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ON CERTAIN NONLOCAL PROBLEMS FOR A PARABOLIC SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

1. Introduction

In the present paper we give the solution of some nonlocal problems for a parabolic system of partial differential equations. Nonlocal problems of the types (N1), (N2), (N3) admit broad physical interpretations, e.g. in diffusion processes. These problems are closely related to some problems already investigated in literature, e.g. in [1], [4]. The methods used there consist in reducing the corresponding nonlocal problem to a boundary problem in the half-plane for a system of equations.

The cardinal inconvenience of the method proposed by Eidelman [4] consists in the fact that, even in the case of one equation, this reduction of the starting-problem leads to a limit problem for a system of equations.

The method presented in this paper makes possible the reduction of some nonlocal problems to a system of Volterra or Fredholm integral equations. We shall make use of the results established by the author in papers [2] and [3], in particular of the construction and properties of the matrix function $M(x,t)$. We recall that if D is a matrix such that $\operatorname{re}\lambda > 0$ for any eigenvalue λ of the matrix D, then by $M(x,t)$ for $x \in R$, $t > 0$, we shall understand the sum

(1.1) $$M(x,t) = I + 2\sum_{k=1}^{\infty} \exp\left[-k^2\pi^2 tD\right]\cos k\pi x.$$

The vector-function $\vec{u}(x,t)$ is said to satisfy the system of equations (*) in Q if

(*) $$\frac{\partial \vec{u}}{\partial t} = D\frac{\partial^2 \vec{u}}{\partial x^2} + \vec{f}(x,t),$$

where

$$D = \left[d_{ij}\right]_{i,j=1,\dots,n}, \qquad \vec{u}(x,t) = \begin{bmatrix} u_1(x,t) \\ \cdots\cdots \\ \cdots\cdots \\ u_n(x,t) \end{bmatrix}$$

$$\vec{f}(x,t) = \begin{bmatrix} f_1(x,t) \\ \cdots\cdots \\ f_n(x,t) \end{bmatrix}$$ in the domain $Q = (0,1)\times(0,T)$, $T < \infty$.

The matrix $D = \left[d_{ij}\right]$ is assumed to be such that the system of equations (*) is parabolic in the sense of Petrovski, i.e. that

(1.2) re $\lambda > 0$ for any eigenvalue λ of the matrix D.

Introduce the following notations:

(1.3) $$\vec{u}_1(x,t) = -D\int_0^t M_x(x,t-s)\vec{g}_1(s)ds$$

(1.4) $$\vec{u}_2(x,t) = D\int_0^t M_x(x-1,t-s)\vec{g}_2(s)ds$$

(1.5) $$\vec{u}_3(x,t) = \frac{1}{2}\int_0^1 \left[M(x-s,t)-M(x+s,t)\right]\vec{g}_3(s)ds$$

$$(1.6)\qquad \vec{u}_4(x,t) = \frac{1}{2}\int_0^t\int_0^1 \left[M(x-s,t-\eta)-M(x+s,t-\eta)\right] \vec{g}(s,\eta)\, ds\, d\eta$$

where $\vec{g}$, $\vec{g}_1$, $\vec{g}_2$, $\vec{g}_3$ are some fixed functions.

2. Solution of the problems (N1) and (N2)

Consider the problem (N1):

Find a vector-function $\vec{u}(x,t)$ satisfying the system of equations (*) in the domain Q, continuous in $\bar{Q}$ and such that

$$(2.1)\qquad \vec{u}(0,t) = \vec{f}_1(t) \qquad t \in (0,T)$$

$$(2.2)\qquad \vec{u}(x,0) = \vec{f}_3(x) \qquad x \in (0,1).$$

It is required, moreover, that at a point $\varkappa \in (0,1)$ the equalities

$$(2.3)\qquad \begin{aligned} \vec{u}(1,t) - \vec{u}(\varkappa,t) &= \vec{h}(t) \\ \vec{f}_3(1) - \vec{f}_3(\varkappa) &= \vec{h}(0) \end{aligned} \qquad t \in (0,T)$$

should hold, where $\vec{f}(x,t)$, $\vec{f}_1(t)$, $\vec{f}_3(x)$, $\vec{h}(t)$ are given functions.

