DEMONSTRATIO MATHEMATICA
Vol. XIV No 3 1981

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AN ALGEBRAIC NOMOGRAMMABILITY TEST FOR FUNCTIONS OF n VARIABLES

1. One of the fundamental problems in theoretical nomography is that of reducing functions of n variables to a canonical form. In the case of $n = 3$ this problem has been thoroughly discussed by M. Warmus [5] and E. Otto [4]. In [4] (cf. § 31 pp.293-301) an old idea of E.Duporcq [3] was successfully developed. In another paper [2] M. Czyżykowski adapted the methods used in [4] to the problem of reduction of a function of $n = 4$ variables to Soreau form. In this paper an extension of the results of [4] and [2] to arbitrary $n \geq 3$ is presented.

2. First we shall list the principal notions and definitions used throughout this paper.

Let K be a number field, Ω_i $(i = 1,2,\dots,n)$ - arbitrary sets, and let Ω denote their Cartesian product: $\Omega = \bigtimes_{i=1}^{n} \Omega_i$. F will always denote a function of n variables x_i $(i = 1,2,\dots,n)$:

$$F : \Omega \ni (x_1,x_2,\dots,x_n) \to F(x_1,x_2,\dots,x_n) \in K. \tag{1}$$

D e f i n i t i o n 1. We say that F is reducible to Soreau form or shortly, to S-form in its domain Ω if such functions

(2) $$\bar{X}_i^k : \Omega_i \ni x_i \to \bar{X}_i^k(x_i) \in K, \quad i,k = 1,2,\dots,n$$

exist that for each $(x_1,x_2,\dots,x_n) \in \Omega$

(S) $$F(x_1,x_2,\dots,x_n) = \det\left[\bar{X}_i^k(x_i)\right]_{n\times n},$$

where the indices i,k number the rows and columns of the matrix $\left[\bar{X}_i^k(x_i)\right]_{n\times n}$., respectively. It is obvious that not every function F described by (1) is reducible to S-form.

Let now Φ_u and Φ_v denote two arbitrary non-empty sets, and let Φ be their Cartesian product, $\Phi = \Phi_u \times \Phi_v$. Consider a function G such that

(3) $$G : \Phi \ni (u,v) \to G(u,v) \in K.$$

We shall define the rank of G (cf. [5]) with respect to the variables $u \in \Phi_u$ and $v \in \Phi_v$.

D e f i n i t i o n 2. The rank $R(G)$ of the function G (3) is not higher than m, $R(G) \leq m$, if such functions

(4) $$\begin{cases} U_i : \Phi_u \ni u \to U_i(u) \in K, & i = 1,2,\dots,m, \\ V_i : \Phi_v \ni v \to V_i(v) \in K, & i = 1,2,\dots,m, \end{cases}$$

exist, that

(5) $$G(u,v) = \sum_{i=1}^{m} U_i(u)V_i(v).$$

D e f i n i t i o n 2a. The rank $R(G)$ of G is higher than m, $R(G) > m$, if it is not true that the rank $R(G)$ is not higher than m, i.e.

$$\left[R(G) > m\right] \Leftrightarrow \left[\sim(R(G) \leq m)\right].$$

D e f i n i t i o n 3. The function G has the rank $R(G) = 0$, if $G(u,v) \equiv 0$, i.e.

$$\left[R(G) = 0\right] \Longleftrightarrow \left[G = 0\right].$$

D e f i n i t i o n 4. The rank $R(G)$ of G is equal to m if $R(G) \leqslant m$ and $R(G) > m-1$. Then we write $R(G) = m$.

From these definitions we can see that G is of rank 1 if G has the form

$$G(u,v) \equiv U_1(u)V_1(v),$$

where neither U_1 nor V_1 is identically equal to zero, i.e. there is such an $(u_1,v_1) \in \Phi$ that $U_1(u_1) \neq 0$ and $V_1(v_1) \neq 0$.

