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VARIETIES OF ORTHOMODULAR LATTICES WITH A STRONGLY FULL SET OF STATES

1. Introduction

In this paper we study some equational classes of orthomodular lattices. The lattice of varieties of orthomodular lattices has been studied by Bruns and Kalmbach ([1], [2]). Varieties studied here are contained between the variety TSFSS of orthomodular lattices with a full set of two-valued states and the variety SFSS generated by orthomodular lattices with a strongly full set of states. We show that any of these varieties is not finitely based. We show also, using Birkhoff's Theorem, that the class of orthomodular lattices with a full set of two-valued states forms a variety.

2. Basic definitions and properties

As in [1], an orthomodular lattice (abbreviated oml) is considered as an universal algebra $(L; \wedge, \vee, ', 0, 1)$ with the binary lattice operations $\wedge$ and $\vee$, the unary orthocomplementation operation $'$, and the two nullary operations (constants) 0 and 1, the smallest and largest element of the lattice. If some subalgebra of L is a Boolean algebra, then we call it a Boolean subalgebra. We write $a \perp b$, if $a \leq b'$ and aCb if a and b commute (i.e. the subalgebra generated by set $\{a,b\}$ is a Boolean subalgebra).

Recall that a state on an oml L is a map $m : L \to \langle 0,1 \rangle \subseteq R$ such that $m(1) = 1$ and if $a,b \in L$, $a \perp b$, then $m(a \vee b) =$

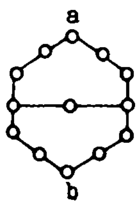
= m(a) + m(b). A two-valued (or dispersion free) state is one assuming only the values 0 and 1. If m is a state, then $a \leq b$ implies $m(a) \leq m(b)$. A set $\{m_t \mid t \in T\}$ of states on L is said to be full (strongly full or strongly ordered) if for any $a,b \in L$

$$\left[\bigwedge_{t\in T} m_t(a) \leq m_t(b)\right] \Longrightarrow a \leq b$$

$$\left(\left[\bigwedge_{t\in T} (m_t(a) = 1 \Longrightarrow m_t(b) = 1)\right] \Longrightarrow a \leq b, \text{ respectively}\right).$$

Any strongly full set of states is full. The converse is false (see the oml FNS_n in this paper). On the other hand if a set of two-valued states is full, then it is strongly full.

The class of omls with a full set of states we denote by FSS. The class of omls with a strongly full set of (two-valued) states we denote by SFSS (TSFSS respectively). Thus, by definition $TSFSS \subseteq SFSS \subseteq FSS$. Oml L_{28} presented below on the Greechie diagram



$$m(a) = 1 \Longrightarrow m(b') = 1 \quad \text{but} \quad a \nleq b'.$$

is an element of SFSS - TSFSS. All omls FNS_n are elements of FSS-SFSS.

D e f i n i t i o n 1. A Boolean block - embedding of an oml L is a map $f : L \longrightarrow B$, where B is a Boolean algebra and for any $a,b \in L$, the following conditions hold:

(Bbe 1) $f(a') = [f(a)]'$

(Bbe 2) $a \perp b \Longleftrightarrow f(a) \perp f(b)$

(Bbe 3) $a \perp b \Longrightarrow f(a \vee b) = f(a) \vee f(b)$.

L e m m a 1. Let L be an oml; B be a Boolean algebra and let f be a map $f : L \to B$. Then f is a Boolean block - embedding iff for any Boolean subalgebra $A \subseteq L$, the restriction f to A is a monomorphism of Boolean algebras.

D e f i n i t i o n 2. A partial field of sets (see [4], [5]) is a non empty family M of subsets of a set X, such that for any $A,B \in M$ the following conditions hold:

(PFS 1) $A \in M \Rightarrow X \setminus A \in M$

(PFS 2) $[A,B \in M,\ A \cap B = \emptyset] \Rightarrow A \cup B \in M$.

L e m m a 2. If a partial field M of subsets of a set X forms a lattice under inclusion, then it is an orthomodular lattice where $A' = X \setminus A$, $0 = \emptyset$, $1 = X$, and for any $A,B \in M$, $A \perp B$ iff $A \cap B = \emptyset$.

L e m m a 3. Let L be an oml. Then the following conditions are equivalent:

(1) L has a full set of two-valued states.

(2) There exists a Boolean block - embedding of L.

(3) L is isomorphic to a partial field of sets.

The proofs of the above lemmas are straightforward and we can omit them.

3. The variety TSFSS

In this section we prove the following

T h e o r e m 1. The class TSFSS of orthomodular lattices with a full set of two-valued states forms a variety.

