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ON DIFFERENTIABILITY OF THE SOLUTION OF SOME FUNCTIONAL EQUATION OF THE DYNAMIC PROGRAMMING

1. Let us consider the functional equation of the dynamic programming (see [1])

$$(1)\quad f(x) = \max_{0<y<x}\left[g(y) + h(x-y) + f((a-b)y + bx)\right], \quad f(0) = 0$$

where g and h are the given functions, a and b are the given numbers, f is the unknown function.

We assume that

(a) g and h are functions defined and continuous on the interval $[0,+\infty)$

(b) $g(0) = h(0) = 0$

(c) $a,b \in (0,1)$

(d) $\sum_{n=0}^{\infty} m(c^n x) < +\infty$, where $m(x) = \max_{0\le y\le x} \max\left\{|g(y)|, |h(y)|\right\}$, $c = \max(a,b)$.

With the equation (1) there is connected the sequence of approximations

$$(2)\begin{cases} f_1(x) = \max_{0\le y\le x}\left[g(y) + h(x-y)\right] \\ f_n(x) = \max_{0\le y\le x}\left[g(y) + h(x-y) + f_{n-1}((a-b)y + bx)\right] \quad n=2,3,\dots \end{cases}$$

Let $F_n : [0,+\infty) \to 2^{[0,+\infty)}$ $(n = 0,1,2,\dots)$ be point-to-set transformations given by the formulas

$$(3)\begin{cases} F_0(x) = \{y \in [0,x] \mid f(x) = g(y) + h(x-y) + f[(a-b)y + bx]\} \\ F_1(x) = \{y \in [0,x] \mid f_1(x) = g(y) + h(x-y)\} \\ F_n(x) = \{y \in [0,x] \mid f_n(x) = g(y)+h(x-y)+f_{n-1}[(a-b)y+bx]\} \end{cases}$$
$n=2,3\dots$

In this paper we shall deal with problems connected with differentiability of the solution of the equation (1) in the case when the sets given by formulas (3) are one-element sets for every $x \geq 0$.

2. Let $X = Y = R^1$.

We shall cite some definitions and theorems necessary in this paper.

Definition 1. A point-to-set transformation $F: X \to 2^Y$ is called upper semi-continuous at a point $x_0 \in X$ if the fact that $\{x_n\} \subset X$, $x_n \to x_0$, $y_n \in F(x_n)$ implies the existence of a subsequence $\{y_n\} \subset Y$ convergent to some $y_0 \in F(x_0)$.

Definition 2. A point-to-set transformation $F: X \to 2^Y$ is called lower semi-continuous at a point $x_0 \in X$ if for every sequence $\{x_n\} \subset X$, $x_n \to x_0$ and for every $y_0 \in F(x_0)$ there exists a sequence $\{y_n\} \subset Y$ such that $y_n \in F(x_n)$ and $y_n \to y_0$.

Definition 3. A point-to-set transformation $F: X \to 2^Y$ lower semi-continuous and upper semi-continuous at a point $x_0 \in X$ will be called continuous at the point x_0.

Theorem 1. If a transformation $F: X \to 2^Y$ is continuous and a function $\rho: G_F \to R^1$ is continuous, then the transformation $\tilde{F}: X \to 2^Y$ given by the formula $\tilde{F}(x) = \{y \in F(x) \mid p(x,y) = \max_{z \in F(x)} p(x,z)\}$ is upper semi-continuous (where $G_F = \{(x,y) \in X \times Y \mid x \in X,\ y \in F(x)\}$ is the graph of a transformation F).

