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A DECOMPOSITION OF ELEMENTS IN A GROUP AND ITS APPLICATIONS

In this note we prove a theorem concerning the decomposition of elements in a group into a product of n elements from a normal subset and provide some of its applications in the study of properties of free products of groups. We will also introduce the concept of quasisimple groups.

We shall use the following notations:

A group G generated by elements $a_1,a_2,\ldots$ and with relations $R_1,R_2,\ldots$ will be denoted by

$$G = \langle a_1,a_2,\ldots ; R_1,R_2,\ldots\rangle .$$

By C_j we mean the class of conjugate elements of G, by x_j the representant of C_j and by $A = BUB^{-1}$ the normal set of G i.e. invariant with respect to inner automorphisms.

Theorem 1. If $A = BUB^{-1}$ is a normal subset of a group G, then the following conditions are equivalent:
(i) an element $g\in G$ is a product of n elements from A,
(ii) there are elements $g_1,g_2,\ldots,g_n \in G$ such that $g = g_1g_2 \ldots g_n$ and there is a system of representants x_{i_j} of classes C_{i_j} from A such that

(1) $$g_1x_{i_1}g_2 \ldots x_{i_{n-1}}g_n \in A.$$

Theorem 1 results from the following two lemmas.

L e m m a 1. If $A = BUB^{-1}$ is a normal subset of a group G then the following conditions are equivalent:
(i) the element $g \in G$ is a product of n elements from A,
(ii) there are elements $g_1, g_2, \ldots, g_n \in G$ such that $g = g_1 g_2 \ldots g_n$ and there is a system of elements $a_1, a_2, \ldots, a_{n-1} \in A$ such that $g_1 a_1 g_2 \ldots a_{n-1} g_n \in A$.

P r o o f . Let $g = a_1 a_2 \ldots a_n$, $a_i \in A$. If we put $g_i = a_i$ $(i = 1,2,\ldots,n)$ then there is a system $a_1^{-1}, a_2^{-1}, \ldots, a_{n-1}^{-1}$ such that $g_1 a_1^{-1} g_2 \ldots a_{n-1}^{-1} g_n = a_n \in A$.

Conversely, if there are elements $g_1, g_2, \ldots, g_n \in G$ such that $g_1 g_2 \ldots g_n = g$ and $g_1 a_1^{-1} g_2 \ldots a_{n-1}^{-1} g_n = a_n \in A$ then by normality of the set A we have

$$g_1 g_2 \ldots g_n = (a_{n-1})^{g_n} (a_{n-2})^{g_{n-1} g_n} \ldots$$
$$\ldots (a_1)^{g_2 \ldots g_n} (a_n)^{g_1 g_2 \ldots g_n}$$

$(a^x = x^{-1} a x)$. Therefore the element g is a product of n elements of the set A.

L e m m a 2. If for each system $g_1, g_2, \ldots, g_n \in G$ $(g = g_1 g_2 \ldots g_n)$ and for a_i from the normal set $A = BUB^{-1}$ the element

$$g_1 a_1^{-1} g_2 \ldots a_{n-1}^{-1} g_n \tag{2}$$

does not belong to A, then the element

$$g_1 (a_1^{-1})^{s_1} g_2 \ldots (a_{n-1}^{-1})^{s_{n-1}} g_n \tag{3}$$

does not belong to A fot each $s_i \in G$, too.

P r o o f . Since the set A is a normal set, following expressions are equivalent:

$$g_1a_1^{-1}g_2 \dots a_{n-1}^{-1}g_n \notin A,$$

$$a_1^{-1}g_2a_2^{-1}g_3 \dots a_{n-1}^{-1}g_ng_1 \notin A,$$

$$g_2a_2^{-1}g_3 \dots a_{n-1}^{-1}g_ng_1 \notin a_1A,$$

$$a_2^{-1}g_3 \dots a_{n-1}^{-1}g_ng_1g_2 \notin a_1^{g_2}A,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

(4) $$g_1g_2\dots g_n \notin a_{n-1}^{g_n}a_{n-2}^{g_{n-1}g_n} \dots a_1^{g_2g_3\dots g_n}A.$$

By analogous transformations we see that (3) is equivalent to

(5) $$(g_1g_2\dots g_n)^x \notin \left(a_{n-1}^{g_n}\right)^x\left(a_{n-2}^{g_{n-1}g_n}\right)^x \dots \left(a_1^{g_2g_3\dots g_n}\right)^x A$$

for each $x \in G$. Hence $g_1g_2\dots g_n \notin C_{n-1}C_{n-2}\dots C_1A$ which implies (5) and also (3). This ends the proof of Lemma 2.

