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A CONSTRUCTION OF FREE ALGEBRAS AND THE PROBLEM OF EXTENSION OF ALGEBRAS

0. Historical comments and introduction

An interpretation of a theory $\mathcal{T}_1$ in a theory $\mathcal{T}_2$ assigns, roughly speaking, terms and formulas of the language of $\mathcal{T}_1$ to terms and formulas of the language of $\mathcal{T}_2$ in such a way that a consequence of $\mathcal{T}_1$ (a true sentence in $\mathcal{T}_1$) corresponds to a consequence of $\mathcal{T}_2$ (a true sentence in $\mathcal{T}_2$). For instance, if $\mathcal{T}_1$ is the (equational) theory of Boolean rings, $\mathcal{T}_2$ is the (equational) theory of Boolean algebras, and Φ is a mapping from the set of terms of $\mathcal{T}_1$ to the set of terms of $\mathcal{T}_2$ such that

$\Phi(x) = x$ for each variable x, $\Phi(0) = 0$, $\Phi(1) = 1$,

$\Phi(t_1+t_2) = (\Phi(t_1) \wedge (\Phi(t_2))') \wedge ((\Phi(t_1))' \wedge \Phi(t_2))$,

$\Phi(t_1 \cdot t_2) = \Phi(t_1) \wedge \Phi(t_2)$ for all terms t_1, t_2 of $\mathcal{T}_1$,

$\Phi((t)^{-1}) = (\Phi(t))'$ for each term t of $\mathcal{T}_1$

(here $+$, $\cdot$, -1, denote the operations in a Boolean ring, and $\vee$, $\wedge$, $'$ denote the operations in a Boolean algebra), then by extending the mapping Φ to equations in an obvious way we obtain an interpretation of $\mathcal{T}_1$ in $\mathcal{T}_2$. The concept of an interpretation of one theory into another was introduced by A.Tarski, A.Mostowski, R.M.Robinson [27] (cf. J.H.Shoenfield [24]). This concept was also more or less explicitly considered for the case of equational theories (in the sense of [28]) by W.Felscher [4], [5], where the related notion of rational equivalence due to A.I.Mal'cev [17] (ch.9, p.59) has been discussed.

An interpretation θ of a theory $\mathcal{T}_1$ into a theory $\mathcal{T}_2$ defines a mapping $\theta^\vee$ from the class $\mathcal{K}_2$ of models of $\mathcal{T}_2$ to the class $\mathcal{K}_1$ of models of $\mathcal{T}_1$. If $\mathfrak{m}$ is a model of $\mathcal{T}_2$, then the model $\theta^\vee(\mathfrak{m})$ of $\mathcal{T}_1$ has the same universe as $\mathfrak{m}$ and the primitive operations and relations of $\theta^\vee(\mathfrak{m})$ are recovered according to the interpretation. For instance, the interpretation Φ of the equational theory of Boolean rings in the equational theory of Boolean algebras defines the mapping $\Phi^\vee$ given by

$\mathfrak{m} = (A, \vee, \wedge, ', 0, 1) \longmapsto \Phi^\vee(\mathfrak{m}) = (A, +, \cdot, {}^{-1}, 0, 1)$, where

$a_1 + a_2 = (a_1 \wedge (a_2)') \vee ((a_1)' \wedge a_2)$ and

$a_1 \cdot a_2 = a_1 \wedge a_2$ for all $a_1, a_2 \in A$, and

$a^{-1} = a'$ for each $a \in A$.

It appears that the mapping $\Phi^\vee$ is a bijection, hence the class of all Boolean rings and the class of all Boolean algebras are rationaly equivalent in the sense of A.I.Mal'cev [17].

If an interpretation θ of $\mathcal{T}_1$ into $\mathcal{T}_2$ is the inclusion of terms and equations, then the mapping $\theta^\vee$ sends each model of $\mathcal{T}_2$ to its reduct in the sense of Cohn's book [2], p.220. For instance, if $\mathcal{T}_2$ is the equational theory of Boolean algebras, $\mathcal{T}_1$ is the equational theory of distributive lattices with 0,1, then the inclusion of terms and equations of $\mathcal{T}_1$ into $\mathcal{T}_2$ (being obviously an interpretation between these theories) defines the mapping from the class of Boolean algebras to the class of distributive lattices with 0,1 given by

$$\mathfrak{m} = (A, \vee, \wedge, ', 0, 1) \longmapsto \mathfrak{n} = (A, \vee, \wedge, 0, 1),$$

where $\mathfrak{n}$ is a reduct of $\mathfrak{m}$.

The equational theories of groups and abelian groups give rise to a similar example of a mapping defined by interpretation between theories. By a generalized reduct we shall mean here the mapping between classes of models of theories defined by an interpretation between theories.