T h e o r e m 2. If functions g and h fulfil the assumptions (a) - (d) and have derivatives at a point zero, then the function f defined by (1) has a derivative at the point zero and

$$f'(0) = \max\left[\frac{g'(0)}{1-a}, \frac{h'(0)}{1-b}\right].$$

T h e o r e m 3. Let the functions g and h fulfil the assumptions (a) - (d) and let they be continuously differentiable on $[0,+\infty)$. Then for $x > 0$ there exist derivatives in the directions 1 and -1 of the terms of the sequence (2) and

$$(4)\quad \begin{cases} f_1'(x;1) = \max\limits_{y\in F_1(x)} \left\{\frac{y}{x} g'(y) + \frac{x-y}{x} h'(x-y)\right\} \\ f_1'(x;-1) = -\min\limits_{y\in F_1(x)} \left\{\frac{y}{x} g'(y) + \frac{x-y}{x} h'(x-y)\right\} \\ f_n'(x;1) = \max\limits_{y\in F_n(x)} \left\{\frac{y}{x} g'(y) + \frac{x-y}{x} h'(x-y) + \left[(a-b)\frac{y}{x} + b\right] f_{n-1}'\left[(a-b)y+bx;1\right]\right\} \\ f_n'(x;-1) = -\min\limits_{y\in F_n(x)} \left\{\frac{y}{x} g'(y) + \frac{x-y}{x} h'(x-y) + \left[(a-b)\frac{y}{x} + b\right] f_{n-1}'\left[(a-b)y+bx;-1\right]\right\}, \end{cases}$$

where the transformations $F_n : [0,+\infty) \to 2^{[0,+\infty)}$ $(n=1,2,\ldots)$ there are given by formulas (3).

Definition 1 can be found in [2] and [3], Definition 2 in [4] and Definition 3 in [3]. Theorems 1, 2, 3 are proved in [3], [5], [6], respectively.

3. From Theorem 3 we obtain

C o r o l l a r y 1. Let the functions g and h fulfil the assumptions (a) - (d), let them be continuously differentiable on $[0,+\infty)$ and let the sets images $F_n(x)$ $(n = 0,1,2,\ldots)$ given by formulas (3) be one-element sets for $x > 0$. Then for $x > 0$ there exist derivatives of terms of the sequence (2) and

$$(5)\quad \begin{cases} f_1'(x) = \max\left[g'(y_1(x)),\ h'(x-y_1(x))\right] \\ f_n'(x) = \max\left\{g'(y_n(x)) + af_{n-1}'\left[(a-b)y_n(x) + bx\right],\right. \\ \qquad\qquad \left. h'(x-y_n(x)) + bf_{n-1}'\left[(a-b)y_n(x) + bx\right]\right\}, \end{cases}$$

where $\{y_n(x)\} = F_n(x)$ $n = 1,2,\ldots$

P r o o f . According to Theorem 3 (see note 2 in [6]) the derivatives of terms of the sequence (2) for $x > 0$ will be given by formulas

$$(6)\quad \begin{cases} f_1'(x) = \dfrac{y_1(x)}{x} g'(y_1(x)) + \dfrac{x-y_1(x)}{x} h'(x-y_1(x)) \\ f_n'(x) = \dfrac{y_n(x)}{x} g'(y_n(x)) + \dfrac{x-y_n(x)}{x} h'(x-y_n(x)) + \\ \qquad + \left[(a-b)\dfrac{y_n(x)}{x} + b\right] f_{n-1}'\left[(a-b)y_n(x)+bx\right] \quad n=2,3,\ldots \end{cases}$$

Taking into account the differentiability of g and h we obtain f_1

$y_1(x) = 0$ and then $g'(y_1(x)) - h'(x-y_1(x)) \le 0$,

$0 < y_1(x) < x$ and then $g'(y_1(x)) - h'(x-y_1(x)) = 0$,

$y_1(x) = x$ and then $g'(y_1(x)) - h'(x-y_1(x)) \ge 0$,

i.e. $f_1'(x) = \max\left[g'(y_1(x)),\ h'(x-y_1(x))\right]$.

Analogously in the case f_n' $(n = 2,3,\ldots)$ we have $y_n(x) = 0$ and then

$$g'(y_n(x)) - h'(x-y_n(x)) + (a-b)f_{n-1}'\left[(a-b)y_n(x)+bx\right] \le 0,$$

$0 < y_n(x) < x$ and then

$$g'(y_n(x)) - h'(x-y_n(x)) + (a-b)f_{n-1}'\left[(a-b)y_n(x)+bx\right] = 0,$$

$y_n(x) = x$ and then

$$g'(y_n(x)) - h'(x-y_n(x)) + (a-b)f'_{n-1}\left[(a-b)y_n(x)+bx\right] \geq 0,$$

i.e. for $n = 2,3,\ldots$

$$f'_n(x) = \max\Big\{g'(y_n(x)) + af'_{n-1}\left[(a-b)y_n(x)+bx\right],\ h'(x-y_n(x)) + bf'_{n-1}\left[(a-b)y_n(x)+bx\right]\Big\}.$$