By means of the substitution

$$(2.4)\qquad \vec{v}(x,t) = \vec{u}(x,t) - x\int_0^t \left[\vec{f}(1,s) - \vec{f}(0,s)\right] ds - \int_0^t \vec{f}(0,s)\, ds$$

we reduce the problem to the problem (N1*):

$$(2.5)\qquad \frac{\partial \vec{v}}{\partial t} = D\,\frac{\partial \vec{v}}{\partial x^2} + \vec{g}(x,t)$$

with the conditions

$$(2.6)\qquad \vec{v}(0,t) = \vec{g}_1(t) \qquad t \in^{*} (0,T)$$

$$(2.7)\qquad \vec{v}(x,0) = \vec{g}_3(x) \qquad x \in (0,1),$$

where

$$(2.8)\qquad \begin{cases} \vec{g}(x,t) = \vec{f}(x,t) - x\left[\vec{f}(1,t) - \vec{f}(0,t)\right] - \vec{f}(0,t) \\ \vec{g}(0,t) = \vec{g}(1,t) = 0 \\ \vec{g}_1(t) = \vec{f}_1(t) - \int_0^t \vec{f}(0,s)ds \quad t \in (0,T) \\ \vec{g}_3(x) = \vec{f}_3(x) \quad x \in (0,1). \end{cases}$$

The condition (2.3) takes the form

$$(2.9)\qquad \vec{v}(1,t) - \vec{v}(\varkappa,t) = \vec{H}(t) \qquad t \in (0,T),$$

where

$$\vec{H}(t) = \vec{h}(t) - (1-\varkappa)\int_0^t \left[\vec{f}(1,s)-\vec{f}(0,s)\right]ds.$$

By Theorem 8 of paper [3] the solution of this problem can be set in the form

$$(2.10)\qquad \vec{v}(x,t) = D\int_0^t M_x(x-1,t-s)\,\vec{g}_2(s)ds + \vec{F}(x,t),$$

where

$$(2.11)\qquad \vec{F}(x,t) = -D\int_0^t M_x(x,t-s)\,\vec{g}_1(s)ds + \frac{1}{2}\int_0^1 \left[M(x-s,t) - M(x+s,t)\right]\vec{g}_3(s)ds + \frac{1}{2}\int_0^t\int_0^1 \left[M(x-s,t-\eta)-M(x+s,t-\eta)\right]g(s,\eta)ds\,d\eta,$$

$$(2.12)\qquad \vec{g}_2(t) = \vec{v}(1,t) \qquad t \in (0,T)$$

provided that the given functions satisfy the respective assumptions.

Hence we may state the following

T h e o r e m 1. If

1^o $\vec{f}_1(t)$ is piecewise of class C^1 for $t \in (0,T)$,

2^o $\vec{h}(t)$ is piecewise of class C^1 for $t \in (0,T)$,

3^o $\vec{f}_3(x)$ is the sum of its Fourier series, $x \in (0,1)$,

4^o $\vec{f}(x,t)$ is of class C^2 in $\bar{Q}$ (hence $\vec{g}(x,t)$ satisfies the assumptions of theorem 8 of paper [2]), then there exists a solution of the problem (N1).

P r o o f . Combining formulas (2.9) and (2.10) we infer that the function $\vec{g}_2(t)$ must satisfy the following system of Volterra integral equations of second kind

$$(2.13)\quad \vec{g}_2(t) = D\int_0^t M_x(x-1,t-s)\,\vec{g}_2(s)ds + \vec{H}(t) + \vec{F}(x,t).$$

It is easy to show that if A is the operator mapping the space $L^2<0,T>$ into itself, defined by the formula

$$(A\,\vec{v})(t) \overset{df}{=} D\int_0^t M_x(x-1,t-s)\vec{v}(s)ds,$$

then

$$\|(A^k\, v)\, t\|_{E_n} \le \frac{C^k\sqrt{T}\,T^{k-1}}{(k-1)!}\|v\|_{L_2} \qquad k = 1,2,\dots$$

for some constant $C > 0$.

