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ON n-HAMILTONIAN GRAPHS OF MINIMAL SIZE

Our terminology follows that of Harary [3]. We consider finite, undirected graphs without loops and multiple edges.

Let $G = (V,E)$ be a graph of p vertices and $0 \leqslant n \leqslant p-3$. G is said to be n-Hamiltonian iff for every set $U \subseteq V$ of at most n elements, the graph $G-U$ is Hamiltonian.

n-Hamiltonian graphs were introduced and investigated first by Chartrand, Kapoor and Lick [1]. They proved in particular that the size of an n-Hamiltonian graph of the order p is not less than $\frac{p(n+2)}{2}$. Introducing for every p and n an n-Hamiltonian graph of exactly $\left\lceil\frac{p(n+2)}{2}\right\rceil$ edges we shall show that this bound is the best possible.

Theorem 1. For every $p \geqslant 4$ and n, $0 \leqslant n \leqslant p-3$, there exists an n-Hamiltonian graph of the order p and of the size $\left\lceil\frac{p(n+2)}{2}\right\rceil$.

Proof. Let C_p be a graph with the set of vertices $V_p = \{0,1,\dots,p-1\}$ and having as edges all pairs of the form $\{i,i+1\}$ for $i = 0,1,\dots,p-1$ (all arithmetic in V_p is done modulo p), i.e. C_p is a (simple) cycle of the length p. Clearly C_p is 0-Hamiltonian and has $\left\lceil\frac{p(0+2)}{2}\right\rceil = p$ edges.

Case 1. $n = 1$.

Denote by $G_1(p)$ the graph obtained from C_p by:

(1) adding to C_p all edges of the form $\{k,p-k\}$

(2) adding the edge $\{0, \frac{p}{2}\}$ if p is even or the edges $\{0, \frac{p-1}{2}\}$ and $\{0, \frac{p+1}{2}\}$ if p is odd (see Fig.1).

One can effortlessly convince himself that in both cases $G_1(p)$ is 1-Hamiltonian and its size equals $\lceil\frac{3p}{2}\rceil$.

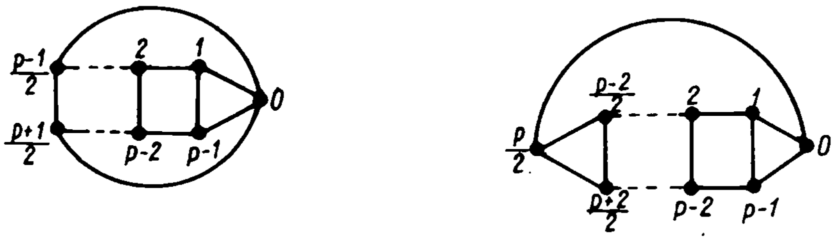

Fig.1

The k-th power G^k of a graph G is the graph obtained from G by joining all pairs of vertices at distances at most k in G.

<u>Case 2.</u> n is even.

Let $n = 2k$ and denote $G_n(p) = (C_p)^{k+1}$. Clearly $G_n(p)$ is $2k+2$-regular and thus its size equals $\lceil\frac{p(n+2)}{2}\rceil$.

For every two vertices $i,j \in V_p$ denote by $\mathrm{dist}(i,j)$ the distance in C_p between i and j. Clearly

$$\mathrm{dist}(i,j) = \min(i-j, j-i)$$

($i-j$ and $j-i$ are calculated modulo p). Notice that $\mathrm{dist}(i,j) = i-j$ means that there are exactly $i-j-1$ vertices of C_p set after j and before i in the natural cyclic ordering of V_p. For convenience we shall write "between j and i" instead of "after j and before i in the ... etc". Notice that "between j and i" and "between i and j" are different things.

Let us take a set $A \subseteq V_p$, $|A| = s \geqslant p-2k$. Denote $A = a_1, a_2, \ldots, a_s$ and assume $a_1 < a_2 < \ldots < a_s$. Let $U = V - A$. Clearly $|U| = p-s \leqslant 2k = n$. We have to find a spanning cycle in $G_n(p) - U$.