In the case when the transformations given by formulas (3) are one-elements for $x > 0$ we shall use the denotation $F_n(x) = \{y_n(x)\}$ $(n = 0,1,2,\ldots)$ where $y_n\colon [0,+\infty) \to [0,+\infty)$. In this case the upper semi-continuity of the transformations F_n coincides with their continuity and hence it follows that the functions $y_n(x)$ $(n = 0,1,2,\ldots)$ are continuous on $[0,+\infty)$. The following lemma holds.

L e m m a 1. Let the sets given by formulas (3) be one-element sets for $x \geq 0$. Then the sequence $\{y_n(x)\}$ $(n = 1,2,\ldots)$ is uniformly convergent to $y_0(x)$ on every interval $[0,\bar{x}]$, $\bar{x} > 0$.

P r o o f . Let $\{x_n\} \subset [0,\bar{x}]$, $x_n \to x_0$. In view of the compactness of the set $[0,\bar{x}]$ and by the definition of the sequence $\{y_n(x)\}$ $(n = 1,2,\ldots)$ there follows the existence of a subsequence $\{y_{n_k}(x_{n_k})\}$ of a sequence $\{y_n(x)\}$ convergent to some $\bar{y} \in [0,\bar{x}]$.

By virtue of the continuous convergence of a sequence of the continuous functions $\{f_n(x)\}$ to the continuous function $f(x)$ we have $\lim f_{n_k}(x_{n_k}) = f(x_0)$ i.e. $f(x_0) = g(y) + h(x_0-y) + f\left[(a-b)y + bx_0\right]$ and hence we obtain $\{\bar{y}\} = \{y_0(x_0)\} = F_0(x_0)$. Taking into account that $F_0(x_0)$ is the one-element set we obtain $y_n(x_n) \to y_0(x_0)$. Thus we have proved the continuous convergence $\{y_n(x)\}$ on $[0,\bar{x}]$.

Now we shall prove

L e m m a 2. If the assumptions of Corollary 1 are fulfilled, there exists a number $L > 0$ such that then for every sequence $\{x_n\} \subset [0,\bar{x}]$ (where $\bar{x} > 0$ is arbitrary and fixed) is $|f'_n(x_n)| \leq L$.

P r o o f . If $x = 0$ then (see [5]) the sequence $\{f_n'(0)\}$ is convergent and bounded. Let $\bar{x} > 0$ and let $\{x_n\} \subset [0, \bar{x}]$ be an arbitrary and fixed sequence. Let us denote

$$M = \max_{0 \le x \le \bar{x}} \max \left\{ \max_{0 \le y \le x} |h'(x-y)|, \max_{0 \le y \le x} |g'(y)| \right\} .$$

We shall show by induction that

$$|f_n'(x_n)| \le (1+c+\ldots+c^{n-1})M \quad \text{for every } n.$$

For n = 1 we have

$$|f_1'(x_1)| \le \max \left\{ |g'(y_1(x_1))|, |h'(x_1 - y_1(x_1))| \right\} \le M.$$

Let us assume that

$$|f_k'(x_k)| \le (1+c+\ldots+c^{k-1})M, \quad \text{where } k \ge 1.$$

We shall prove the following inequality

$$|f_{k+1}'(x_{k+1})| \le (1+c+\ldots+c^k) M.$$

Indeed

$$f_{k+1}'(x_{k+1})| \le \max \Big\{ \big|g'(y_{k+1}(x_{k+1})) + af_k'[(a-b)y_{k+1}(x_{k+1}) + bx_{k+1}]\big|,$$
$$\big|h'(x_{k+1} - y_{k+1}(x_{k+1})) + bf_k'[(a-b)y_{k+1}(x_{k+1}) + bx_{k+1}]\big| \Big\} \le$$