It follows that

$$\|A^k\, v\|_{L_2} \le \frac{C^kT^k}{(k-1)!}\|v\|_{L_2} \qquad k = 1,2,\dots$$

that is

$$\|A^k\| \le \frac{C^kT^k}{(k-1)!} \qquad k = 1,2,\dots$$

Hence we conclude that there exists exactly one solution of the equation (2.13) in the space $L^2 \langle 0,T \rangle$ and is given in the form

$$\vec{g}_2(t) = \vec{H}(t) + \vec{F}(\varkappa,t) + \sum_{k=1}^{\infty} A^k(\vec{H}(\cdot) + \vec{F}(\varkappa,\cdot))(t). \tag{2.14}$$

This solution is continuous and piecewise of class C^1, which follows from assumptions 1^0-2^0. By assumptions 3^0-4^0 and theorem 8 of paper [3] it follows that formula (2.10) defines a solution of the problem (N1*). Hence, by virtue of formula (2.4) we get the solution of the proposed problem (N1), Q.E.D.

Consider now the problem (N2):

Find a vector-function $\vec{u}(x,t)$ satisfying the system of equations (*) in the rectangle Q, continuous in its closure $\bar{Q}$, such that

$$\vec{u}_x(0,t) = \vec{f}_1(t) \qquad t \in (0,T) \tag{2.15}$$

$$\vec{u}(x,0) = \vec{f}_3(x) \qquad x \in (0,1) \tag{2.16}$$

$$\vec{u}_x(1,t) - \vec{u}(\varkappa,t)\mu(t) = \vec{h}(t) \qquad (0,1), \tag{2.17}$$

where $\vec{h}(t)$ is a given vector-function and $\mu(t)$ is a given scalar function.

Applying, as in the case of problem (N1), the substitution (2.4) we reduce our problem to the problem (N2*):

$$\frac{\partial \vec{v}}{\partial t} = D\frac{\partial^2 \vec{v}}{\partial x^2} + \vec{g}(x,t) \tag{2.18}$$

with the conditions

$$\vec{v}(0,t) = \vec{g}_1(t) \qquad t \in (0,T) \tag{2.19}$$

$$\vec{v}(x,0) = \vec{g}_3(x) \qquad x \in (0,1), \tag{2.20}$$

where

$$\overline{g}(x,t) = \overline{f}(x,t) - x\left[\overline{f}(1,t) - \overline{f}(0,t)\right] - \overline{f}(0,t)$$

$$\overline{g}(0,t) = \overline{g}_1(1,t) = 0$$

$$\overline{g}_1(t) = \overline{f}_1(t) - \int_0^t \left[\overline{f}(1,s) - \overline{f}(0,s)\right] ds, \qquad t \in (0,T)$$

$$\overline{g}_3(x) = \overline{f}_3(x) \qquad\qquad x \in (0,1).$$

The condition (2.17) takes the form

$$(2.21) \qquad \overline{v}_x(1,t) - \mu(t)\overline{v}(\varkappa,t) = \overline{H}(t),$$

where

$$(2.22) \qquad \overline{H}(t) = \overline{h}(t) - (1 - \mu(t)\varkappa) \int_0^t \left[\overline{f}(1,s) - \overline{f}(0,s)\right] ds + - \mu(t) \int_0^t \overline{f}(0,s) ds.$$

By Theorem 9 of paper [3] the solution of the problem (N2) can be set in the form

$$(2.23) \qquad \overline{v}(x,t) = D \int_0^t M(x-1,t-s)\overline{g}_2(s) ds + \overline{F}(x,t),$$

where

$$(2.24) \qquad \overline{F}(x,t) = -D \int_0^t M(x,t-s)\overline{g}_1(s) ds + $$

$$+ \frac{1}{2} \int_0^1 \left[M(x-s,t) + M(x+s,t)\right] \overline{g}_3(s) ds +$$

$$+ \frac{1}{2} \int_0^t \int_0^1 \left[M(x-s,t-\eta) + M(x+s,t-\eta)\right] \overline{g}(s,\eta) ds\, d\eta,$$

$$\overline{g}_2(t) = \overline{u}_x(1,t), \qquad t \in (0,T)$$

provided that the given functions satisfy the respective assumptions.

Let us now state the

T h e o r e m 2. If

1^0 $\vec{f}_1(t)$ is piecewise of class C^1 for $t \in (0,T)$

2^0 $\vec{h}(t)$ is piecewise of class C^1 for $t \in (0,T)$

3^0 $\mu(t)$ is continuous and bounded for $t \in (0,T)$

4^0 $\vec{f}_3(x)$ is the sum of its Fourier series, $x \in (0,1)$

5^0 $\vec{f}(x,t)$ is of class C^2 in $\overline{Q}$ (hence $\vec{g}(x,t)$ satisfies the assumptions of Theorem 9 of paper [2]),

then there exists a solution of the problem (N2).