If $\mathcal{K}_1$ and $\mathcal{K}_2$ are equational classes in the sense of A.Tarski [28] (i.e. equationaly defined classes of algebras), then a generalized reduct $\theta^{\vee}$ from $\mathcal{K}_2$ into $\mathcal{K}_1$ induces a mapping $\theta^{\wedge}$ from $\mathcal{K}_1$ into $\mathcal{K}_2$ sending each algebra π of $\mathcal{K}_1$ to an algebra $\theta^{\wedge}(\pi) = F(A)/Q$ in $\mathcal{K}_2$; here A is the underlying set of π, F(A) is a free algebra generated by the set A in the class $\mathcal{K}_2$, and Q is a suitable congruence (cf. J.Słomiński [25], [26]). An example of the mapping between equational classes induced by a generalized reduct is the mapping sending each group to the result of its abelianization (cf. [23]). Using another language one can say that the algebra $\theta^{\wedge}(\pi)$ is a free algebra generated by the algebra $\pi = (A, \ldots)$.

The mappings induced by generalized reducts $\theta^{\vee}$ play an important rôle in algebra, because they provide in many cases a solution of the following problem of extension of algebras: does for a given algebra π in a class $\mathcal{K}_1$ exist an algebra m in a class $\mathcal{K}_2$ such that π can be homomorphically embedded in $\theta^{\vee}(m)$ (cf. A.I.Mal'cev [16], J.Łoś [18], B.H.Neumann [20]). J.Słomiński ([25], [26]) showed that the problem has a positive solution for an algebra π if and only if π can be homomorphically embedded in $\theta^{\vee}(\theta^{\wedge}(\pi))$. Using this idea he also proved in [25] that each distributive lattice with 0,1 can be homomorphically embedded in some Boolean algebra and each semigroup can be homomorphically embedded in some ring.

The notions mentioned above were described in an uniform way for the case of equational classes by F.W.Lawvere in his thesis summarized in [11], [12]. In Lawvere's approach the language of category theory is applied other related categorical approaches to universal algebra are in [1], [3], [8], [9], [10], [13], [14], [19], [29]. The following dictionary shows how some notions of universal algebra are translated into Lawvere's language.

Universal algebra language	Lawvere's language
an equational theory $\mathcal{T}$	an algebraic theory $\underline{T}$, i.e. a category with distinguished product families
an interpretation θ of an equational theory $\mathcal{T}_1$ into an equational theory $\mathcal{T}_2$	a functor $J:\underline{T}_1 \to \underline{T}_2$ preserving distinguished product families, where $\underline{T}_1$ and $\underline{T}_2$ are algebraic theories
an equational class $\mathcal{K}$	an algebraic category $\mathrm{Alg}(\underline{T})$, i.e. the category of certain set-valued functors defined on an algebraic theory $\underline{T}$
a generalized reduct $\theta^{\vee}$ from an equational class $\mathcal{K}_2$ into an equational class $\mathcal{K}_1$	an algebraic functor, i.e. the functor $\mathcal{J}:\mathrm{Alg}(\underline{T}_2) \to \mathrm{Alg}(\underline{T}_1)$ induced by a functor $J:\underline{T}_1 \to \underline{T}_2$ preserving distinguished product families
the mapping $\theta^{\wedge}$ from an equational class $\mathcal{K}_1$ into an equational class $\mathcal{K}_2$ induced by a generalized reduct $\theta^{\vee}$	a left adjoint to an algebraic functor $\mathcal{J}:\mathrm{Alg}(\underline{T}_2) \to \mathrm{Alg}(\underline{T}_1)$

In the paper we follow the Lawvere's approach by giving constructions of a free algebra in an algebraic category, of left adjoint to an algebraic functor, and some conditions for the positive solution of the problem of extension of algebras.

1. Notation

1.1. We shall use the following notation:

w.r.t. will serve as an abbreviation for "with respect to",

$?$, $?_1$, $?_2$ are symbols of variables, N is the set of all non-negative integers: by a non-negative integer we shall mean the finite cardinal number in von Neumann sense, i.e. $0 = \emptyset$, $n+1 = \{0,1,\dots,n\}$.

$\underline{N}^+$ is the set of all positive integers $\{1,2,\dots\}$.

$\underline{n}$ is the set $\{1,\dots,n\}$, in particular $\underline{0} = \emptyset$.

If S and A are sets, then by a family $(a_s \mid s \in S)$ of elements of A we shall mean the function $s \longmapsto a_s$ from S into A.

If x is an element of X, then q_x^X will denote the function from $\underline{1}$ into X given by $1 \longmapsto x$, in particular if $i \in \underline{n}$, then $q_i^n : \underline{1} \longrightarrow \underline{n}$ sends 1 to i.

