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ON DECOMPOSITIONS OF QUASI-LEIBNIZ D-R ALGEBRAS

Introduction

In the present paper we consider D-R algebras in the sense of [2] and [5], which satisfy an additional condition, namely the so-called Quasi-Leibniz condition (shortly: QL-condition). We have distinguished three types of such algebras, namely E1, E2, E3 (see [1], [2]). In this paper we shall investigate algebras of the type E3 and theirs decompositions onto direct sums. We shall characterize such decompositions and give conditions for their existence. First let us recall some definitions.

Let here and in the sequel X be a commutative algebra over a field K of scalars. We assume that the multiplication in X is not trivial. Denote by $L(X)$ the set of all linear operators acting in X. For a given operator $A \in L(X)$ its domain will be denoted by $\mathcal{D}_A$ ($\mathcal{D}_A$ is a subalgebra of X). The set of all right invertible operators acting in X will be denoted by $R(X)$ (cf. [3]).

Definition 1 (cf. [2]). A right invertible operator $A \in L(X)$ is said to be a QL-operator if there exists $d \in K$ such that

$$(1) \qquad D(x\cdot y) = Dx\cdot y + x\cdot Dy + d\cdot Dx\cdot Dy \quad \text{for all } x,y \in \mathcal{D}_D.$$

In [2] we have shown that the constant d in (1) is uniquely determined by D. We will denote this constant by d_D. Denote by $QL(X)$ the set of all QL-operators in X and by $QL^*(X)$ the set of all QL-operators in X, having theirs universal constant d_D different from zero, i.e.

(2) $$QL(X) = \{A \in L(X):\ A \text{ is a QL-operator in } X\},$$

(3) $$QL^*(X) = \{A \in QL(X):\ d_A \in K^*\},$$

where $K^* = K \setminus \{0\}$.

It is easy to show that for each $\alpha \in K^*$ and $D \in QL(X)$ we have (cf. [2])

(4) $$\alpha D \in QL(X),$$

(5) $$d_{\alpha D} = \frac{d_D}{\alpha}.$$

Definition 2 (cf. [2]). A multiplicative operator $A \in L(X)$ is said to be an M-operator if its domain $\mathcal{D}_A$ is a subalgebra of X and $(A-I) \in R(X)$, i.e.

(6) $$A(x \cdot y) = Ax \cdot Ay \quad \text{for all } x, y \in \mathcal{D}_A,$$

(7) $$\exists R \in L(X), \mathcal{D}_R = x \text{ and } DR = I,$$

where I denotes the identity operator in X.

Denote the set of all M-operators in X by $\mathcal{M}(X)$.

Theorem 1. There is one-to-one correspondence between $QL^*(X)$ and $\mathcal{M}(X) \times K^*$ given by the formulas

(8) $$\Phi:\ QL^*(X) \to \mathcal{M}(X) \times K^*,\quad \Phi(D) = (I + d_D \cdot D,\ d_D),$$

(9) $$\Phi^{-1}:\ \mathcal{M}(X) \times K^* \to QL^*(X),\quad \Phi^{-1}(A, d) = \frac{1}{d}(A - I).$$

P r o o f . Denote the components of the map Φ by Φ_1 and Φ_2 respectively. We have

$$(10) \qquad \Phi(D) = (\Phi_1(D), \Phi_2(D)) = (I + d_D \cdot D, d_D).$$

Let now $D \in QL^*(X)$ and $x,y \in \mathcal{D}_D = \mathcal{D}_{\Phi_1(D)}$. Then we obtain

$$(11) \quad \Phi_1(D)(x\cdot y) = (I + d_D\cdot D)(x\cdot y) = x\cdot y + d_D(Dx\cdot y + x\cdot Dy + d_D\cdot Dx\cdot Dy) =$$
$$= \Phi_1(D)x\cdot \Phi_1(D)y.$$

Of course, the operator $\Phi_1(D) - I = d_D\cdot D$ is right invertible, which proves the first part of Theorem 1.

Let now $A \in \mathcal{M}(X)$ and $d \in K^*$. Then the operator $\Phi^{-1}(A,d) = \frac{1}{d}(A-I)$ is right invertible, because A is an M-operator.