In the function F (1) we can distinguish the variable x_i for any $i = 1,2,\dots,n$. Then, we can define a function G_i of two variables $u = x_i$ and $v = (x_1,\dots,x_{i-1},x_{i+1},\dots x_n)$ in the domain

$$\Phi = \Phi_u \times \Phi_v = \Omega_i \times \Big(\underset{j\neq i}{\times}\ \Omega_j\Big),$$

by the relations

$$\text{(3a)}\qquad \begin{cases} G_i \ :\ \Phi \ni (u,v) \longrightarrow G_i(u,v) \in K, \\ G_i((x_i),(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)) = F(x_1,x_2,\dots,x_n), \end{cases}$$

$$\text{for all } (x_1,x_2,\dots x_n) \in \Omega .$$

D e f i n i t i o n 5. We say that F in (1) is of rank m (or, alternatively, of rank $\leqslant m$, $> m$) with respect to the variable x_i, if the function G_i defined by (3a) is of rank m ($\leqslant m$ or $> m$) with respect to $u = x_i$ and $v = (x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$.

D e f i n i t i o n 6. The function F in (1) is called nomographic (cf. M. Warmus [5]) if

1°. F can be reduced to Soreau form S, and

2°. F is of rank higher than 1 with respect to each variables x_i, $i = 1,2,\ldots,n$.

In this paper a necessary and sufficient condition for a nomographic function of n variables is given.

3. In order to formulate in a concise way our first theorem, we make use of the following assumptions and notations:

i) The ordered set $\{1,2,\ldots,n\}$ will be denoted by I.

ii) It is assumed that $\bar{\bar{\Omega}}_i \geqslant 2$ for all $i \in I$. Using this assumption we can select from each Ω_i two distinct elements $a_i \neq b_i$, $a_i, b_i \in \Omega_i$.

iii) The value $F(x_1,x_2,\ldots,x_n)$ of the function F at the point $(x_1,x_2,\ldots,x_n)$ will be also denoted by $F\left[(x_s)_{s\in I}\right]$.

iv) For each $(i,k) \in I^2$, X_i^k will be the restriction of F to the domain $\Omega_{ik} = \bigtimes_{s\in I} \bar{\Omega}_s$, where

$$(6) \qquad \bar{\Omega}_s = \begin{cases} \Omega_s & \text{for } s = i \\ \{b_s\} & \text{for } (s \neq i) \wedge (s = k) \\ \{a_s\} & \text{for } (s \neq i) \wedge (s \neq k). \end{cases}$$

Thus we can express the values of X_i^k in terms of F in the following way

$$(7) \quad X_i^k(x_i) \overset{\text{def}}{=} F\left[(z_s)_{s\in I}\right], \quad \text{where} \quad z_s = \begin{cases} x_s & \text{for } s = i \\ b_s & \text{for } (s\neq i)\wedge(s=k) \\ a_s & \text{for } (s\neq i)\wedge(s\neq k). \end{cases}$$

v) We denote further

$$(8a) \qquad F_0 \overset{\text{def}}{=} F\left[(a_s)_{s\in I}\right] = F(a_1,a_2,\ldots,a_n)$$

and also

(8b) $$F_i \overset{\text{def}}{=} F\left[(z'_s)_{s\in I}\right], \quad \text{where} \quad z'_s = \begin{cases} b_s & \text{for } s = i \\ a_s & \text{for } s \neq i. \end{cases}$$

If $t_1 \neq t_2$ $(t_1, t_2 \in I)$ then we define

(8c) $$F_{t_1t_2} \overset{\text{def}}{=} F_{t_2t_1} \overset{\text{def}}{=} F\left[(z''_s)_{s\in I}\right], \quad \text{where} \quad z''_s = \begin{cases} b_s & \text{for } (s=t_1)\vee(s=t_2) \\ a_s & \text{for } (s\neq t_1)\wedge(s\neq t_2). \end{cases}$$

For $t_1 < t_2 < t_3$ $(t_1, t_2, t_3 \in I)$ we define

(8d) $$F_{t_1t_2t_3} \overset{\text{def}}{=} F\left[(z'''_s)_{s\in I}\right], \quad \text{where} \quad z'''_s = \begin{cases} b_s & \text{for } s=t_j \ (j=1,2,3) \\ a_s & \text{for } (s\neq t_1)\wedge(s\neq t_2)\wedge(s\neq t_3). \end{cases}$$

From (8a-8d) it follows that for all $i,k \in I$ we have

(9) $$X_i^k(a_i) = \begin{cases} F_0 & \text{for } i = k \\ F_k & \text{for } i \neq k, \end{cases}$$

and

(10) $$X_i^k(b_i) = \begin{cases} F_k & \text{for } i = k \\ F_{ik} & \text{for } i \neq k. \end{cases}$$

vi) We shall assume that

(11) $$F_0 \neq 0,$$

(12) $$\bigwedge_{t_1\in I} \bigwedge_{t_2\in I} \left[(t_1 \neq t_2) \Rightarrow (F_{t_1t_2} \neq 0)\right],$$