If the distances in C_p between every two consecutive elements of the sequence $(a_1, a_2, \ldots, a_s, a_1)$ are not greater than $k+1$, then the sequence itself is a spanning cycle in $G_{2k}(p) - U$. Suppose this is not the case and let $\mathrm{dist}(a_1,a_s) > k+1$. Consequently, there are at least $k+1$ elements of U between a_s and a_1. Since $|U| \leq 2k$, there are at most $k-1$ elements of U between a_1 and a_s. Thus, for every $i = 1,2,\dots,s-2$ we have $\mathrm{dist}(a_i,a_{i+2}) \leqslant k+1$ and by definition of $G_{2k}(p)$ a_i is adjacent to a_{i+2}. Now it is clear that the sequence $(a_1,a_3,a_5,\dots,a_s,\dots,a_4,a_2,a_1)$ is a spanning cycle in $G_n(p) - U$.

Case 3. n is odd, $n > 1$.

Let $n = 2k+1$ and denote $G_{2k}(p) = (V_p,E_{2k})$. Furthermore let

$$D = \left\{\left\{0, \left\lfloor \tfrac{p}{2} \right\rfloor\right\}, \left\{1, \left\lfloor \tfrac{p}{2} \right\rfloor + 1\right\}, \dots, \left\{\left\lfloor \tfrac{p}{2} \right\rfloor - 1, 2\left\lfloor \tfrac{p}{2} \right\rfloor - 1\right\}, \left\{\left\lfloor \tfrac{p}{2} \right\rfloor, 2\left\lfloor \tfrac{p}{2} \right\rfloor\right\}\right\}.$$

Notice that for even p's $\left\{0, \left\lfloor \frac{p}{2} \right\rfloor\right\} = \left\{\left\lfloor \frac{p}{2} \right\rfloor, 2\left\lfloor \frac{p}{2} \right\rfloor\right\}$. Hence $|D| = \left\lceil \frac{p}{2} \right\rceil$ in both even and odd cases, and therefore the size of the graph $G_{2k+1}(p) = (V_p, E_{2k} \cup D)$ equals $\left\lceil \frac{p(2k+3)}{2} \right\rceil = \left\lceil \frac{p(n+2)}{2} \right\rceil$

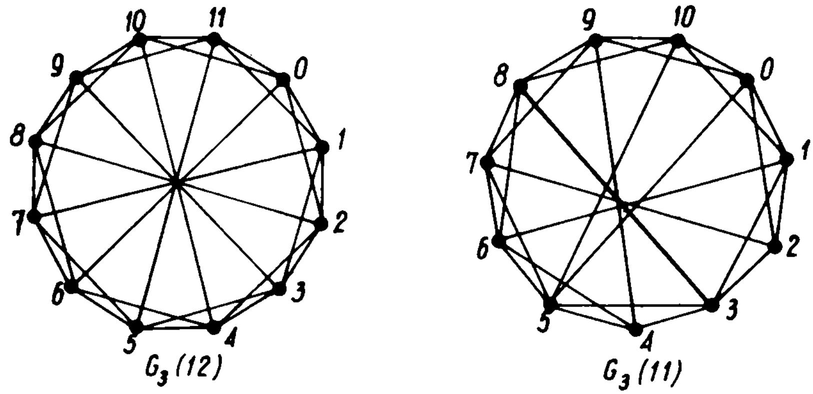

Fig.2

Let us take a set $A \subseteq V_p$, $A = \{a_1,a_2,\dots,a_s\}$ and assume $s \geqslant p-2k-1$ and $a_1 < a_2 < \dots < a_s$. If $s \geqslant p-2k$ or every two consecutive elements of the sequence $(a_1,a_2,\dots,a_s,a_1)$

are in C_p at the distance not greater than $k+1$, or for every i between 1 and $s-2$ $dist(a_i,a_{i+2}) \leq k+1$, then A is the set of vertices of a cycle in $G_{2k}(p)$ and thus in $G_{2k+1}(p)$, too.

Now, assume $s = p-2k-1$. Denote $U = V - A$. Clearly $|U| = 2k+1$. Suppose $dist(a_s,a_1) = k+2$ and for some i, $dist(a_i,a_{i+2}) = k+2$.

There are two cases to be considered:

(a) there exists l, $1 \leq l \leq s-2$, such that $dist(a_l,a_{l+2}) = k+2$ and $1 < dist(a_l,a_{l+1}) < k+1$. See fig.4a.

(b) there exists l such that $dist(a_l,a_{l+2}) = k+2$ and $dist(a_l,a_{l+1}) = k+1$. See fig.4b.