Let $x,y \in \mathcal{D}_A = \mathcal{D}_{\Phi^{-1}(A,d)}$. Then

$$(12) \quad \Phi^{-1}(A,d)(x\cdot y) = \frac{1}{d}(A-I)(x\cdot y) = \frac{1}{d}(A(x\cdot y) - x\cdot y) = \frac{1}{d}(Ax\cdot Ay - x\cdot y) =$$
$$= \frac{1}{d}(Ax\cdot Ay - x\cdot Ay - Ax\cdot y + x\cdot y + Ax\cdot y - x\cdot y + Ay\cdot x - x\cdot y) =$$
$$= \frac{1}{d}(Ax-x)(Ay-y) + \frac{1}{d}(Ax-x)y + \frac{1}{d}(Ay-y)x =$$
$$= \Phi^{-1}(A,d)x\cdot y + x\cdot \Phi^{-1}(A,d)y +$$
$$+ d\Phi^{-1}(A,d)x\cdot \Phi^{-1}(A,d)y.$$

D e f i n i t i o n 3 (cf. [2], [4]). If D is a QL-operator in the algebra X, then we say that the pair (X,D) is a QL-algebra (or quasi-Leibniz D-R algebra). If R is a fixed right inverse of D, then the QL-algebra (X,D) will be denoted by (X,D,R). We say that a QL-algebra (X,D) is generated by an operator $A \in \mathcal{M}(X)$ if

(13) $$D = \phi^{-1}(A, d_D),$$

where ϕ^{-1} is as in (9).

Proposition 1. If $A \in \mathcal{M}(X)$ is a homomorphism of a subalgebra $\mathcal{D}_A$ with unit e into X and the QL-algebra (X,D) is generated by A, then

(14) $$De = 0.$$

Proof follows immediately from (13).

Since now we shall consider only such QL-algebras where $\mathcal{D}_D$ is a subalgebra with unit e of X.

Definition 4. A given QL-algebra (X,D) with unit $e \in \mathcal{D}_D$ is said to be of the type Reg^k if

(15) $$D^i e \in \mathcal{D}_D \quad \text{for} \quad i = 0,1,2,\dots,k.$$

If a QL-algebra (X,D) is of the type Reg^k for all $k=0,1,\dots$, then we say that (X,D) is of the type Reg^∞.

Open question: Does exist a QL-algebra of the type Reg^k, which is not of the type Reg^∞?

For given subsets A, B of an algebra X we denote by $\langle A\rangle_B$ the set

(16) $$\langle A\rangle_B = \Big\{ y \in X: \ \exists_{n\in N} \ \exists_{\substack{a_1,\dots,a_n \in A \\ b_1,\dots,b_n \in B}} \ y = \sum_{j=1}^{n} a_j b_j \Big\}.$$

If A is a finite subset of X, i.e. $A = \{a_1,\dots,a_n\}$, then we write

(17) $$\langle A\rangle_B = \langle a_1,\dots,a_n\rangle_B.$$

Corollary. If $B = X$ and $A = \{a_0\}$, then $\langle a_0\rangle_X$ is a two-sided ideal generated by a_0; in other words

(18) $$\langle a_0\rangle_X = \{za_0 : z \in X\}.$$

If $B = X$ then we will denote the set $\langle x_0\rangle_X$ shortly by $\langle x_0\rangle$.

Definition 5 (cf. [2]). A given QL-algebra (X,D) is said to be QQL-decomposable if there exist non trivial subalgebras X_1, X_2 of X such that

(19) $$X = X_1 \oplus X_2, \quad \mathcal{D}_D = (X_1 \cap \mathcal{D}_D) \oplus (\mathcal{D}_D \cap X_2)$$

and

(20) $$D: X_j \longrightarrow X_j \quad \text{for} \quad j = 1,2.$$

A QQL-decomposable QL-algebra (X,D) is said to be QL-decomposable if $(X_1, D|_{X_1})$ and $(X_2, D|_{X_2})$ are QL-algebras.

Theorem 2. If a QL-algebra (X,D) is QQL-decomposable onto algebras X_1 and X_2 then

(21) $$X = \langle e_1\rangle \oplus \langle e_2\rangle,$$

where e_1, e_2 are units of X_1 and X_2 respectively.