It is easy to show that TSFSS is closed under taking of subalgebras and products. As a consequence of two udermentioned Lemmas we obtain that TSFSS is closed under homomorphic images.

L e m m a 4. Let a partial field of sets M forms an oml; $D,A,B \in M$, such that $A \cap B \subseteq D$. If $h : M \to L$ is a homomorphism of M to some oml L, such that $h(D) = 0$, then $h(A) \perp h(B)$.

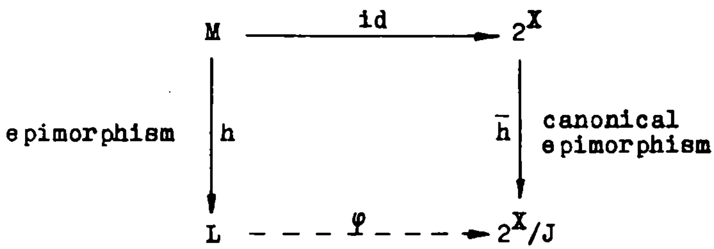
P r o o f . It is a well known fact that A is the sum of mutually orthogonal elements $A = (A \wedge B) \vee (A \wedge B') \vee A_1$, where $A_1 = A \wedge (A' \vee B) \wedge (A' \vee B')$. Thus we have:

$h(A) = h(A \wedge B) \vee h(A \wedge B') \vee h(A_1)$. Since $A \wedge B \subseteq D$, and $h(D) = 0$ then $h(A \wedge B) = 0$. We show that $h(A_1) = 0$. In fact:

$A_1 \wedge D' \subseteq A_1 \cap D' \subseteq A \cap D' \subseteq B'$. Thus $A_1 \wedge D' \subseteq A_1 \wedge B' = \emptyset$. Now $h(D') = 1$ and $h(A_1) = h(A_1) \wedge h(D') = h(A_1 \wedge D') = h(\emptyset) = 0$. Therefore we have $h(A) = h(A \wedge B')$. Similarly $h(B) = h(A' \wedge B)$ and so $h(A) \perp h(B)$.

Lemma 5. Let a partial field M of subsets of a set X be an oml. Let h be an epimorphism from M onto an oml L and $\bar{h}$ be a canonical epimorphism from the Boolean algebra 2^X of all subsets of a set X onto the quotient Boolean algebra $2^X/J$ where $J = \left\{ A \subseteq X \mid \bigvee_{A_0 \in M} A \subseteq A_0 \;\; h(A_0) = 0 \right\}$ is an ideal of the Boolean algebra 2^X generated by $h^{-1}(\{0\})$. Then there exists a Boolean block - embedding $\varphi : L \to 2^X/J$.

Proof. The situation described above is ilustrated by the diagram



First, we prove, that if $h(A) = h(B)$, then $\bar{h}(A) = \bar{h}(B)$. We denote $A \wedge B$ by P and $A \vee B$ by Q. Then $h(P) = h(A) = h(B) = h(Q)$, and $P \subseteq Q$. Therefore $P' \cap Q \in M$ and $h(P' \cap Q) = 0$. Hence $\bar{h}(P' \cap Q) = 0$ and $\bar{h}(P) = \bar{h}(Q)$. Since $P \subseteq A \subseteq Q$, then $\bar{h}(A) = \bar{h}(P)$. Similary $\bar{h}(B) = \bar{h}(P)$. Therefore $\bar{h}(A) = \bar{h}(B)$.

Now, we define

$$\varphi(h(A)) =: \bar{h}(A).$$

We shall show that φ is a Boolean block - embedding, i.e. that the conditions (Bbe 1) - (Bbe 3) hold.

(Bbe 1). $\varphi([h(A)]') = \varphi(h(A')) = \bar{h}(A') = [h(A)]' = [\varphi(h(A))]'$.

(Bbe 2). "$\Longrightarrow$" If $h(A) \perp h(B)$, then $h(A \wedge B') = h(A)$ and $h(A' \wedge B) = h(B)$. Thus $\varphi(h(A)) = \bar{h}(A \wedge B')$ and $\varphi(h(B)) = \bar{h}(A' \wedge B)$. Since $A \wedge B' \perp A' \wedge B$, then $\bar{h}(A \wedge B') \perp \bar{h}(A' \wedge B)$. Therefore $\varphi(h(A)) \perp \varphi(h(B))$.

(Bve 2). "$\Longleftarrow$" If $\bar{h}(A) \perp \bar{h}(B)$, then $\bar{h}(A \cap B) = \emptyset/J$, i.e. $A \cap B \in J$. Hence there exists $D \in M$ such that $h(D) = 0$ and $A \cap B \subseteq D$. Thus, by Lemma 4, $h(A) \perp h(B)$.