Note that the set

$$K_m = \{g \in G : |g| = m\} \quad \text{(see [1])}$$

is a normal subset of a group G which satisfies the assumptions of Theorem 1.

Now we shall prove the following theorem.

T h e o r e m 2. Let $G = \prod^*_{s\in S} A_s$ be the free product of family of groups $\{A_s\}$, $s \in S$ such that at least one of them has an element whose order is different from 2 and at least one of them has an element of finite order. Then the set K_mK_m is not a subgroup of G for any finite $m > 1$, while $K_\infty K_\infty = G$.

P r o o f . Suppose that the group A_{s_0} contains elements of order m, let $C_i \subset K_m$ be any class of conjugate elements of order m, and let $x_i \in A_{s_0}$ be a representative of class C_i.

We will consider the cases: (i) $m > 2$, (ii) $m = 2$. In both cases we will find an element $g \in K_m^4$ such that $g \in K_m^2$.

<u>Case (i)</u>. Let $g = x_i x_i^{a_s} x_i x_i^{a_s} \in K_m^4$ with $a_s \in A_s$ and $s \neq s_0$.

In order to investigate the decomposition of an element g into product of two elements of order m we must check according to Theorem 1 whether we can find a decomposition of the element g into g_1 and g_2 $(g = g_1 g_2)$ in such a way that $g_1 x_j g_2 \in K_m$ where x_j runs over all representatives of the classes of conjugate elements from the set K_m.

It is easy to verify that the following elements

$$(x_i x_j) a_s^{-1} x_i a_s x_i a_s^{-1} x_i a_s,$$

$$x_i a_s^{-1} (x_i x_j) a_s x_i a_s^{-1} x_i a_s,$$

$$x_i a_s^{-1} x_i a_s (x_i x_j) a_s^{-1} x_i a_s,$$

$$x_i a_s^{-1} x_i a_s x_i a_s^{-1} (x_i x_j) a_s$$

are not of order m no matter whether $x_j \neq x_i^{-1}$ or $x_j = x_i^{-1}$.

By Theorem 1 $g \notin K_m K_m$ so $K_m^4 \not\subset K_m^2$ and $K_m K_m \nless G$.

<u>Case (ii)</u>. Considering the element

$$g = (a_s^{-1} x_i a_s)(x_i a_s x_i a_s^{-1} x_i a_s x_i a_s^{-1} x_i)(a_s^{-1} x_i a_s x_i a_s^{-1} x_i a_s)(x_i a_s x_i a_s^{-1} x_i)$$

with $a_s \in A_s$ and $a_s^2 \neq 1$, in the same way, we can prove that $K_2 K_2 \nless G$.

Now we shall show that $K_\infty K_\infty = G$.

If $|g| = \infty$ then we have $g = g^2 g^{-1} \in K_\infty K_\infty$.

If $|g| < \infty$ then $g = a_{s_i} \in A_{s_i}$. We have two cases:

1°) $a_{s_i}^2 = 1$,

2°) $a_{s_i}^2 \neq 1$.

If $a_{s_i}^2 = 1$ then

$$= a_{s_i} = \left(a_{s_i} a_{s_{i+1}} a_{s_i} a_{s_{i+1}}\right)\left(a_{s_{i+1}}^{-1} a_{s_i}^{-1} a_{s_{i+1}}^{-1}\right) \in K_\infty K_\infty,$$

because from the assumption there is an element $a_{s_{i+1}}$ such that $a_{s_{i+1}}^2 \neq 1$. If $a_{s_i}^2 \neq 1$ then also

$$g = a_{s_i} = \left(a_{s_i} a_{s_{i+1}} a_{s_i}\right)\left(a_{s_i}^{-1} a_{s_{i+1}}^{-1}\right) \in K_\infty K_\infty.$$

Therefore in all the cases we have $g \in K_\infty K_\infty$ i.e. $G \subseteq K_\infty K_\infty$. Thus $G = K_\infty K_\infty$.