(13) $$\bigwedge_{i\in I} (F_i = 0).$$

vii) Let us finally assume that for each $t_1 < t_2 < t_3$ ($t_j \in I$, $j = 1,2,3$), there are elements $T_{t_2t_3t_1}$ and $T_{t_3t_1t_2}$ of the field K satisfying the following system of equations

(14) $$T_{t_2t_3t_1} + T_{t_3t_1t_2} = F_{t_1t_2t_3}$$

(15) $$T_{t_2t_3t_1} \cdot T_{t_3t_1t_2} = - F_{t_1t_2} \cdot F_{t_1t_3} \cdot F_{t_2t_3} / F_0$$

and for $t_1 < t_2 < t_3 < t_4$ ($t_j \in I$, $j = 1,2,3,4$) the additional system of equations

(16) $$T_{t_2t_3t_1} T_{t_4t_1t_2} T_{t_3t_4t_1} T_{t_4t_2t_3}$$

(17) $$T_{t_3t_1t_2} T_{t_2t_4t_1} T_{t_4t_1t_3} T_{t_3t_4t_2}$$

$$\left.\begin{matrix} (16) \\ (17) \end{matrix}\right\} = \frac{F_{t_1t_2} F_{t_1t_3} F_{t_1t_4} F_{t_2t_3} F_{t_2t_4} F_{t_3t_4}}{(F_0)^2} .$$

There are $2\binom{n}{3}$ equations (14)-(15) and $2\binom{n}{4}$ equations (16)-(17).

4. Using the above assumptions and notation we are able to formulate the following theorem:

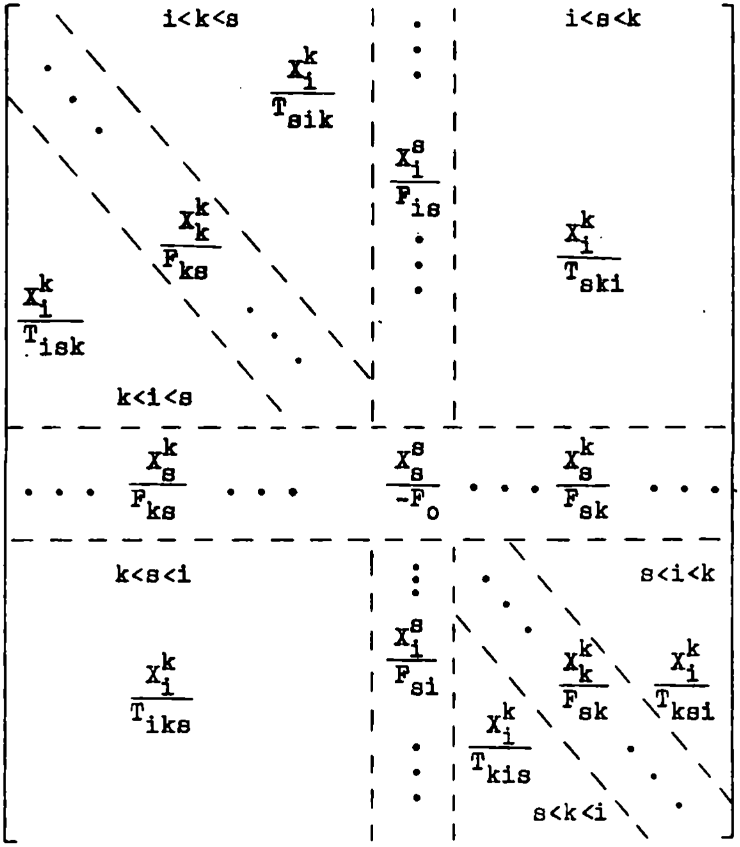
T h e o r e m 1. Under the assumptions i)-vii) a necessary and sufficient condition for the function F in (1) to be nomographic, is that for each $s \in I$ the following identity holds in Ω

(18a) $$F(x_1, x_2, \ldots, x_n) \equiv M_s \det W_s ,$$

where

(18b) $$M_s = -\Bigl(\prod_{m \in I \setminus \{s\}} F_{ms}\Bigr) / (F_0)^{n-2} ,$$

(18c) $W_s \equiv$



Now follows the proof of this theorem.