1.2. For all unexplained terms concerning category theory we refer the reader to S. Mac Lane [15]. If $\mathcal{A}$ is a category, then $\mathrm{Ob}\,\mathcal{A}$ denotes the class of all objects of $\mathcal{A}$ and $\mathcal{A}(X,Y)$ denotes the set of all arrows of $\mathcal{A}$ with domain X and codomain Y. The composition of arrows $f:X \longrightarrow Y$ and $g:X \longrightarrow Z$ will be denoted by $g \circ f$. If A is an object of a category, then id_A will denote the identity arrow for A. The opposite category of a category $\mathcal{A}$ will be denoted by $\mathcal{A}^{op}$. By a functor we mean a covariant functor.

Set denotes the category of sets and $\mathcal{N}$ denotes the full subcategory of set with $\mathrm{Ob}\,\mathcal{N} = \{\underline{n} \mid n \in \underline{N}\}$. If X and Y are sets, then we whall write sometimes X^Y instead of $\mathrm{Set}(Y,X)$.

The symbol $\mathcal{A}(?_1,?_2)$ denotes the hom-bifunctor from $\mathcal{A}^{op} \times \mathcal{A}$ to Set; if f is an arrow of $\mathcal{A}$ and A is an object of $\mathcal{A}$, then $\mathcal{A}(f,A)$ will denote the value $\mathcal{A}(f,id_A)$.

The symbol $\mathrm{Set}^{\mathcal{A}}$ denotes the category whose objects are set-valued functors defined on a category $\mathcal{A}$ and whose arrows are natural transformations.

The notion of a colimit object of a diagram (functor) and the notion of the universal cone from a diagram to a colimit object are meant as in Mac Lane's book [15], p.67.

2. Algebraic theories

2.1. By an algebraic theory we shall mean an ordered pair $\underline{\mathcal{T}} = (\mathcal{T}, I)$ such that

1) $\mathbb{T}$ is a category and $I: \mathbb{N}^{op} \to \mathbb{T}$ is a functor bijective on objects; the object $I(\underline{n})$ will be denoted by $[\underline{n}]$ for each $n \in N$ and the arrow $I(\jmath^{\underline{m}}_{i})$ will be denoted by pr^m_i for each $m \in \underline{N}^+$, $i \in \underline{m}$,

2) the object $[\underline{0}]$ is a terminal object in $\mathbb{T}$; the unique arrow $g:[\underline{n}] \to [\underline{0}]$ will be denoted by $!^n$,

3) the family $(pr^m_i:[\underline{m}] \to [\underline{1}] \mid i \in \underline{m})$ is a product family in $\mathbb{T}$ for each $m \in \underline{N}^+$, i.e. for each family $(f_i:[\underline{n}] \to [\underline{1}] \mid i \in \underline{m})$ of arrows of $\mathbb{T}$ there exists a unique arrow $h:[\underline{n}] \to [\underline{m}]$ in $\mathbb{T}$ such that $pr^m_i \circ h = f_i$ for each $i \in \underline{m}$; this unique arrow h will be denoted by $\langle f_i : i \in \underline{m} \rangle$.

2.2. Let $\underline{\mathbb{T}} = (\mathbb{T}, I)$ be an algebraic theory. By an algebraic congruence on $\underline{\mathbb{T}}$ we shall mean a congruence on the category $\mathbb{T}$ (cf. Mac Lane's book [15], p.52) such that the following condition holds:

(c) if $f,g \in \mathbb{T}([\underline{n}],[\underline{m}])$ and $pr^m_i \circ f \; R_{[\underline{n}],[\underline{1}]} \; pr^m_i \circ g$ for all $i \in \underline{m}$, then $f \; R_{[\underline{n}],[\underline{m}]} g$.

If no confusion arises, we shall omit subscripts in $R_{[\underline{n}],[\underline{m}]}$. It is easy to verify that if R is an algebraic congruence on $\underline{\mathbb{T}}$, then the pair $\underline{\mathbb{T}}/R = (\mathbb{T}/R, I/R)$ is an algebraic theory, where $\mathbb{T}/R$ is the quotient category and I/R is the functor given by

$\underline{n} \mapsto [\underline{n}]$ for each object $\underline{n}$ of $\mathbb{N}^{op}$,

$f \mapsto \{g: g \; R \; I(f)\}$ for each arrow f of $\mathbb{N}^{op}$.

2.3. We shall now describe the construction of algebraic theories $\underline{\mathbb{T}}[\Omega]$ and $\underline{\mathbb{T}}([\Omega;E])$. Let $\Omega = (\Omega_n \mid n \in \underline{N})$ be a family of sets with $\Omega_n \cap \Omega_m = \emptyset$ for $n \neq m$ and let $V = \{x_i \mid i \in N^+\}$ be a set, called the set of variables, such that $x_i \neq x_j$ for $i \neq j$ and $V \cap \bigcup_{n \in \underline{N}} \Omega_n = \emptyset$. We define Ω-terms by induction as follows:

1) each element of $V \cup \Omega_0$ is an Ω-term,

2) if $(t_i \mid i \in \underline{n})$ is a family of Ω-terms and $\omega \in \Omega_n$, then the expression of the form $\omega(t_1, \ldots, t_n)$ is an Ω-term; we denote this Ω-term by $\omega(t_i : i \in \underline{n})$.