D e f i n i t i o n 1. We say that a group G has the property W (see [1]) if for each $m \in \{|g| : g \in G\}$ the set K_m^2 is a subgroup of group G.

C o r o l l a r y 2.1. The group G which fulfils the assumptions of Theorem 2 does not have the property W.

C o r o l l a r y 2.2. If the group G is a free product of more than two cyclic groups of order 2 then the set K_2K_2 is not a subgroup of G.

Indeed,

$$G = \prod_{s \in S}^{*} A_s = A_{s_0} = \prod_{\substack{s \in S \\ s \neq s_0}}^{*} A_s = A_1 * A_2,$$

where

$$A_1 = A_{s_0}, \quad A_2 = \prod_{\substack{s \in S \\ s \neq s_0}}^{*} A_s,$$

fulfils the assumptions of Theorem 2.

T h e o r e m 3. Among groups which are free products only the group $G = \langle a,b; a^2, b^2 \rangle$ has the property W.

P r o o f . By Theorem 2 it is enough to prove that the group $\langle a,b;a^2,b^2\rangle$ has the property W, but this follows from [2] (see Theorem 2).

The subgroups of free products have the form

$$U = F * \prod_{s_i \in S}^{*} C^{-1} A_{s_i} C,$$

where F is the free group.

From Theorem 2 we have

C o r o l l a r y 2.3. If $F \neq 1$ and $A_{s_i} \neq 1$ then in the group U the sets K_m^2 $(1 < m < \infty)$ are not subgroups while $K^2 = U$.

Therefore we see that the free products apart from $\langle a,b;a^2,b^2\rangle$ do not have the property W and also their subgroups other than factors of the free product do not have this property.

Among normal subgroups of the group G can specify these which have the form K_mK_m where $K_m = \{g \in G: |g| = m\}$.

D e f i n i t i o n 2. We will say that a group G is a quasi-simple group if it does not have proper normal subgroups of form K_mK_m.

From Definition 2 and from Theorem 2 we have

T h e o r e m 4. The group $G = \prod_{s \in S}^{*} A_s$ different from $Z_2 * Z_2$ is a auasi-simple group.

From Definition 2 we see that the simple groups are also quasi-simple.

There are 2-groups which are quasi-simple groups, for example the group $G = \langle a,b,c;\ a^4=b^2=c^2=1,\ c^{-1}bc = ba^2,\ ab=ba, ac=ca\rangle$ is a quasi-simple group.

D e f i n i t i o n 3. We say that a group G has the property W^n if there is $n \in N$ such that for each $m \in \{|g| : g \in G\}$ the set $K_m^{2^n}$ is a subgroup of group G.

Obviously, finite groups have the property W^n. It is a question whether there exists a group which does not have the property W^n. The following theorem gives the answer.

T h e o r e m 5. The group $G = \prod^{*}_{s\in S} A_s$. with $|S| = \infty$ and $|A_s| < \infty$ does not have the property W^n.

P r o o f . Let us consider the element

$$g = a_{s_{i_1}} a_{s_{i_2}} \dots a_{s_{i_{2^n}}}, \quad a_{s_{i_j}} \in A_{s_{i_j}}, \tag{6}$$

$$i_j \neq i_{j+1}, \quad |a_{s_{i_j}}| = m < \infty \quad (j = 1,2, \dots, 2^n).$$

We must to show that the element (6) is not a product of 2^{n-1} elements of order m. By Theorem 1 it is enough to show that there is no system $x_{i_1}, x_{i_2}, \dots, x_{i_{2^{n-1}-1}}$ (x_{i_j} - - the representant of class C_{i_j} from K_m) such we have

$$g_1 x_{i_1} g_2 \dots x_{i_{2^{n-1}-1}} g_{2^{n-1}} \in K_m \quad (g = g_1 g_2 \dots g_{2^{n-1}}). \tag{7}$$

Obviously, the element (7) has a reduction if $x_{i_k} = a^{-1}_{s_{i_k}}$ ($k = 1,2,\dots,2^{n-1}-1$). But even in this case in the element (7) there will remain still $2^n - 2^{n-1} + 1 \geqslant 2$ of factors. Therefore the element (7) has the order ∞ and does not belong to K_m.

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