5. P r o o f . We shall show first the necessity of the condition (18). Thus, we have to prove that (18a,b,c) follows from the assumption that F is reducible to the S-form (S) in the sense of Definition 6.

Under the assumption that F has the form (S) we can treat the i-th row of the determinant on the r.h.s. of (S) as an element of a linear vector space K^n:

$$\bar{r}_i(x_i) = \left[\bar{X}_i^1(x_i), \bar{X}_i^2(x_i), \dots, \bar{X}_i^n(x_i)\right].$$

Let h denote a linear nonsingular mapping of K^n into itself:

$$h : K^n \ni \bar{r} \longrightarrow h(\bar{r}) = \bar{r}\, A \in K^n,$$

(where A is an $n\times n$ matrix) such that the value of h at the point $\bar{r}_i(x_i)$ is

$$\bar{w}_i(x_i) = h(\bar{r}_i(x_i)) = \left[f_i^1(x_i), f_i^2(x_i), \ldots, f_i^n(x_i)\right].$$

Then for each $i \in I$ the identity

$$\left[\bar{X}_i^1(x_i), \bar{X}_i^2(x_i), \ldots, \bar{X}_i^n(x_i)\right] = \left[f_i^1(x_i), f_i^2(x_i), \ldots, f_i^n(x_i)\right] A^{-1},$$

holds, and we have

$$(19) \qquad \left[\bar{X}_i^k(x_i)\right]_{n\times n} = \left[f_i^k(x_i)\right]_{n\times n} \cdot A^{-1}.$$

We define now h as the linear mapping satisfying the following condition for each $i \in I$:

$$(20) \quad \bar{r}_i(a_i) \xrightarrow{h} \bar{w}_i(a_i) = \left[f_i^1(a_i), f_i^2(a_i), \ldots, f_i^n(a_i)\right], \text{ where } f_i^k(a_i) = \delta_{ik}$$

(δ_{ik} denotes here a Kronecker's delta). From Eqs. (19) and (20) we obtain now that

$$\left[\bar{X}_i^k(a_i)\right]_{n\times n} = J\, A^{-1},$$

where J is a $n\times n$ unit matrix. From Eqs. (S) and (8a) we get

$$(21) \qquad \det A^{-1} = \det\left[\bar{X}_i^k(a_i)\right]_{n\times n} = F_0.$$

It follows now from Eqs. (S), (19) and (21) that

$$(22)\qquad F(x_1,x_2,\dots,x_n) \equiv \det\left[f_i^k(x_i)\right]_{n\times n}\cdot F_0.$$

We will be able now to express the values $X_i^k(x_i)$ defined by (7) in terms of the values $f_i^k(x_i)$ appearing in the identity (22). For brevity's sake, we shall write X_i^k instead of $X_i^k(x_i)$. First, we see from Eqs. (7), (20) and (22) that for each $k \in I$

$$(23)\qquad X_k^k \equiv X_k^k(x_k) \equiv f_k^k(x_k)\cdot F_0.$$

From the above identity and from Eqs. (10) and (13) we obtain

$$(24)\qquad f_k^k(b_k) = 0, \quad \text{for each } k \in I.$$

In the same way as for (23) we get for each $i,k \in I$ the identity

$$(25)\qquad X_i^k \equiv X_i^k(x_i) \equiv -f_i^k(x_i)f_k^i(b_k)\cdot F_0 \quad \text{for } i \neq k,$$

which, together with Eq. (10) gives for all $i \neq k$, $i,k \in I$

$$(26)\qquad F_{ik} = -f_i^k(b_i)f_k^i(b_k)\cdot F_0.$$

Now using the assumption (12) together with (26) we see that for all $i,k \in I$

$$(27)\qquad f_i^k \overset{\text{def}}{=} f_i^k(b_i) \neq 0, \quad \text{for } i \neq k.$$

In a similar way as for (23) and (25) we deduce that for all $t_1 < t_2 < t_3$, $t_1, t_2, t_3 \in I$

$$(28)\qquad F_{t_1t_2t_3} = \left(f_{t_1}^{t_2}\cdot f_{t_2}^{t_3}\cdot f_{t_3}^{t_1} + f_{t_1}^{t_3}\cdot f_{t_2}^{t_1}\cdot f_{t_3}^{t_2}\right)F_0,$$

Putting

$$(29)\quad T_{t_2t_3t_1} \overset{\text{def}}{=} f_{t_1}^{t_2} f_{t_2}^{t_3} f_{t_3}^{t_1} F_o, \quad T_{t_3t_1t_2} \overset{\text{def}}{=} f_{t_1}^{t_3} f_{t_2}^{t_1} f_{t_3}^{t_2} F_o$$

and inserting these values into (28) we obtain all $\binom{n}{3}$ equations of the form (14).

