$, where $\alpha : \mathbb{R} \to \mathbb{R}$ is a non-zero polynomial, i.e. in the case if an CGFk-process is a (α,k)-computation, the condition (24) has the form

$$g = \int_0^k \alpha(s) g ds.$$

If $g \neq 0$, then this equality is equivalent to the equality

$$\int_0^k \alpha(s) ds = 1$$

given in the papers [1] and [3].

3. <u>Relations between CGFk-processes and k-computable functions</u>

T h e o r e m 5. If x is CGFk-process and if there exists a number $T > 0$ such that for every $t > k$, $s \in \langle 0;k \rangle$, $v \in \mathbb{R}$ and $u \in \mathbb{R}$

(26) $$G(t+T,v) = G(t,v) \quad \text{and} \quad F(s,t+T,u) = F(s,t,u)$$

then the function x_T (see (7)) is also a CGFk-process.

P r o o f . The function x_T is obviously continuous. In view of (6) we have for every $t > k$

$$x(t+T) = G\left(t+T, \int_0^k F(s,t+T, x(t+T-k+s))ds\right).$$

Hence, in virtue of (26) and (7)

$$x_T(t) = G\left(t, \int_0^k F(s,t,x_T(t-k+s))ds\right)$$

for all $t > k$, which completes the proof.

T h e o r e m 6. If x_1 and x_2 are CGFk-processes and the condition (26) holds for $T = T_1$ and $T = T_2$, then

$$(27) \qquad \left[(x_1 | \langle T_1;T_1+k\rangle)_{T_1} = (x_2 | \langle T_2;T_2+k\rangle)_{T_2}\right] \Rightarrow$$

$$\Rightarrow \left[(x_1 | \langle T_1;+\infty))_{T_1} = (x_2 | \langle T_2;+\infty))_{T_2}\right].$$

P r o o f . According to (7) we have

$$h_1(t) = (x_1 | \langle T_1;T_1+k\rangle)_{T_1}(t) = x_1(t+T_1) \quad \text{for} \quad t \in \langle 0;k\rangle$$

$$h_2(t) = (x_2 | \langle T_2;T_2+k\rangle)_{T_2}(t) = x_2(t+T_2) \quad \text{for} \quad t \in \langle 0;k\rangle$$

$$H_1(t) = (x_1 | \langle T_1;+\infty))_{T_1}(t) = x_1(t+T_1) \quad \text{for} \quad t \in \langle 0;+\infty)$$

and

$$H_2(t) = (x_2 | \langle T_2;+\infty))_{T_2}(t) = x_2(t+T_2) \quad \text{for} \quad t \in \langle 0;+\infty).$$

It follows from Theorem 5 that the function H_1 is an CGFK-process with the initial state h_1 and the function H_2 is an CGFk-process with the initial state h_2. Consequently, in virtue of Theorem 1, the equality $h_1 = h_2$ implies $H_1 = H_2$. Q.E.D.

If the conditions (26) hold for all $T \geqslant 0$, i.e. if the functions F and G are constant with respect to the variable t then from Theorem 6 we get for any CGFk-process x the following conditions

$$(28) \qquad \left[(x | \langle T_1;T_1+k\rangle)_{T_1} = (x | \langle T_2;T_2+k\rangle)_{T_2}\right] \Rightarrow$$

$$\Rightarrow \left[(x | \langle T_1;+\infty))_{T_1} = (x | \langle T_2;+\infty))_{T_2}\right]$$

for every $T_1 \geq 0$ and $T_2 \geq 0$. This condition is a necessary and sufficient condition of k-computability in the sense of the paper [5]. Consequently, we have

C o r o l l a r y 2. If the conditions (26) hold for all $T \geq 0$, then any CGFk-process x is k-computable.

We observe that in the linear case, if $G(t,v) = v$ and $F(s,t,u) = \alpha(s)\cdot u$ where $\alpha : \mathbb{R} \to \mathbb{R}$ is a non-zero polynomial, the conditions (26) obviously hold for all $T \geq 0$. Hence any (α,k)-computation is k-computable.

The k-computable functions posses many interesting properties (see [5]). In the special case when the function F and G are constant with respect to the variable t it follows from Corollary 2 that CGFk-processes possesses also these properties.

Note that investigations, given in papers [6] and [7], concerning (Z,Q)-computability also concern k-computability because any k-computable function is (Z,Q)-computable in the case if $Z = \langle 0;k)$ and $Q = \langle 0;+\infty)$.

REFERENCES

[1] W. Ż a k o w s k i : On a concept of (α,k)-computation, Bull. Acad. Polon. Sci., Ser. Sci. Techn. 27 (1979) 841-844.

[2] M. K i m m e l : Integro-differential equations in cellular population dynamics, Proc. Internat. Conf. "Functional Differential Systems and Related Topics", Błażejewko, 19-26 May, 1979.

[3] W. Ż a k o w s k i : (α,k)-computations in continuous time, Demonstratio Math., 12 (1979) 1071-1087.

[4] W. Ż a k o w s k i : Defining polynomials for the functions sint and cost, [in Polish], ICS PAS Reports, 371 (1979).

[5] B. K o n i k o w s k a : Continuous machines, Information and Control 22 (1973) 353-372.

[6] W. Żakowski: A generalization of the notions of a machine and computability, Information and Control 33 (1977) 166-176.
[7] E. Stankiewicz: On some properties of (Z,Q)-machine and its computations, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astron. Phys. 25 (1977) 205-208.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW
Received August 11th, 1980.