(Bbe 3). If $h(A) \perp h(B)$ then $h(A) = h(A \wedge B')$ and $h(B) = h(A' \wedge B)$. Hence $h(A) \vee h(B) = h(A \wedge B') \vee h(A' \wedge B) = h((A \wedge B') \vee (A' \wedge B)) = h((A \wedge B') \cup (A' \wedge B))$. Thus: $\varphi(h(A) \vee h(B)) = \bar{h}((A \wedge B') \cup (A' \wedge B)) = \bar{h}(A \wedge B') \vee \bar{h}(A' \wedge B) = \varphi(h(A \wedge B')) \vee \varphi(h(A' \wedge B)) = \varphi(h(A)) \vee \varphi(h(B))$.

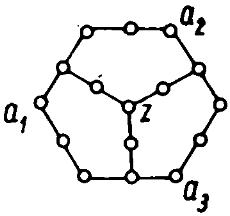
4. The omls FNS_n

All varieties of omls studied in Bruns, Kalmbach [1] and [2] are finitely based. The main result of this section is the following

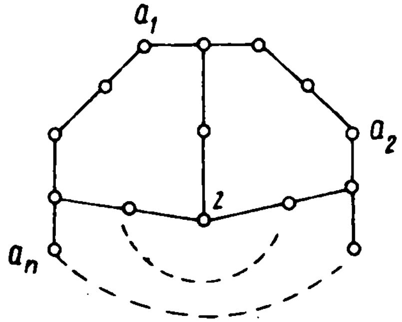
Theorem 2. Let V be a variety of omls such that $TSFSS \subseteq V \subseteq SFSS$, where $SFSS$ is a variety generated by omls with a strongly full set of states. Then V is not finitely based.

To prove this theorem (using the method of Model Theory - see ([3], Thm 4.1.12)) we define for any natural number $n \geq 3$, the oml FNS_n. We sill show that for any $n \geq 3$, FNS_n is not in the variety $SFSS$ and that the ultraproduct of the FNS_n's corresponding to a nonprincipal ultrafilter on $\{3,4,5,\ldots\}$ is contained in the variety TSFSS.

Definition 3. The FNS_n is an oml of length 3 with 5n+1 atoms presented below on the Greechie diagram:



FNS_3



FNS_n

L e m m a 6. For any $n \geqslant 3$ the oml FNS_n is not contained in the variety *SFSS*.

P r o o f . For simplicity we give the proof for the case $n = 3$. For other n's the proof is similar.

Let $z_i =: x_i \vee (x_i' \wedge x_j')$, $u_i =: x_j \vee (x_i' \wedge x_j')$, $i = 1,2,3$; $j = i+1 \pmod 3$. Let $z = z(x_1,x_2,x_3) =: z_1 \wedge z_2 \wedge z_3$; $u = u(x_1,x_2,x_3) =: u_1 \wedge u_2 \wedge u_3$. We prove that the equality

$$\text{(SF)} \qquad z(x_1,x_2,x_3) = u(x_1,x_2,x_3)$$

is true in all omls with a strongly full set of states. First, observe that if m is a state, then $m(z_i) = m(x_i) + m(x_i' \wedge x_j')$ and $m(u_i) = m(x_j) + m(x_i' \wedge x_j')$. Now let $m(z) = 1$. Then $m(z_1) = m(z_2) = m(z_3) = 1$. Hence $3 = \sum_{i=1}^{3} m(z_i) = \sum_{i=1}^{3} m(x_i) + m(x_i' \wedge x_j') = \sum_{i=1}^{3} m(u_i)$. Therefore $m(u_1) = m(u_2) = m(u_3) = 1$. Thus $z \leqslant u_i$, $i = 1,2,3$, and hence $z \leqslant u$. Similarly $u \leqslant z$. Now observe that in FNS_3 $z(a_1,a_2,a_3) \neq 0$ and $u(a_1,a_2,a_3) = 0$. Therefore $FNS_3 \notin$ *SFSS*.