Now we shall show that (29) satisfy also Eqs. (15) for all $(t_1,t_2,t_3) \in I^3$ such that $t_1 < t_2 < t_3$. From (26) and (27) we get

$$T_{t_2t_3t_1}T_{t_3t_1t_2} = f_{t_1}^{t_2} f_{t_2}^{t_3} f_{t_3}^{t_1} f_{t_1}^{t_3} f_{t_2}^{t_1} f_{t_3}^{t_2} (F_o)^2 =$$

$$= \left(-f_{t_1}^{t_2} f_{t_2}^{t_1} F_o\right)\cdot\left(-f_{t_1}^{t_3} f_{t_3}^{t_1} F_o\right)\cdot\left(-f_{t_2}^{t_3} f_{t_3}^{t_2} F_o\right)/F_o =$$

$$= -F_{t_1t_2}F_{t_1t_3}F_{t_2t_3}/F_o.$$

In a similar way we obtain all $\binom{n}{4}$ equations of the form (16) for all $(t_1,t_2,t_3,t_4) \in I^4$ such that $t_1 < t_2 < t_3 < t_4$:

$$T_{t_2t_3t_1}T_{t_4t_1t_2}T_{t_3t_4t_1}T_{t_4t_2t_3} =$$

$$= f_{t_1}^{t_2} f_{t_2}^{t_3} f_{t_3}^{t_1}\cdot F_o\cdot f_{t_1}^{t_4} f_{t_2}^{t_1} f_{t_4}^{t_2}\cdot F_o\cdot f_{t_1}^{t_3} f_{t_3}^{t_4} f_{t_4}^{t_1}\cdot F_o\cdot f_{t_2}^{t_4} f_{t_3}^{t_2} f_{t_4}^{t_3}\cdot F_o =$$

$$= \left(-f_{t_1}^{t_2} f_{t_2}^{t_1}F_o\right)\cdot\left(-f_{t_1}^{t_3} f_{t_3}^{t_1}F_o\right)\cdot\left(-f_{t_1}^{t_4} f_{t_4}^{t_1}F_o\right)\cdot\left(-f_{t_2}^{t_3} f_{t_3}^{t_2}F_o\right)\cdot$$

$$\cdot\left(-f_{t_2}^{t_4} f_{t_4}^{t_2}F_o\right)\cdot\left(-f_{t_3}^{t_4} f_{t_4}^{t_3}F_o\right)\cdot (F_o)^{-2} =$$

$$= F_{t_1t_2}F_{t_1t_3}F_{t_1t_4}F_{t_2t_3}F_{t_2t_4}F_{t_3t_4}\cdot(F_o)^{-2},$$

as well as all $\binom{n}{4}$ equations of the form (17).

Finally we show that the identity (18a-c) holds for F. Starting from Eq. (22) and using (23) and (25) we get first the values of $f_i^k(x_i)$ for all $i,k \in I$:

$$(30)\qquad \begin{cases} f_i^k(x_i) = -X_i^k/(f_k^i F_o) & \text{for } i \neq k, \\ f_k^k(x_k) = X_k^k/F_o. \end{cases}$$

Then, denoting

$$(31)\qquad u_{ik} \overset{def}{=} \begin{cases} -f_k^i F_o & \text{for } i \neq k, \\ F_o & \text{for } i = k, \end{cases}$$

we can rewrite the identity (22) as

$$(32)\qquad F(x_1,x_2,\dots,x_n) \equiv \det\left[\frac{X_i^k}{u_{ik}}\right]_{n\times n} \cdot F_o.$$