Proof. Since $X = X_1 \oplus X_2$, we have $X_1 \cdot X_2 = 0$. We shall show that

(22) $$e = e_1 + e_2.$$

Assume that $e = c_1 + c_2$ for some elements $c_1 \in X_1$, $c_2 \in X_2$. Then $c_1 + c_2 = e = e^2 = c_1^2 + c_2^2 + 2c_1c_2 = c_1^2 + c_2^2$ and $c_1 = c_1^2$, $c_2 = c_2^2$. On the other hand we have that for all elements $x_1 \in X_1$ and $x_2 \in X_2$

(23) $$x_1 = x_1 \cdot e = x_1 \cdot c_1 + x_1 \cdot c_2 = x_1 \cdot c_1,$$

(24) $$x_2 = x_2 \cdot e = x_2 \cdot c_1 + x_2 \cdot c_2 = x_2 \cdot c_2,$$

which implies that

(25) $$c_1 = e_1 \quad \text{and} \quad c_2 = e_2.$$

It follows from (23-25) that $X_1 \subset \langle e_1 \rangle$ and $X_2 \subset \langle e_2 \rangle$.

For each $x \in X$ we have

(26) $$x = x_1 + x_2,$$

where $x_1 \in X_1$, $x_2 \in X_2$, which implies that

$$e_1 \cdot x = e_1 \cdot x_1 = x_1 \in X_1, \quad e_2 \cdot x = e_2 \cdot x_2 = x_2 \in X_2,$$

so

$$\langle e_1 \rangle \subset X_1, \quad \langle e_2 \rangle \subset X_2 \quad \text{and} \quad X_1 = \langle e_1 \rangle,\ X_2 = \langle e_2 \rangle.$$

Of course, from Theorem 2, we have for both X_1 and X_2 that

(27) $$X_1 = \langle e_1 \rangle_X = \langle e_1 \rangle_{X_1},$$

(28) $$X_2 = \langle e_2 \rangle_X = \langle e_2 \rangle_{X_2}.$$

P r o p o s i t i o n . If the only idempotents in a QL-algebra (X,D) are e and 0 then (X,D) is not decomposable.

P r o o f . Let (X,D) be QQL-decomposable onto X_1 and X_2 with units e_1 and e_2 respectively. Then we have

(29) $$e = e^2 = e_1^2 + e_2^2 = e_1 + e_2.$$

Hence $e_1 = 0$ or $e_2 = 0$, which contradicts to our assumptions that X_1 and X_2 are not trivial algebras.

An immediate consequence of (1) is the equality

(30) $$De(x + d \cdot Dx) = 0 \quad \text{for all} \quad x \in \mathcal{D}_D \quad (\text{cf. } [1]).$$

The equality (30) gives three possible cases

E1: $De = 0$,

E2: $D = -\frac{1}{d} I$,

E3: $De \neq 0$ and $D \neq -\frac{1}{d} I$ and $De(x + d \cdot Dx) = 0$ for all $x \in \mathcal{D}_D$.

Let (X,D) be a QL-algebra. Denote by DE the set

$$DE = \{y \in X: De \cdot y = 0\}. \tag{31}$$

P r o p o s i t i o n 3. For a given QL-algebra (X,D) the set DE is a two-sided ideal in X.

P r o o f . If $x \in X$, $y \in DE$ then $(x \cdot y) \cdot De = x(y \cdot De) = 0$. Also, if $x, y \in DE$ then $(x+y) \in DE$.

T h e o r e m 3. If (X,D) is a QL-algebra and $DE = \{0\}$ then

$$D = -\frac{1}{d} I. \tag{32}$$

P r o o f . Since $DE = \{0\}$ then from (30) we have

$$De = -\frac{1}{d} e. \tag{33}$$

Let $x \in \mathcal{D}_D$. Then we obtain

$$\begin{aligned} Dx &= D(x \cdot e) = Dx \cdot e + x \cdot De + d \cdot Dx \cdot De = \\ &= Dx + x\left(-\frac{1}{d} e\right) + d \cdot Dx \cdot \left(-\frac{1}{d} e\right) = \\ &= Dx - \frac{1}{d} x - Dx = -\frac{1}{d} x. \end{aligned} \tag{34}$$