P r o o f . Combining formulas (2.21) and (2.23) we find that the function $\vec{g}_2(t)$ must satisfy the equation

$$(2.25)\quad \vec{g}_2(t) = \mu(t)\, D \int_0^t M(\varkappa-1,t-s)\vec{g}_2(s)ds + f(\varkappa,t) + \vec{H}(t).$$

By an argument similar to that of Theorem 1 we easily show that there exists a unique solution $\vec{g}_2(t)$, piecewise of class C^1, of the equation (2.25). This solution defines, by means of formulas (2.23) and (2.24), a solution of the problem (N2).

3. Solution of the problem (N3)

Consider the problem (N3):

Find a vector-function $\vec{u}(x,t)$ satisfying the system of equations (*) and such that

$$(3.1)\quad \vec{u}(0,t) = \lim_{x\to 0^+} \vec{u}(x,t) = \vec{f}_1(t) \quad \text{for} \quad t \in (0,T)$$

$$(3.2)\quad \vec{u}(1,t) = \lim_{x\to 1^-} \vec{u}(x,t) = \vec{f}_2(t) \quad \text{for} \quad t \in (0,T).$$

It is required, moreover, that for a fixed $T_0 \in (0,T)$ the equality

(3.3) $\vec{u}(x,0) - \mu(x)\,\vec{u}(x,T_0) = \vec{h}(x)$ for $x \in (0,1)$

should hold, where $\vec{f}$, $\vec{f}_1$, $\vec{f}_2$, $\vec{h}$ are given vector-functions and $\mu(x)$ is a scalar function such that

(3.4) $$0 < \mu(x) \le 1.$$

Theorem 3. If the eigenvalues of the matrix D are real, then the problem (N3) has at most one solution.

Proof. If $\vec{u}_1(x,t)$ and $\vec{u}_2(x,t)$ were distinct solutions of the problem (N3), then $\vec{u}(x,t) = \vec{u}_1(x,t) - \vec{u}_2(x,t)$ would be a solution of the following homogeneous problem:

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} = D\dfrac{\partial \vec{u}}{\partial x^2}, & (x,t) \in Q \\ \vec{u}(0,t) = \vec{u}(1,t) = 0 \\ \vec{u}(x,0) - \mu(x)\,\vec{u}(x,T_0) = 0. \end{cases}$$

Let K be a real nonsingular matrix such that the matrix $B = K\,D\,K^{-1}$ is of the canonical Jordan form. Setting $K\,\vec{u} = \vec{v}$ we find that $\vec{v}(x,t)$ is a solution of the following problem:

$$\begin{cases} \dfrac{\partial \vec{v}}{\partial t} = B\,\dfrac{\partial^2 v}{\partial x^2} \\ \vec{v}(0,t) = \vec{v}(1,t) = 0 \\ \vec{v}(x,0) - \mu(x)\,\vec{v}(x,T_0) = 0. \end{cases}$$

Considering one block of the matrix B we have

$$\begin{bmatrix} \frac{\partial v_1}{\partial t} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial v_n}{\partial t} \end{bmatrix} = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & & 1 \\ & & & \lambda \\ & & & \end{bmatrix} \begin{bmatrix} \frac{\partial^2 v_1}{\partial x^2} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial^2 v_n}{\partial x^2} \end{bmatrix}$$

If $v_k(x,0) \not\equiv 0$, then it would follow from condition (3.4) that $v_k(x,T_0) \geqslant v_k(x,0)$ for $x \in (0,1)$. But this is in contradiction with the maximum principle applied to the last equation in this block. Therefore $v_k(x,t) \equiv 0$. Substituting $v_k(x,t) \equiv 0$ into the last but one equation in this block, we find, applying once more the maximum principle for the heat equation, that $v_{k-1}(x,t) \equiv 0$. Repeating this argument as many times as necessary we infer that $\vec{v}(x,t) \equiv 0$, whence $\vec{u}(x,t) = K^{-1}\,\vec{v}(x,t) \equiv 0$, which completes the proof of the theorem.

We now turn to the problem of existence of a solution of problem (N3).