$$\le \max \left\{ |g'(y_{k+1}(x_{k+1}))|, |h'(x_{k+1} - y_{k+1}(x_{k+1}))| \right\} +$$

$$+ c\,|f_k'[(a-b)y_{k+1}(x_{k+1}) + bx_{k+1}]| \le M + c(1+c+\ldots+c^{k-1}) M,$$

where the last inequality follows by the arbitrariness of the sequence $\{x_m\}$. Thus for an arbitrary sequence $\{x_n\} \subset [0,\bar{x}]$

$$|f_n'(x_n)| \leq (1+c+\ldots+c^{n-1})\, M \leq \frac{1}{1-c}\, M.$$

Hence, if we shall denote $L = \frac{1}{1-c} M$ we shall obtain the thesis.

Now we shall prove five theorems.

T h e o r e m 4. If the assumptions of Corollary 1 are fulfilled, then the sequence $\{f_n'(x)\}$ is convergent in any interval $[0,\bar{x}]$, $\bar{x} > 0$.

P r o o f . In view of Lemma 2 there follows the existence of finite limits $A_0 = \overline{\lim_{n\to\infty}}\, f_n'(x)$ and $B_0 = \underline{\lim_{n\to\infty}}\, f_n'(x)$ for $x \in [0,\bar{x}]$. Let $K_0 = \{n_k\}$ and $S_0 = \{n_s\}$ be these subsequences of natural numbers for which $A_0 = \lim f_{n_k}'(x)$ and $B_0 = \lim f_{n_s}'(x)$. By virtue of the fact that for $x > 0$ ($\{f_n'(0)\}$ is convergent, see [5]) we have

$$f_n'(x) = \frac{y_n(x)}{x}\, g'(y_n(x)) + \frac{x - y_n(x)}{x}\, h'(x - y_n(x)) +$$

$$+ \left[(a-b)\frac{y_n(x)}{x} + b\right] f_{n-1}'\left[(a-b)y_n(x) + bx\right]$$

by the continuity of g', h' and by the uniform convergence of $\{y_n(x)\}$ to $y_0(x)$ there follows the existence of the limit

$$A_1 = \lim_{n_k\to\infty} f_{n_k-1}'\left[(a-b)y_{n_k}(x)+bx\right].$$

Analogously one can show the existence of the limit

$$B_1 = \lim_{n_s\to\infty} f_{n_s-1}'\left[(a-b)y_{n_s}(x)+bx\right].$$

Since

$$f_n'(x) = \max\Big\{g'(y_n(x)) + af_{n-1}'\big[(a-b)y_n(x) + bx\big],$$
$$h'(x - y_n(x)) + bf_{n-1}'\big[(a-b)y_n(x) + bx\big]\Big\}$$

then

$$A_0 = \max\big[g'(y_0(x)) + aA_1,\ h'(x - y_0(x)) + bA_1\big]$$

and

$$B_0 = \max\big[g'(y_0(x)) + aB_1,\ h'(x - y_0(x)) + bB_1\big].$$

Hence it follows that the following alternative is true

$$A_0 = g'(y_0(x)) + aA_1 \quad \text{and} \quad B_0 \geq g'(y_0(x)) + aB_1$$

or

$$A_0 = h'(x-y_0(x)) + bA_1, \text{ and} \quad B_0 \geq h'(x-y_0(x)) + bB_1.$$

From this alternative the following one follows

$$A_0 - B_0 \leq a(A_1-B_1) \qquad \text{or} \quad A_0 - B_0 \leq b(A_1-B_1)$$

and hence we obtain the inequality

$$A_0 - B_0 \leq \max\big[a(A_1 - B_1),\ b(A_1 - B_1)\big].$$

Analogously we obtain the inequality

$$A_0 - B_0 \geq -\max\big[a(A_1 - B_1),\ b(A_1 - B_1)\big].$$

This and the previous inequalities can be written in the form

$$|A_0 - B_0| \leq \max\big\{a|A_1 - B_1|,\ b|A_1 - B_1|\big\} \leq c|A_1 - B_1|.$$