Since all X_i^k $(i,k \in I)$ were defined by (7) as particular values of F, we must now find only the elements of the $n\times n$ matrix $[u_{ik}]$ $(i,k \in I)$. They must also be expressed in terms of F in n different ways, corresponding to n possible values of s in the condition (18a,b,c). Using Eq. (31) we obtain for all $k \in I\setminus\{s\}$

$$(33)\qquad \frac{u_{sk}}{u_{ss}} : \frac{u_{kk}}{u_{ks}} = -F_{sk}/F_o.$$

For all $i,k \in I\setminus\{s\}$, such that $i < k$ we have

$$(34)\qquad \frac{u_{ik}}{u_{is}} : \frac{u_{kk}}{u_{ks}} = \frac{f_k^i}{f_s^i} : \frac{-1}{f_s^k} = \frac{f_s^k\, f_i^s\, f_k^i\, F_o}{-f_s^i\, f_i^s\, F_o} =$$

$$= \begin{cases} T_{ksi}/F_{si} & \text{for } s<i<k, \\ T_{ski}/F_{is} & \text{for } i<s<k, \\ T_{sik}/F_{is} & \text{for } i<k<s \end{cases}$$

and for $k < i$ we get

$$(35)\qquad \frac{u_{ik}}{u_{is}} : \frac{u_{kk}}{u_{ks}} = \begin{cases} T_{kis}/F_{si} & \text{for } s<k<i, \\ T_{iks}/F_{si} & \text{for } k<s<i, \\ T_{isk}/F_{si} & \text{for } k<i<s. \end{cases}$$

Now put

$$p_k^s \overset{\text{def}}{=} \frac{u_{kk}}{u_{ks}},$$

then, making use of Eqs. (31) and (33)-(35) we obtain for all $i,k \in I$, $k \neq s$ the values

$$u_{ik} = \begin{cases} u_{is}\, p_k^s & \text{for } i = k \neq s \\ u_{is}\, p_k^s\, F_{ks}/(-F_o) & \text{for } k < i = s \\ u_{is}\, p_k^s\, F_{sk}/(-F_o) & \text{for } i = s < k \\ u_{is}\, p_k^s\, T_{ksi}/F_{si} & \text{for } s < i < k \\ u_{is}\, p_k^s\, T_{ski}/F_{is} & \text{for } i < s < k \\ u_{is}\, p_k^s\, T_{sik}/F_{is} & \text{for } i < k < s \\ u_{is}\, p_k^s\, T_{kis}/F_{si} & \text{for } s < k < i \\ u_{is}\, p_k^s\, T_{iks}/F_{si} & \text{for } k < s < i \\ u_{is}\, p_k^s\, T_{isk}/F_{is} & \text{for } k < i < s \end{cases}$$

Inserting the above values of the coefficients u_{ik} into (32) we get

$$F(x_1,x_2,\dots,x_n) \equiv F_0 \det \overline{W}_s,$$

where

$$\overline{W}_s \equiv \begin{bmatrix} \frac{X_1^1}{u_{1s}p_1^s} & \cdots & \frac{X_1^k}{u_{1s}p_k^s \frac{T_{s1k}}{F_{1s}}} & \cdots & \frac{X_1^s}{u_{1s}} & \cdots & \frac{X_1^n}{u_{1s}p_n^s \frac{T_{sn1}}{F_{1s}}} \\ \frac{X_2^1}{u_{2s}p_1^s \frac{T_{2s1}}{F_{2s}}} & \cdots & \frac{X_2^k}{u_{2s}p_k^s \frac{T_{s2k}}{F_{2s}}} & \cdots & \frac{X_2^s}{u_{2s}} & \cdots & \frac{X_2^n}{u_{2s}p_n^s \frac{T_{sn2}}{F_{2s}}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{X_k^1}{u_{ks}p_1^s \frac{T_{ks1}}{F_{ks}}} & \cdots & \frac{X_k^k}{u_{ks}p_k^s} & \cdots & \frac{X_k^s}{u_{ks}} & \cdots & \frac{X_k^n}{u_{ks}p_n^s \frac{T_{snk}}{F_{ks}}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{X_s^1}{-u_{ss}p_1^s \frac{F_{1s}}{F_o}} & \cdots & \frac{X_s^k}{-u_{ss}p_k^s \frac{F_{ks}}{F_o}} & \cdots & \frac{X_s^s}{u_{ss}} & \cdots & \frac{X_s^n}{-u_{ss}p_n^s \frac{F_{sn}}{F_o}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{X_n^1}{u_{ns}p_1^s \frac{T_{n1s}}{F_{sn}}} & \cdots & \frac{X_n^k}{u_{ns}p_k^s \frac{T_{nks}}{F_{sn}}} & \cdots & \frac{X_n^s}{u_{ns}} & \cdots & \frac{X_n^n}{u_{ns}p_n^s} \end{bmatrix}$$