R e m a r k . Just as above we can show that if p is a permutation on the set $\{1,2,\dots,n\}$, $q =: p^{-1}$, then the equality

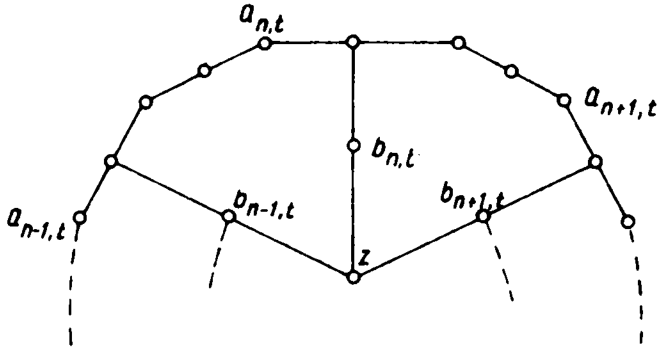
$$\bigwedge_{i=1}^{n} (x_i \vee (x_i' \wedge x_{p(i)}')) = \bigwedge_{i=1}^{n} (x_i \vee (x_i' \wedge x_{q(i)}'))$$

is true in all omls with a strongly full set of states.

L e m m a 7. Let FNS_n, $n = 3,4,5,\dots$ are olms given by Definition 3 and let F be a nonprincipal ultrafilter on the set $\{3,4,5,\dots\}$. Then the ultraproduct

$$FAN =: \prod_{n=3}^{\infty} FNS_{n/F}$$ belongs to the variety TSFSS.

P r o o f . It is easy to show that the FAN is an oml of cardinality continuum of length 3. We denote by A the underlying set of FAN. Then $A = \bigcup_{t\in T} A_t$, where T is a set of cardinality continuum; for any $t \in T$, A_t is the underlying set of some subalgebra FAN_t of FAN, isomorphic to oml schematically presented on the Greechie diagram below. Moreover, if $t \neq s$, then $A_t \cap A_s = \{0, z, z', 1\}$.



We shall describe the full set of two-valued states on FAN_t

	z	$b_{n,t}$	$a_{n,t}$
$m_{1,t}$	1	0	0
$m_{2,t}$	1	0	1
$m_{3,t}$	0	0	0
$m_{4,k,t}$	1	0	$\begin{cases} 1 & n \leq k \\ 0 & n > k \end{cases}$
$m_{5,k,t}$	0	$\begin{cases} 0 & n = k, k+1 \\ 1 & n \neq k, k+1 \end{cases}$	$\begin{cases} 0 & n \leq k+1 \\ 1 & n > k+1 \end{cases}$

	z	$b_{n,t}$	$a_{n,t}$
$m_{6,k,t}$	0	$\begin{cases} 1 & n=k \\ 0 & n \neq k \end{cases}$	0
$m_{7,k,t}$	0	$\begin{cases} 0 & n=k \\ 1 & n \neq k \end{cases}$	$\begin{cases} 1 & n \leqslant k-1 \\ 0 & n > k \end{cases}$
$m_{3,k,t}$	0	$\begin{cases} 0 & n=k \\ 1 & n \neq k \end{cases}$	$\begin{cases} 0 & n \leqslant k \\ 1 & n > k \end{cases}$

for any integer k

Using the above states it is easy to construct the full set of two-valued states on FAN.

R e m a r k . Since the variety of omls is arithmetical, then we have proved that the oml FAN generates a variety which has no finite base.

5. Open problems and conjectures

The class SFSS of omls with a strongly full set of states is obviously closed under subalgebras. The SFSS is also closed under products. If $\{L_t\}_{t \in T}$ is a family of omls from SFSS and for any $t \in T$, $\{m_{t,s}\}_{s \in S_t}$ is a strongly full set of states on L_t, then we obtain a family $S = \{\bar{m}_{t,s} \mid t \in T,\ s \in S_t\}$ of states on the product $\bar{L} =: \prod_{t \in T} L_t$, where $\bar{m}_{t,s}(a) =:$ $=: m_{t,s}(a(t))$ for $a \in \bar{L}$. The set S is strongly full but it need not be the set of all states on $\bar{L}$.

If $h : L \longrightarrow L_1$ is an epimorphism of orthomodular lattices and the set $\{m_a\}_{a \in A}$ is the set of all states on L, then put

$$A_0 =: \left\{a \in A \,\Big|\, \bigwedge_{x \in L} \left[h(x) = 0 \Longrightarrow m_a(x) = 0\right]\right\}.$$

It is easy to show that if g is a state on L_1, then $m(x) =:$ $=: g(h(x))$ is a state on L, i.e. $m = m_a$ for some $a \in A$.

Moreover $a \in A_0$. Therefore every state on L_1 is of the form $g_a(h(x)) =: m_a(x)$ for $a \in A_0$.

These two remarks give us some information about ultraproducts and homomorphic images of omls from the class SFSS. But we cannot prove or disprove the following two conjectures:

(1) The class SFSS is closed under ultraproducts (i.e. forms a quasivariety)

(2) The class SFSS is closed under homomorphic images (i.e. forms a variety).

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