T h e o r e m 4. If (X,D) is a QL-algebra and $DE = X$, then

$$De = 0. \tag{35}$$

P r o o f . Since $DE = X$, we have for $y = e$,

$$De \cdot y = De = 0.$$

Since now we assume that (X,D) is a QL-algebra such that

(36) $$0 \neq DE \neq X.$$

T h e o r e m 5. If (X,D) is a QL-algebra and $0 \neq DE \neq X$ then

(37) $$DE = \langle e + d\,De \rangle ,$$

(38) $$\ker D \subset DE,$$

(39) $$DE \text{ is an algebra with unit } (e + d\,De),$$

(40) $$\langle De \rangle \text{ is an algebra with unit } (-d\,De),$$

(41) $$\langle De \rangle \text{ is a two-sided ideal in } X,$$

(42) $$X = \langle De \rangle \oplus De.$$

P r o o f . In order to prove formula (37) suppose that $y \in \langle e + d\,De \rangle$. Then $De \cdot y = x \cdot (e + d\,De)De = 0$ for an x, which implies $y \in DE$.

Let now $y \in DE$. Then $y = ye = y\,(e + d \cdot De - d \cdot De) = y(e + dDe)$. In order to prove formula (38) suppose that $x \in \ker D$. Then from (30) we have $De \cdot x = De(x + d \cdot Dx) = 0$. In order to prove formula (39) observe that DE is closed with respect to the operations in X. Hence it is enough to show that

$$(e + dDe)^2 = (e + dDe).$$

From (30) we have for $x = e$ that

(43) $$De = -d\,(De)^2.$$

Thus

(44) $$-dDe = (-dDe)^2$$

and

$$(45)\qquad (e + dBe)^2 = e^2 + 2dDe + d^2(De)^2 = (e + dDe).$$

In order to show formula (40), observe that $\langle De \rangle$ is closed respect to operations in X and apply (44). Formula (41) follows just from the definition of $\langle De \rangle$.

For the proof of formula (42) consider an arbitrary $x \in X$. We have the decomposition

$$(46)\qquad x = x_1 + x_2,$$

where $x_1 \in \langle De \rangle$, $x_2 \in DE$ and

$$(47)\qquad x_1 = x\cdot(-dDe),$$

$$(48)\qquad x_2 = x\cdot(e + dDe).$$

Hence

$$(49)\qquad X = \langle De \rangle + DE.$$

Let now $y \in \langle De \rangle \cap DE$. Then

$$(50)\qquad y = y\cdot(-dDe) = y\cdot(e + dDe) = y\cdot(-dDe)\cdot(e + dDe) = 0,$$

so $\langle De \rangle \cap DE = \{0\}$ and

$$(51)\qquad X = \langle De \rangle \oplus DE.$$

P r o p o s i t i o n 4. If (X,D) is a QL-algebra of the type Reg^1 then

$$(52)\qquad \mathcal{D}_D \cap DE = \langle e + dDe \rangle \mathcal{D}_D,$$

$$(53)\qquad \mathcal{D}_D \cap \langle De \rangle = \langle De \rangle \mathcal{D}_D.$$

P r o o f . To prove formula (52) suppose that $y \in \langle e + dDe \rangle_{\mathcal{D}_D}$. Then $y = x\cdot(e + dDe)$ for an $x \in \mathcal{D}_D$. Since $e, De, e + dDe \in \mathcal{D}_D$, we have $x\cdot(e + dDe) \in \mathcal{D}_D$. Thus $y \in DE \cap \mathcal{D}_D$ and $\langle e + dDe \rangle \subset \mathcal{D}_D \cap DE$. The converse inclusion is obvious. In the same way we prove (53).

T h e o r e m 6. If (X,D) is a QL-algebra of the type Reg^1 and $0 \neq DE \neq X$ then $(DE, D|_{DE})$ is a QL-algebra with unit $(e + dDe)$.

P r o o f . Theorem 5 implies that DE is an algebra with unit $(e + dDe)$. From Proposition 4 we have that

$$\mathcal{D}_D \cap DE = \langle e + dDe \rangle \mathcal{D}_D = \mathcal{D}_{D|_{DE}}.$$

To prove Theorem 6 it is enough to show that D and its right inverse R are invariant on DE:

$$D: DE \longrightarrow DE \tag{54}$$

and

$$R: DE \longrightarrow DE. \tag{55}$$

If $x \in \mathcal{D}_D \cap DE$ then from (30) we have

$$0 = De(x + dDx) = De\, x + De\, dDx = d\, De\, Dx.$$

Therefore $Dx \in DE$ and (54) is true.