Taking into account the existence of limits A_1 and B_1 we can analogously prove the existence of limits

$$A_2 = \lim f'_{n_k-2}\left[(a-b)y_{n_k-1}(x_1^A) + bx_1^A\right]$$

and

$$B_2 = \lim f'_{n_s-2}\left[(a-b)y_{n_s-1}(x_1^B) + bx_1^B\right],$$

where $x_1^A = (a-b)y_{n_k}(x) + bx$ and $x_1^B = (a-b)y_{n_s}(x) + bx$. Hence we obtain the inequality

$$|A_o - B_o| \le c^2|A_2 - B_2|.$$

By iteration we have the inequality

$$|A_o - B_o| \le c^p|A_p - B_p|,$$

where

$$A_p = \lim f'_{n_k-p}\left[(a-b)y_{n_k-p+1}(x_{p-1}^A) + bx_{p-1}^A\right],$$

$$B_p = \lim f'_{n_s-p}\left[(a-b)y_{n_s-p+1}(x_{p-1}^B) + bx_{p-1}^B\right],$$

$$x_{p-1}^A = (a-b)y_{n_k-p+2}(x_{p-2}^A) + bx_{p-2}^A,$$

$$x_{p-1}^B = (a-b)y_{n_s-p+2}(x_{p-2}^B) + bx_{p-2}^B.$$

From Lemma 2 there follows the existence of a number $L > 0$ such that $|A_p - B_p| \le L$ for $x \in [0,\bar{x}]$ and for every p, i.e. $|A_o - B_o| \le c^pL$.

Thus for every $\varepsilon > 0$ there exists p_o that for $p > p_o$ $|A_o - B_o| < \varepsilon$ and hence $A_o = B_o$.

T h e o r e m 5. Let the functions g and h fulfil the assumptions (a) - (d) and let them be differentiable on $[0,+\infty)$. If the equation

(7) $$G(x) = \max\left\{g'(s(x)) + aG\left[(a-b)s(x) + bx\right],\ h'(x-s(x)) + bG\left[(a-b)s(x) + bx\right]\right\}$$

for $x > 0$ and for the fixed function $s(x)$ such that $0 \le s(x) \le x$ has a solution in the class G of bounded functions on every interval $[0,\bar{x}]$, $\bar{x} > 0$, then this solution is unique.

P r o o f . Let G and $\bar{G}$ be solutions of the equation (7). The following alternative is true

$$G(x) = g'(s(x)) + aG\left[(a-b)s(x) + bx\right]$$

and

$$\bar{G}(x) \ge g'(s(x)) + a\bar{G}\left[(a-b)s(x) + bx\right]$$

or

$$G(x) = h'(x-s(x)) + bG\left[(a-b)s(x) + bx\right]$$

and

$$\bar{G}(x) \ge h'(x-s(x)) + b\bar{G}\left[(a-b)s(x) + bx\right]$$

i.e.

(8) $$G(x)-\bar{G}(x) \le \max\left\{a\left[G((a-b)s(x)+bx) - \bar{G}((a-b)s(x)+bx)\right],\ b\left[G((a-b)s(x)+bx) - \bar{G}((a-b)s(x)+bx)\right]\right\}.$$

On the other hand by analogous reasoning, we have

(9) $$G(x)-\bar{G}(x) \ge -\max\left\{a\left[G((a-b)s(x)+bx) - \bar{G}((a-b)s(x)+bx)\right],\ b\left[G((a-b)s(x)+bx) - \bar{G}((a-b)s(x)+bx)\right]\right\}.$$

In view of the inequalities (8) and (9) we have

(10) $|G(x)-\bar{G}(x)| \leq \{\max a|G((a-b)s(x)+bx)-\bar{G}((a-b)s(x)+bx)|,$

$b|G((a-b)s(x)+bx)-\bar{G}((a-b)s(x)+bx)|\} \leq$

$\leq c \max|G((a-b)s(x)+bx)-\bar{G}((a-b)s(x)+bx)|.$

Let us denote $u(x) = \sup_{0 \leq z \leq x} |G(z) - \bar{G}(z)|$. Then by (10) we have $|G(x) - \bar{G}(x)| \leq c\,u(cx)$.