$T(\Omega)$ will denote the set of all Ω-terms.

If $(t_j \mid j \in \underline{m})$ is a family of Ω-terms and t is an Ω-term, then the result of the simultaneous substitution of t_j for occurrences of variable x_j in t, denoted by $[x_j/t_j : j \in \underline{m}]t$, is defined by induction as follows:

1° $[x_j/t_j : j \in \underline{m}]\omega = \omega$ for each $\omega \in \Omega_0$ and

$$[x_j/t_j : j \in \underline{m}]x_k = \begin{cases} t_k & \text{if } k \in \underline{m}, \\ x_k & \text{otherwise;} \end{cases}$$

2° if an Ω-term t is of the form $\omega(t'_1, \ldots, t'_n)$, where $\omega \in \Omega_n$ and $(t'_i \mid i \in \underline{n})$ is a family of Ω-terms, then

$$[x_j/t_j : j \in \underline{m}]t = \omega([x_j/t_j : j \in \underline{m}]t'_i : i \in \underline{n}).$$

The Ω-terms and the simultaneous substitution give rise to the category $\mathbb{T}[\Omega]$ whose objects are the sets $\underline{0}, \underline{1}, \underline{2}, \ldots, \underline{n}, \ldots$ and whose arrows from $\underline{n}$ to $\underline{m}$ for $m > 0$ are families $((t_j, n) \mid j \in \underline{m})$, where t_j for each $j \in \underline{m}$ is an Ω-term such that $\max\{i \mid x_i \text{ occures in } t_j \text{ or } i = 0\} \leq n$; we also assume that there is the unique arrow from $\underline{n}$ to $\underline{0}$ denoted by $!^n$. The composition of arrows in $\mathbb{T}[\Omega]$ is defined by using simultaneous substitution, e.g. for $f = (t, m) : \underline{m} \longrightarrow \underline{1}$ and $g = ((t_j, n) \mid j \in \underline{m}) : \underline{n} \longrightarrow \underline{m}$

$$f \circ g = ([x_j/t_j : j \in \underline{m}]t, n) : \underline{n} \longrightarrow \underline{1}.$$

In the case $h = ((t_k, 0) \mid k \in \underline{n}) : \underline{0} \longrightarrow \underline{n}$ and $!^m : \underline{m} \longrightarrow \underline{0}$ we assume that

$$h \circ !^m = ((t_k, m) \mid k \in \underline{n}) : \underline{m} \longrightarrow \underline{n}.$$

The category $\mathbb{T}[\Omega]$ and the functor $I : \mathbb{N}^{op} \longrightarrow \mathbb{T}[\Omega]$ given by

$\underline{n} \longmapsto \underline{n}$ for each object $\underline{n}$ and by

$f \longmapsto ((x_{f(i)}, n) \mid i \in m)$ for an arrow $f : \underline{m} \longrightarrow \underline{n}$ of $\mathbb{N}$

form an algebraic theory, denoted by $\underline{\mathbb{T}}[\Omega]$.

Let E be a set of ordered pairs of Ω-terms, called a set of Ω-equations, and let $\sim_E$ be the smallest algebraic congruence on the algebraic theory $\underline{T}[\Omega]$ such that

$$\text{if } (t,t') \in E, \text{ then } (t,m) \sim_E (t',m),$$

where $m = \max\{j \mid x_j$ occures in t or x_j occures in t' or $j = 0\}$.

The quotient algebraic theory $\underline{T}[\Omega]/\sim_E$ will be denoted by $\underline{T}[\Omega;E]$ and $(t,m)/\sim_E$ will denote the set $\{(t',m) \mid (t',m) \sim_E (t,m)\}$.

3. <u>Algebraic categories</u>

3.1. Let $\underline{T} = (T,I)$ be an algebraic category. By a $\underline{T}$-algebra we shall mean an ordered pair $\alpha = (A,G)$, where A is a set and $G: T \to Set$ is a functor such that $G(I(f)) = Set(f,A)$ for each arrow f of the category $\mathbb{N}$, or equivalently, the following two conditions hold:

1) $G([\underline{\Omega}]) = A^{\Omega}$,

2) $G(pr_i^m) = Set(q_i^m, A)$ for each $m \in \underline{N}^+$, $i \in \underline{m}$.