Putting in (30) Rx instead of x we have

$$0 = De\, Rx + d\, DRx = De\, Rx.$$

Then $Rx \in DE$ and (55) is true.

T h e o r e m 7. Let (X,D) be a QL-algebra of the type Reg^1 and $0 \neq DE \neq X$. Then (X,D) is QL-decomposable onto QL-algebras of the types E1 and E2 if and only if

$$D^2e = -\frac{1}{d}\, De. \tag{56}$$

P r o o f . Necessity. If $X = X_1 \oplus X_2$, $e = e_1 + e_2$ and $(X_1, D|_{X_1})$ is of the type E1, $(X_2, D|_{X_2})$ is of the type E2 then

$$(57)\qquad D^2e = D(De_1 + De_2) = D(0 + De_2) = D\left(-\frac{1}{d}e_2\right) = = -\frac{1}{d}De_2 = -\frac{1}{d}De.$$

Sufficiency. Let $D^2e = -\frac{1}{d}De$.

Put

$$(58)\qquad X_1 = \langle e + dDe \rangle,$$

$$(59)\qquad X_2 = \langle -dDe \rangle.$$

$(X_1, D|_{X_1})$ and $(X_2, D|_{X_2})$ are QL-algebras with units, which follows from (56) and Theorems (5)-(6). From Theorem 5 we have that $X = X_1 + X_2$ which proves sufficiency.

T h e o r e m 8. Let (X,D) be a QL-algebra of the type Reg^{∞} and $\ker D \neq \{0\}$ and $\{0\} \neq DE \neq X$. Then (X,D) is QQL-decomposable onto algebras X^I, X_I, with units e^I, e_I respectively, such that

$$(60)\qquad \ker D_{|X^I} = \{0\},$$

$$(61)\qquad \ker D \subset X_I,$$

$$(62)\qquad (X_I, D|_{X_I}) \text{ is a QL-algebra.}$$

P r o o f . Define the following sequences:

$$(63)\qquad e_{(1)} = e + dDe, \qquad X_{(1)} = DE = \langle e_{(1)} \rangle_X,$$

$$(64)\qquad e^{(1)} = -dDe, \qquad X^{(1)} = \langle De \rangle = \langle e^{(1)} \rangle_X,$$

(65) $$e_{(k+1)} = e_{(k)} + dDe_{(k)} \quad \text{for } k = 1,2,\dots$$

(66) $$e^{(k+1)} = -dDe_{(k)} \quad \text{for } k = 1,2,\dots$$

(67) $$X_{(k)} = \langle e_{(k)} \rangle X_{(k-1)} \quad \text{for } k = 2,3,\dots$$

(68) $$X^{(k)} = \langle e^{(k)} \rangle X_{(k-1)} \quad \text{for } k = 2,3,\dots$$

From the assumption we have that $X \neq X_{(1)} \neq \{0\}$, which implies that also $X \neq X^1 \neq \{0\}$.

If $X_{(k+1)} = \{0\}$ for a $k \geqslant 1$ then from Theorem 3 we have that $D = -\frac{1}{d} I$ on $X_{(k)}$, which implies that $\ker D = \{0\}$. If $X_{(k+1)} = X_{(k)}$ then from Theorem 4 we have that $De_{(k)} = \{0\}$ and putting

(69) $$X^I = \bigoplus_{j=1}^{k} X^{(j)},$$

(70) $$X_I = X_{(k)}$$

we obtain the required decomposition (cf. [3]).

Let now $X_{(k)} \neq \{0\}$ and $X_{(k)} \neq X_{(k+1)}$ for each $k = 1,2,\dots$. Putting

(71) $$X^I = \bigoplus_{k=1}^{\infty} X^{(k)},$$

(72) $$X_I = \bigcap_{k=1}^{\infty} X_{(k)}$$

we obtain the desired decomposition (cf. [3]). The units in X^I and X_I are

(73) $$e^I = \bigoplus_{k=1}^{\infty} e^{(k)},$$

(74) $$e_I = e - e^I,$$

respectively.

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