By iterations we shall obtain the inequality

$$|G(x) - \bar{G}(x)| \leq c^n u(c^n x).$$

If in the last inequality we pass to the limit with $n \to \infty$, we shall obtain for every $x \in [0,\bar{x}]$ the equality $G(x) = \bar{G}(x)$.

T h e o r e m 6. If a function f being the solution of the equation (1) has the continuous derivative, then at a point $x_0 > 0$ this derivative is given by the formula

(11) $f'(x_0) = \max[g'(s(x_0)) + af'((a-b)s(x_0) + bx_0),$

$h'(x_0-s(x_0)) + bf'((a-b)s(x_0) + bx_0)],$

where s is the arbitrary selector for F_0 (given by formula (3)) and g, h fulfil the assumptions (a) - (d) and are differentiable on $[0,+\infty)$.

P r o o f . Let s be an arbitrary fixed selector for F_0, i.e. $s : [0,+\infty) \to [0,+\infty)$ and $s(x) \in F_0(x)$ for every $x \geq 0$ (where F_0 is given by formula (3)).

Then s_1 defined by formula $s_1(x) = \frac{s(x_0)}{x_0} x$ $(x_0 > 0)$ will be a selector for the transformation F such that $F(x) = [0,x]$ for $x \geq 0$.

Let us denote $T(f,y) = g(y) + h(x-y) + f[(a-b)y+bx]$.

Since s_1 is a selector for F but not necessarily a selector for F_0, we have

$$\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} \geq \frac{T\left[f, \frac{s(x_0)}{x_0}(x_0+\Delta x)\right] - T\left[f, s(x_0)\right]}{\Delta x}$$

when $\Delta x > 0$ and

$$\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} \leq \frac{T\left[f, \frac{s(x_0)}{x_0}(x_0+\Delta x)\right] - T\left[f, s(x_0)\right]}{\Delta x}$$

when $\Delta x < 0$.

Tending to the limit in above inequalities when $\Delta x \to 0^+$ and $\Delta x \to 0^-$ respectively, we obtain the inequalities

$$f'_+(x_0) \geq \frac{s(x_0)}{x_0} g'(s(x_0)) + \frac{x_0 - s(x_0)}{x_0} h'(x_0 - s(x_0)) + \left[(a-b)\frac{s(x_0)}{x_0} + b\right] f'_+\left[(a-b)s(x_0) + bx_0\right],$$

$$f'_-(x_0) \leq \frac{s(x_0)}{x_0} g'(s(x_0)) + \frac{x_0 - s(x_0)}{x_0} h'(x_0 - s(x_0)) + \left[(a-b)\frac{s(x_0)}{x_0} + b\right] f'_-\left[(a-b)s(x_0) + bx_0\right].$$

By the arbitrariness of a selector and by the continuity of a derivative we have

$$\max_{y \in F_0(x_0)} \left\{\frac{y}{x_0} g'(y) + \frac{x_0-y}{x_0} h'(x_0-y) + \left[(a-b)\frac{y}{x_0} + b\right] f'\left[(a-b)y+bx_0\right]\right\} =$$

$$= \min_{y \in F_0(x_0)} \left\{\frac{y}{x_0} g'(y) + \frac{x_0-y}{x_0} h'(x_0-y) + \left[(a-b)\frac{y}{x_0} + b\right] f'\left[(a-b)y+bx_0\right]\right\}.$$

Since the functions g, h and f have the derivatives for $x > 0$, then depending on $s(x_0)$ we have the following cases:

$s(x_0) = 0$ and then $h'(x_0)+bf'(bx_0) \geq g'(0) + af'(bx_0)$,

$s(x_0) = x_0$ and then $g'(x_0)+af'(ax_0) \leq h'(0) + bf'(ax_0)$,

$0 < s(x_0) < x_0$ and then $g'(s(x_0)) + af'\left[(a-b)s(x_0) + bx_0\right] =$

$= h'(x_0-s(x_0)) + bf'\left[(a-b)s(x_0) + bx_0\right]$,

i.e. $f'(x_0) = \max\Big\{g'(s(x_0)) + af'\left[(a-b)s(x_0) + bx_0\right],$

$h'(x_0-s(x_0)) + bf'\left[(a-b)s(x_0) + bx_0\right]\Big\}$.