If $f \in T([\underline{n}],[\underline{1}])$ and a function from $\underline{n}$ to A is presented in the form of $(a_i \mid i \in \underline{n})$, then we shall write sometimes $G(f)\cdot(a_i \mid i \in \underline{n})$ for $(G(f)((a_i \mid i \in \underline{n})))(1)$.

3.2. Let $\underline{T} = (T,I)$ be an algebraic theory. A useful example of a $\underline{T}$-algebra is the $\underline{T}$-algebra $\alpha_{\underline{n}} = (A_{\underline{n}}, G_{\underline{n}})$, where $A_{\underline{n}} = T([\underline{n}],[\underline{1}])$ and the values $G_{\underline{n}}(f)$ for $f \in T([\underline{k}],[\underline{m}])$ are equal to the composition of the following functions:

$$A_{\underline{n}}^{\underline{k}} \xrightarrow{\varrho_{\underline{n},\underline{k}}} T([\underline{n}],[\underline{k}]) \xrightarrow{T([\underline{n}],f)} T([\underline{n}],[\underline{m}]) \xrightarrow{\varrho^{-1}_{\underline{n},\underline{m}}} A_{\underline{n}}^{\underline{m}};$$

here $\varrho_{\underline{n},\underline{k}}$ is given by

$$(f_i:[\underline{n}] \to [1] \mid i \in \underline{k}) \mapsto \langle f_i : i \in \underline{k} \rangle,$$

and $\varrho^{-1}_{n,m}$ is given by

$h \mapsto (pr_i^m \circ h : i \in \underline{m})$ for each $h \in T([\underline{n}],[\underline{m}])$.

In an obvious way $\varrho_{\underline{n},\underline{m}} \circ \varrho^{-1}_{\underline{n},\underline{m}} = id_{\mathbb{T}([\underline{n}],[\underline{m}])} \cdot \varrho^{-1}_{\underline{n},\underline{m}} \circ \varrho_{\underline{n},\underline{m}} = id_{Set(\underline{m},A_{\underline{n}})}$.

3.3. We shall denote by $Alg(\mathbb{T})$ the category whose objects are all $\mathbb{T}$-algebras and whose arrows from a $\mathbb{T}$-algebra $\alpha = (A,G)$ to a $\mathbb{T}$-algebra $\mathcal{B} = (B,L)$ are the ordered triples $\varphi = (\alpha, f, \mathcal{B})$, where f is a function from the set A to the set B such that the family $(Set(\underline{n},f) \mid [\underline{n}] \in Ob\mathbb{T})$ is a natural transformation from the functor G to the functor L.

It is easy to verify that each natural transformation α from the functor G to the functor L determines in the unique way a function $g: A \to B$ such that $\alpha_{[\underline{n}]} = Set(\underline{n},g)$ for each $n \in \underline{N}$.

The category $Alg(\mathbb{T})$ is called an algebraic category corresponding to $\mathbb{T}$.

We define the forgetful functor $U: Alg(\mathbb{T}) \to Set$ by
$\alpha \mapsto A$ for each $\mathbb{T}$-algebra $\alpha = (A,G)$,
$\varphi \mapsto f$ for each arrow $\varphi = (\alpha, f, \mathcal{B})$ of $Alg(\mathbb{T})$.

3.4. The $\mathbb{T}$-algebra $\alpha_{\underline{n}} = (A_{\underline{n}}, G_{\underline{n}})$ defined in 3.2 has the following property. Let $\eta_{\underline{n}}: \underline{n} \to A_{\underline{n}}$ be a function given by $i \mapsto pr^n_i$ $(i \in \underline{n})$, and let $\alpha = (A,G)$ be a $\mathbb{T}$-algebra, while $u: \underline{n} \to A$ is an arbitrary function. For the function u there exists a unique arrow (α_n, h, α) in $Alg(\mathbb{T})$ such that $h \circ \eta_{\underline{n}} = u$. The function $h: A_{\underline{n}} \to A$ is given by

$$f \mapsto (G(f)(u))(1) \quad \text{for each} \quad f \in \mathbb{T}([\underline{n}],[\underline{1}]) = A_{\underline{n}}.$$

3.5. We shall show how to represent a category of equationaly defined algebras by some isomorphic algebraic category.

Let the family Ω be as in 3.2. We recall that an Ω-algebra (cf. Cohn's book [2]) is an ordered pair $\mathcal{A} = (A, Op)$, where A is a set and $Op = (op_\omega \mid \omega \in \bigcup_{n \in \underline{N}} \Omega_n)$ is a family of functions $op_\omega: Set(\underline{n},A) \to A$ $(\omega \in \Omega_n,\ n \in \underline{N})$. Each Ω-algebra $\mathcal{A} = (A, Op)$ determines in the unique way the function $[\![?]\!]$ from the set $T(\Omega)$ to the set $Set(Set(V,A),A)$, called the interpretation of Ω-terms in $\mathcal{A}$, such that the following conditions hold:

1) $[\![x_i]\!](v) = v(x_i)$ for each variable x_i, and each $v \in Set(V,A)$, $[\![\omega]\!](v) = op_\omega(q)$ for each $\omega \in \Omega_0$, and each $v \in Set(V,A)$, where q is a unique function from $\underline{0}$ to A,

2) if an Ω-term t is of the form $\omega(t_1,\dots,t_n)$, where $\omega \in \Omega_n$ and $t_1,\dots,t_n$ are Ω-terms, then $[\![t]\!](v) = op_\omega([\![t_1]\!](v),\dots,[\![t_n]\!](v))$ for each $v \in Set(V,A)$.