We see that all elements of the i-th row of $\det \overline{W}_s$ have a common factor equal to F_{is}/u_{is} for $i \in I \setminus \{s\}$ and to $-F_o/u_{ss}$ for $i = s$. At the same time all elements of the k-th column have the same common factor $1/p_k^s$. Thus we obtain

(36) $F(x_1,x_2,\dots,x_n) =$

$$= F_o \prod_{i\in I\setminus\{s\}} (F_{is}/u_{is}) \prod_{k\in I\setminus\{s\}} (1/p_k^s)(-F_o/u_{ss}) \det W_s$$

where det W_s has the same form as in (18c). We can compute now the coefficient M_s of det W_s appearing on the r.h.s of (36). First we obtain

(37) $$\prod_{i\in I\setminus\{s\}} u_{is} = (-1)^{n-1}(F_o)^{n-1} \prod_{i\in I\setminus\{s\}} f_s^i,$$

and

(38) $$\prod_{k\in I\setminus\{s\}} p_k^s = (-1)^{n-1}\left(\prod_{k\in I\setminus\{s\}} f_s^k\right)^{-1}.$$

These values inserted together with (31) into (36) give

$$M_s = \left(-\prod_{i\in I\setminus\{s\}} F_{is}\right)/(F_o)^{n-2}$$

which is the same as in (18a), thus completing the proof that (18a,b,c) is a necessary condition for the part 1° of Definition 6.

The proof of sufficiency of (18a,b,c) for the part 1° of Definition 6 is extremely simple. Namely multiplying an arbitrary i-th row or k-th column of the s-th determinant on the r.h.s. of (18c) by the number M_s we obtain from (18a) the identity (S).

6. Finally we shall show that the rank of our function F (1) satisfying i)-vii) is higher than 1 with respect to each variable x_i, i.e. that from i)-vii) follows that F satisfies

part 2° of Definition 6. To this end we shall need the following lemma (cf. Warmus [5]).

L e m m a 1. If for any function G_i, $i \in I$ defined by (3a) there are such elements $u_1, u_2 \in \Phi_u$ and $v_1, v_2 \in \Phi_v$ that

$$\begin{vmatrix} G_i(u_1,v_1) & G_i(u_1,v_2) \\ G_i(u_2,v_1) & G_i(u_2,v_2) \end{vmatrix} \neq 0, \tag{39}$$

then G_i is of rank higher than 1 with respect to the variable x_i.

P r o o f . From assumption (39) one can see that the rank of G_i is higher than 0, since $G_i \neq 0$. If we assumed the rank of G_i to be equal to 1 i.e. G_i have the form

$$G_i(u,v) \equiv U_1(u)V_1(v), \quad \text{where} \quad U_1 \neq 0, \quad V_1 \neq 0,$$

the determinant in (39) would be equal to 0 for all $u \in \Phi_u$ and $v \in \Phi_v$, which would contradict our assumption (39). Thus the rank of G_i cannot be 1 nor 0, so it must be higher than 1, $R(G_i) > 1$.

Making use of this lemma we shall show that the rank of F under the assumptions i)-vii) is higher than 1 with respect to each variable x_i, $i \in I$. Putting

$$u_1 = a_i, \qquad u_2 = b_i$$

$$v_1 = (a_1,\dots,a_{i-1},a_{i+1},\dots,a_{k-1},a_k,a_{k+1},\dots,a_n)$$

$$v_2 = (a_1,\dots,a_{i-1},a_{i+1},\dots,a_{k-1},b_k,a_{k+1},\dots,a_n)$$

and inserting these values into the determinant (39) we obtain using definitions (3a), (8a)-(8d)

$$\begin{vmatrix} G_i(u_1,v_1) & G_i(u_1,v_2) \\ G_i(u_2,v_1) & G_i(u_2,v_2) \end{vmatrix} = \begin{vmatrix} F_0 & F_k \\ F_i & F_{ik} \end{vmatrix} \neq 0$$

since we have assumed (cf. (11)-(13)) that $F_0 \neq 0$, $F_{ik} \neq 0$ and $F_i = F_k = 0$. Thus according to Definition 5 function F is of rank higher than 1 with respect to each variable x_i, $i \in I$.

In this way we have shown that according to Definition 6, the function F in (1) satisfying the assumptions i)-vii) is nomographic if and only if conditions (18a-c) are satisfied.

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Received April 23, 1980.