T h e o r e m 7. Let the assumptions of Corollary 1 be fulfilled. If there exists a derivative of the solution of the equation (1), then

$$(12) \quad f'(x) = \max\Big\{g'(y_0(x)) + af'\left[(a-b)y_0(x)+bx\right], \; h'(x-y_0(x)) + bf'\left[(a-b)y_0(x)+bx\right]\Big\}$$

and when this derivative is bounded then it is a limit of the convergent sequence $\{f_n'(x)\}$.

P r o o f . Since for $x \geq 0$ $F_0(x)$ is one-element set and there exists the derivative f, by Theorem 6 we obtain its form. Let $x_0 > 0$ be an arbitrary fixed point.

Let us consider the cases:

(α) $\quad 0 < y_0(x_0) < x_0$,

(β) $\quad y_0(x_0) = x_0$,

(γ) $\quad y_0(x_0) = 0$.

By the uniformly convergence of $\{y_n(x)\}$ to $y_0(x)$ and by the continuity of $y_n(x)$ $(n=0,1,2,\dots)$ in the case (α) there follows the existence of the neighbourhood $O(x_0)$ of a point x_0 and the existence of a number n_0 such that for $n > n_0$ and $x \in O(x_0)$, $0 < y_n(x) < x$. Hence for $x \in O(x_0)$ and $n > n_0$

$$f_n'(x) = \frac{ah'(x-y_n(x)) - bg'(y_n(x))}{a-b} .$$

Hence there follows the uniformly convergence $\{f_n'(x)\}$ on every closed interval contained in $O(x_0)$. Thus $\lim_{n\to\infty} f_n'(x_0) = f'(x_0)$.

Let us consider the case (β). On account of Theorem 4 there follows the existence of the limit $\lim_{n\to\infty} f_n'(x_0) = A$, whence by (6) there follows the existence of the limit $\lim_{n\to\infty} f_{n-1}'\left[(a-b)y_n(x_0) + bx_0\right] = B$ and

$$A = \max\left[g'(x_0) + aB,\ h'(0) + bB\right].$$

Since we have assumed the existence of the derivative of a function f, then according to Theorem 6 in the neighbourhood of a point x_0 this derivative is given by the formula

$$f'(x) = \max\Big\{g'(y_0(x)) + af'\left[(a-b)y_0(x) + bx\right],\ h'(x-y_0(x)) + bf'\left[(a-b)y_0(x) + bx\right]\Big\}.$$

At a point x_0 we have

$$f'(x_0) = \max\left[g'(x_0) + af'(ax_0),\ h'(0) + bf'(ax_0)\right].$$

Let us denote

$$G(x) = \begin{cases} f'(x) & \text{for } x \neq x_0 \text{ and } x \neq ax_0 \\ A & \text{for } x = x_0 \\ B & \text{for } x = ax_0. \end{cases}$$

This function fulfils the equation

$$G(x) = \max\Big\{g'(y_0(x)) + aG\left[(a-b)y_0(x) + bx\right],\ h'(x-y_0(x)) + bG\left[(a-b)y_0(x) + bx\right]\Big\}.$$

By the uniqueness of the solution of the above equation and by the existence of the derivative of the function f we obtain $A = f'(x_0)$ and $B = f'(ax_0)$ whence $\lim_{n\to\infty} f_n'(x_0) =$

$= f'(x_0)$. Analogously as in the case (β) one can prove that in the case (γ) the theorem also is true.

Theorem 8. Let the assumptions of Corollary 1 be fulfilled. If there exists a continuous derivative of the function f given by the equation (1) and if

$$\frac{g'(0)}{1-a} > \frac{h'(0)}{1-b} \tag{13}$$

then there exists $\bar{x} > 0$ such that for $x \in [0,\bar{x}]$ it is $y_0(x) = x$.

Proof. Let us suppose the contrary. Therefore one of the following cases must holds:

(i) there exists $\bar{x} > 0$ such that for $x \in [0,\bar{x}]$ it is $y_0(x) = 0$

(ii) $\inf\{x \in [0,\bar{x}] \mid 0 < y_0(x) < x\} = 0$, $\bar{x} > 0$.