Let E be a set of Ω-equations (cf. 2.3). We shall say that an Ω-algebra $\mathcal{A}$ satisfies E (or briefly $\mathcal{A}$ is an $(\Omega;E)$-algebra) iff $[\![t]\!](v) = [\![t']\!](v)$ for each $(t,t') \in E$ and each $v \in Set(V,A)$.

Let $Alg(\Omega;E)$ be the category whose objects are all $(\Omega;E)$-algebras and whose arrows from an $(\Omega;E)$-algebra $\mathcal{A} = (A,Op)$ to an $(\Omega;E)$-algebra $\mathcal{A}' = (A',Op')$ are the ordered triples $\varphi = (\mathcal{A},f,\mathcal{A}')$, where $f: A \longrightarrow A'$ is a homomorphism from $\mathcal{A}$ to $\mathcal{A}'$. The category $Alg(\Omega;E)$ is called a category of equationaly defined algebras. Each $(\Omega;E)$-algebra $\mathcal{A} = (A,Op)$ determines in the unique way the $\underline{T}[\Omega;E]$-algebra $\alpha_{(\mathcal{A})} = (A,G)$ such that $G((t,n)/\sim_E)\cdot(a_i \mid i \in \underline{n}) = [\![t]\!]([x_i/a_i : i \in \underline{n}](v))$ for each family $(a_i \mid i \in \underline{n})$ of elements of A and each $v \in Set(V,A)$, where $[\![?]\!]$ is the interpretation of Ω-terms in $\mathcal{A}$ and $[x_i/a_i : i \in \underline{n}](v)$ is defined by

$$([x_i/a_i : i \in n](v))(x_j) = \begin{cases} a_j & \text{if } j \in \underline{n}, \\ v(x_j) & \text{otherwise.} \end{cases}$$

To prove this it is sufficient to note that if $n \geq \max\{j \mid x_j \text{ occures in } t \text{ or } j = 0\}$, then

$$[\![[x_i/t_i : i \in \underline{n}]t]\!](v) = [\![t]\!]([x_i/[\![t_i]\!](v) : i \in \underline{n}](v)).$$

Each $\underline{T}[\Omega;E]$-algebra $\alpha = (A,G)$ determines the Ω-algebra $\mathcal{A}_{[\alpha]} = (A,Op)$ such that for each $\omega \in \Omega_n$ and $n > 0$

$$op_\omega(a_1,\dots,a_n) = G((\omega(x_1,\dots,x_n),n)/\sim_E)\cdot(a_i \mid i \in \underline{n})$$

and for each $\omega \in \Omega_0$

$op_\omega(q) = (G((\omega,0)/\sim_E)(q))(1)$, where q is the unique function from $\underline{0}$ to A. Since the interpretation $[\![?]\!]$ of Ω-terms in $\mathcal{A}_{[\alpha]}$ satisfies the following condition

$$[\![t]\!](v) = G((t,n)/\sim_E)\cdot(v(x_i) \mid i \in \underline{n}),$$

$\mathcal{A}_{[\alpha]}$ is an $(\Omega;E)$-algebra.

By straightforward verification we obtain that

$$\alpha_{(\mathcal{A}_{[\alpha]})} = \alpha \quad \text{and} \quad \mathcal{A}_{[\alpha_{(\mathcal{A})}]} = \mathcal{A},$$

hence $Alg(\underline{T}[\Omega;E])$ and $Alg(\Omega;E)$ are isomorphic categories.