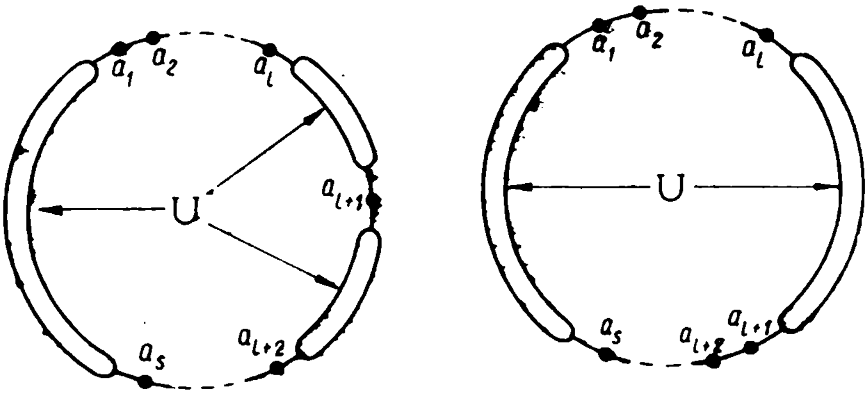

Fig.4a Fig.4b

In the case of (a), $dist(a_{l+1},a_{l+2}) < k+1$ and there are no elements of U either between a_1 and a_l, or between a_{l+2} and a_s. Without loss of generality assume $a_l - a_1 \leq \leq a_s - a_{l+2}$.

Since $k+1 < \lfloor \frac{p}{2} \rfloor$, both $a_1 + \lfloor \frac{p}{2} \rfloor$ and $a_1 + \lfloor \frac{p+1}{2} \rfloor$ belong to the set $\{a_{l+2}, a_{l+3}, \ldots, a_s\}$. At least one of them is adjacent to a_1 (thanks to an edge from D). Denote that one by a_d.

Consider the sequence

$$C_1 = (a_1, a_d, a_{d+2}, \ldots, a_s, \ldots, a_{d+3}, a_{d+1}, a_{d-1}, a_{d-2}, a_{d-3}, \ldots, a_1).$$

Its set of elements equals A and every element (except a_1) occurs in it only once. If $a_d \neq a_{l+2}$ then C_1 is a cycle in $G_{2k+1}(p)$. If $a_d = a_{l+2}$ then, since $dist(a_{l+1}, a_{l+2}) < k+1$ and $dist(a_{l+2}, a_{l+3}) = 1$, we have $dist(a_{l+1}, a_{l+3}) \leq k+1$. Thus, a_{l+1} is adjacent to a_{l+3} in $G_{2k+1}(p)$ and the sequence

$C_2 = (a_1, a_{l+2}, a_{l+4}, \ldots, a_s, \ldots, a_{l+5}, a_{l+3}, a_{l+1}, a_l, \ldots, a_1)$

is a cycle in $G_{2k+1}(p)$.

Now, suppose (b) is the case. Then there are no elements of U between a_1 and a_l, nor between a_{l+1} and a_s. Without loss of generality assume $a_l - a_1 \leq a_s - a_{l+1}$. As in the case of (a), there exists a vertex a_d, $a_{l+1} \leq a_d < a_s$, adjacent to a_1 in $G_{2k+1}(p)$. If $a_d \neq a_{l+1}$ then C_1 is a cycle in request. If $a_d = a_{l+1}$ then by the definition of D $a_l - a_1 = a_s - a_{l+1}$. Consequently a_l or a_{l-1} is adjacent to a_s in $G_{2k+1}(p)$. Denote

$C_3 = (a_1, a_d, a_{d+1}, \ldots, a_s, a_l, a_{l-1}, a_{l-2}, \ldots, a_2, a_1)$ and

$C_4 = (a_1, a_d, a_{d+1}, \ldots, a_s, a_{l-1}, a_l, a_{l-2}, a_{l-3}, \ldots, a_1)$.

If a_l is adjacent to a_s then clearly C_3 is a cycle in $G_{2k+1}(p)$ and if a_{l-1} is adjacent to a_s then C_4 is a cycle. Q.E.D.

The following simple corollary is a consequence of the Theorem, case 2.

C o r o l l a r y 1. For every Hamiltonian graph G and for every integer $k \geq 1$ G^{k+1} is 2k-Hamiltonian.

Further obvious results of similar nature can be obtained using well-known theorems of Sekanina (the cube of every connected graph is Hamiltonian) and Fleischner (the square of every 2-connected graph is Hamiltonian).

C o r o l l a r y 2. For every connected graph G and for every integer $k \geq 1$ $G^{3(k+1)}$ is 2k-Hamiltonian.

C o r o l l a r y 3. For every 2-connected graph C and for every integer $k \geq 1$ $G^{2(k+1)}$ is 2k-Hamiltonian.

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