In the case (i) we have $f(x) = h(x) + f(bx)$ for $x \in [0,\bar{x}]$. By iterations we have $f(x) = \sum_{n=0}^{\infty} h(b^n x)$ for $x \in [0,\bar{x}]$. The series of the derivatives $\sum_{n=0}^{\infty} b^n h'(b^n x)$ is uniformly convergent in the interval $[0,\bar{x}]$. Hence the derivative f' of the solution f is given by formula $f'(x) = \sum_{n=0}^{\infty} b^n h'(b^n x)$ for $x \in [0,\bar{x}]$. By continuity of f' we have $f'(0) = \lim_{x \to 0+} f'(x)$. Hence $\lim_{x \to 0+} f'(x) = \sum_{n=0}^{\infty} b^n h'(0) = \frac{h'(0)}{1-b}$.

On the other hand by Theorem 2 we have $f'(0) = \frac{g'(0)}{1-a}$ contrary to the continuity of a derivative at the point zero.

Let us assume now that the case (ii) is true. Let

$$x_0 \in \{x \in [0,\bar{x}] \mid 0 < y_0(x) < x\} \quad (\bar{x} > 0),$$

i.e. $0 < y_0(x_0) < x_0$.

According to the uniformly convergence of $\{y_n(x)\}$ to $y_0(x)$ there follows the existence of a neighbourhood $O(x_0)$ of the point x_0 such that for $x \in O(x_0)$ and n greater than some n_0 we have $0 < y_n(x) < x$.

Then for $x \in O(x_0)$ and $n > n_0$ we have

$$f'_n(x) = \frac{ah'(x - y_n(x)) - bg'(y_n(x))}{a-b} .$$

Hence by the continuity of g', h' and f' and by the uni-from convergence of $\{y_n(x)\}$ to $y_0(x)$ it follows that for $x \in \overline{O(x_0)}$ ($\overline{O(x_0)}$ is the closure of $O(x_0)$) we have

$$f'(x) = \frac{ah'(x-y_0(x)) - bg'(y_0(x))}{a-b} . \tag{14}$$

Let $\{x_n\}$ be an arbitrary sequence such that $x_n \to 0$,

$$\{x_n\} \subset \{x \in [0,\bar{x}] \mid 0 < y_0(x) < x\} .$$

From the continuity of the derivative f' at the point zero it follows that $\lim_{n\to\infty} f'(x_n) = f'(0)$. According to Theorem 2

$$f'(0) = \frac{g'(0)}{1-a} .$$

Hence and by (14) we have

$$\frac{ah'(0) - bg'(0)}{a-b} = \frac{g'(0)}{1-a}$$

contrary to the inequality (13).

N o t i c e 1. If in Theorem 8 the inequality (13) is replaced by the inequality $\frac{g'(0)}{1-a} < \frac{h'(0)}{1-b}$ without any change of the remaining assumptions, then one can prove the existence of $\bar{x} > 0$ such that for $x \in [0,\bar{x}]$ it is $y_0(x) = 0$

N o t i c e 2. If the assumptions of Corollary 1 are fulfilled, then by Jegorov's theorem (see [7]) we can prove that the sequence $\{f'_n(x)\}$ is almost uniformly convergent on $[0,\bar{x}]$, $\bar{x} > 0$ (the sense of this convergence is following: if in the set $[0,\bar{x}]$ one omits all the points enclosed in correspondingly selected open intervals with arbitrarily small overall length then the sequence $\{f'_n(x)\}$ is uniformly convergent on the remaining set). Hence it follows that in the interval $[0,\bar{x}]$ except the open set with arbitrarily small overall length there exists the continuous derivative of the function f being the limit of the uniformly convergent sequence $\{f'_n(x)\}$.

N o t i c e 3. The case considered in this paper is more general than the one considered in [8] where the author have assumed that the functions g and h are strictly convex on $[0,+\infty)$. If one assumes the strict convexity of the functions g and h, then the transformations given by formulas (3) have one-element sets images for $x \geq 0$.

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