3.6. Let $\underline{T} = (T,I)$ be an algebraic theory and let $\alpha = (A,G)$ be a $\underline{T}$-algebra. By a $\underline{T}$-congruence on α we shall mean an equivalence relation Q on the set A satisfying the following condition: for each $m \in \underline{N}^+$, $f \in T([\underline{m}],[\underline{1}])$, and for all $u, w \in Set(\underline{m},A)$ if $u(i)\ Q\ w(i)$ for each $i \in \underline{m}$, then $(G(f)(u))(1)\ Q\ (G(f)(w))(1)$. Let a/Q denote the set $\{a' \in A \mid a'\ Q\ a\}$ for any $\underline{T}$-congruence Q on a $\underline{T}$-algebra $\alpha = (A,G)$ and any a in A. It is easy to verify that if Q is a T-congruence on α, then the ordered pair $\alpha/Q = (A/Q,G/Q)$ is a $\underline{T}$-algebra, where A/Q is the quotient set and $G/Q: T \to Set$ is the functor given by
$(G/Q)([\underline{n}]) = Set(\underline{n},A/Q)$ for each $n \in \underline{N}$,
$((G/Q)(f))((u(i)/Q \mid i \in \underline{n})) = ((G(f)(u))(j)/Q \mid j \in \underline{m})$ for each $f \in T([\underline{n}],[\underline{m}])$. We shall say that α/Q is a quotient T-algebra.

Moreover, the ordered triple $\iota_Q = (\alpha,h,\alpha/Q)$ is an arrow from α to α/Q in $Alg(\underline{T})$, where $h: A \to A/Q$ is a function given by $h(a) = a/Q$ for each element a of A.

Let $\alpha = (A,G)$ and $\mathcal{B} = (B,L)$ be two $\underline{T}$-algebras and let $\xi = (\alpha,g,\mathcal{B})$ be an arrow from α to $\mathcal{B}$ in $Alg(\underline{T})$. The function $g:A \to B$ gives rise to the $\underline{T}$-congruence Q_ξ on α defined by

$$a\, Q_\xi\, a' \quad \text{iff} \quad g(a) = g(a').$$

We shall call Q_ξ a $\underline{T}$-congruence induced by ξ .

The ordered triple $\nu_\xi = (\alpha/Q_\xi, r, \mathcal{B})$ is an arrow from α/Q_ξ to $\mathcal{B}$ in $Alg(\underline{T})$, where the function $r:A/Q_\xi \to B$ is given by $r(a/Q_\xi) = g(a)$ for each $a \in A$.

4. The construction of free $\underline{T}$-algebras

4.1. By a free $\underline{T}$-algebra generated by a set X we mean a $\underline{T}$-algebra $\alpha_X = (A_X, G_X)$ equipped with a function $\eta_X: X \to A_X$ such that for each $\underline{T}$-algebra $\alpha = (A,G)$ and each function $f:X \to A$ there exists a unique arrow $\varphi: \alpha_X \to \alpha$ in $Alg(\underline{T})$ such that the following condition holds (α) $U(\varphi) \cdot \eta_X = f$, where U is the forgetful functor from $Alg(\underline{T})$ to Set.

We construct a free $\underline{T}$-algebra $\alpha_X = (A_X, G_X)$ generated by a set X in the following two steps:

1° We construct the set A_X as a colimit object of the diagram $\Gamma: E{\downarrow}X \to Set$, where $E{\downarrow}X$ is the comma category given by the inclusion functor $E:\mathbb{N} \to Set$ and the set X, i.e.

$$Ob\, E{\downarrow}X = \bigcup_{n \in \underline{N}} Set(\underline{n}, X), \quad E{\downarrow}X(u,w) = \{(u,h,w): w \cdot h = u\},$$

and the functor Γ is given by
$u \mapsto \underline{T}([\underline{n}],[\underline{1}])$ for each object $u:\underline{n} \to X$ of $E{\downarrow}X$,
$(u,h,w) \mapsto \underline{T}(I(h),[\underline{1}])$ for each arrow (u,h,w) of $E{\downarrow}X$.

We shall use in this step the fact that $E{\downarrow}X$ is a filtered category (cf. [6]), for the definition of a filtered category we refer the reader to Mac Lane's book [15] , p.207.

2° We construct the functor $G_X: \underline{T} \to Set$ as a colimit object of the diagram $\Gamma': E{\downarrow}X \to Set$ given by
$u \mapsto G_{\underline{n}}$ for each object $u:\underline{n} \to X$ of $E{\downarrow}X$ (for $G_{\underline{n}}$ see 3.2),
$(u,h,w) \mapsto \alpha$ for each arrow (u,h,w) of $E{\downarrow}X$, where α is the natural transformation from $G_{\underline{n}}$ to $G_{\underline{m}}$ ($\underline{n}$ and $\underline{m}$ being the domains of u and w respectively) given by

$\alpha_{[\underline{k}]} = \varrho^{-1}_{\underline{m},\underline{k}} \cdot \mathcal{T}(I(h)|,[\underline{k}]) \cdot \varrho_{\underline{n},\underline{k}}$ for each $k \in \underline{N}$
(for the definition of $\varrho^{-1}_{\underline{m},\underline{k}}$ and $\varrho_{\underline{n},\underline{k}}$ see 3.2).