Theorem 4. Assume that:

1^0 $\vec{f}(x,t)$ is of class C^2 in $\bar{Q}$, $\vec{f}(0,t) = \vec{f}(1,t) = 0$

2^0 $\vec{f}_1(t)$, $\vec{f}_2(t)$ are piecewise of class C^2 for $t \in (0,T)$,

3^0 $\vec{h}(x)$ is the sum of its Fourier trigonometric series,

4^0 $\mu(x)$ is continuous and of bounded variation.

Then the problem (N3) has a solution.

Proof. From Theorem 8 of paper [3] it follows that if there exists a solution of the problem (N3) such that $\vec{u}(x,0) = \vec{f}_3(x)$ is the sum of its Fourier trigonometric series and if the given functions $\vec{f}$, $\vec{f}_1$, $\vec{f}_2$ satisfy assumptions 1^0-2^0, then this solution may be represented in the form

(3.5) $$\vec{u}(x,t) = \vec{u}_1(x,t) + \vec{u}_2(x,t) + \vec{u}_3(x,t) + \vec{u}_4(x,t),$$

where $\vec{u}_i(x,t)$, $i=1,2,3,4$, are defined by formulas (1.3)-(1.6) for $\vec{g} = \vec{f}$, $\vec{g}_i = \vec{f}_i$, $i=1,2,3$. From the nonlocal condition (3.3) we infer that $\vec{f}_3(x)$ satisfies the following system of Fredholm integral equations of second kind:

$$(3.6)\quad \vec{f}_3(x) = \frac{1}{2}\mu(x)\int_0^1 \left[M(x-s,T_0)-M(x+s,T_0)\right]\vec{f}_3(s)ds+\vec{F}(x,T_0),$$

where

$$(3.7)\quad \vec{F}(x,t) = \mu(x)\left[\vec{u}_1(x,t)+\vec{u}_2(x,t)+\vec{u}_4(x,t)\right] + \vec{h}(x).$$

Formula (1.1) implies that

$$M(x-s,T_0)-M(x+s,T_0) = 4\sum_{k=1}^{\infty} \exp\left[-k^2\pi^2T_0D\right]\sin k\pi x\cdot\sin k\pi s$$

thus, the kernel of the system of equations (3.6) is continuous and bounded. For the system of equations (3.6) to have a solution in the space of vector-functions continuous for $x \in \langle 0,1\rangle$ it is necessary and sufficient that the system of equations

$$(3.8)\quad \vec{p}(x) = \frac{1}{2}\mu(x)\int_0^1 \left[M(x-s,T_0) - M(x+s,T_0)\right]\vec{p}(s)ds$$

should have only a null solution.

In fact, assume that $\vec{p}(x)$ is an identically nonzero solution of the system of equations (3.8). Then the formula

$$\vec{u}(x,t) = \frac{1}{2}\int_0^1 \left[M(x-s,t) - M(x+s,t)\right]\vec{p}(s)ds$$

defines an identically nonzero solution of the problem (N3), where $\vec{f} = 0$, $\vec{f}_1 = 0$, $\vec{f}_2 = 0$, $\vec{h} = 0$. This follows from equality (3.8) and from the properties of the matrix function $M(x,t)$. However, this is in contradiction with the above established Theorem 3. This means that the unique solution of the system of equations (3.8) is the zero solution. This proves, too, that there exists a unique continuous solution $\overline{f}_3(x)$ of the system (3.6) for arbitrary given functions $\overline{f}$, $\overline{f}_1$, $\overline{f}_2$, $\overline{h}$. It is easy to see that, under the adopted assumptions, this solution determines a solution of the problem (N3). Taking into account (3.6), the uniform convergence of the series

$$\frac{1}{2}\int_0^1 \left[M(x-s,T_0) - M(x+s,T_0)\right] \overline{f}_3(s)ds =$$

$$= 2\sum_{k=1}^{\infty} \exp\left[-k^2\pi^2 T_0 D\right] \sin k\pi x \int_0^1 \overline{f}_3(s) \sin k\pi s\, ds,$$

as well as the properties of the integrals $\overline{u}_1(x,t)$, $\overline{u}_2(x,t)$, $\overline{u}_4(x,t)$ established by the author in paper [3], and the assumption 4°, we infer that $\overline{f}_3(x)$ is for $x \in \langle 0,1 \rangle$ the sum of its Fourier trigonometric series. Hence formula (3.5) represents the solution of the problem (N3).

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