We shall use in this step the "pointwise" construction of colimits in functor categories (cf. Schubert's book [22], p.53); if $D:\mathcal{B} \to Set^{\mathcal{C}}$ is a diagram (functor), then a colimit object of D constructed "pointwise" is the functor $H:\mathcal{C} \to Set^{\mathcal{C}}$ defined as follows:

i) the value $H(C)$ for $C \in Ob\,\mathcal{C}$ is a colimit object of the diagram $D_C:\mathcal{B} \to Set$ given by
$D_C(B) = (D(B))(C)$ for each object B of $\mathcal{B}$,
$D_C(f) = \alpha_C$ for each arrow f of $\mathcal{B}$, where α_C is the C-th component of the natural transformation $\alpha = D(f)$,

ii) the value $H(f)$ for $f \in \mathcal{C}(C,C')$ is the unique function $h:H(C) \to H(C')$ such that $h \cdot q_{B,C} = q_{B,C'} \cdot (D(B)(f))$ for each $B \in Ob\,\mathcal{B}$, where $q_{B,C}:D_C(B) \to H(C)$ are components of the universal cone from D_C to $H(C)$ and $q_{B,C'}:D_{C'}(B) \to H(C')$ are components of the universal cone from $D_{C'}$ to $H(C')$.

We shall also use in this step the fact that "filtered colimits commute with finite products" in Set (cf. [22], p.77), i.e. if $\mathcal{B}$ is a filtered category, $D:\mathcal{B} \to Set$ is a diagram, and C is a colimit object of D, then $C^{\underline{n}}$ is a colimit object of the diagram $D_{(n)}:\mathcal{B} \to Set$ given by
$D_{(\underline{n})}(B) = Set(\underline{n},D(B))$ for each object B of $\mathcal{B}$,
$D_{(\underline{n})}(f) = Set(\underline{n},D(f))$ for each arrow f of $\mathcal{B}$.

Moreover, if $(q_B:D(B) \to C \mid B \in Ob\,\mathcal{B})$ is the universal cone from D to C, then $(Set(n,q_B) \mid B \in Ob\,\mathcal{B})$ is the universal cone from $D_{(\underline{n})}$ to $C^{\underline{n}}$.

4.2. We present the details of the construction in the step 1°. Since $E \downarrow X$ is a filtered category, we construct the colimit object A_X of Γ in the similar way as in [22], p.73, subsection 9.4.2. We define A_X as the quotient set $X^*/\sim$, where

$$X^* = \bigcup_{n \in \underline{N}} \{(f,u) \mid f \in \mathcal{T}([\underline{n}],[\underline{1}]),\ u \in Set(\underline{n},X)\}$$

and $\sim$ is the equivalence relation on the set X^* defined in the following way:
$(f:[\underline{n}] \to [\underline{1}], u:\underline{n} \to X) \sim (g:[\underline{m}] \to [\underline{1}], w:\underline{m} \to X)$ iff there exist $k \in \underline{N}$ and functions $v:\underline{k} \to X$, $h':\underline{n} \to \underline{k}$, $h'':\underline{m} \to \underline{k}$ such that $v \circ h' = u$, $v \circ h'' = w$, and $f \circ I(h') = g \circ I(h'')$. (cf. [21], p.131).

We shall use the following facts:

1) for each object $u:\underline{n} \to X$ of $E \downarrow X$ the u-th component of the universal cone from Γ to $X^*/\sim$ is a function $q_u:\mathbb{T}([\underline{n}],[\underline{1}]) \to X^*/\sim$ given by $q_u(f) = (f,u)/\sim$ for each $f \in \mathbb{T}([\underline{n}],[\underline{1}])$, where $(f,u)/\sim = \{(f',u') \mid (f',u') \sim (f,u)\}$

2) each family $((f_i,u_i:n_i \to X)/\sim \mid i \in \underline{m})$ of elements of $X^*/\sim$ is equal to the family $((f_i',u:\underline{n} \to X)/\sim \mid i \in \underline{m})$, where $n = n_1 + \ldots + n_m$, $u:\underline{n} \to X$ is the unique function such that $u \circ \hat{q}_i = f_i$ for $\hat{q}_i$ given by $j \mapsto n_1 + \ldots + n_{i-1}+j$, and $f_i' = f_i \circ I(\hat{q}_i)$.

4.3. We present the details of the construction in the step 2^0. Since we construct G_X by using "pointwise" construction and "filtered colimits commute with finite products", we define G_X by

a) $G_X([\underline{n}]) = Set(\underline{n}, X^*/\sim) = (A_X)^{\underline{n}}$ for each $n \in \underline{N}$,

b) the value $G_X(s)$ for $s \in \mathbb{T}([\underline{m}],[\underline{k}])$ is defined as the unique function $h:(A_X)^{\underline{m}} \to (A_X)^{\underline{k}}$ such that for each object $u:\underline{n} \to X$ of $E \downarrow X$

$$h \circ Set(\underline{m},q_u) = Set(\underline{k},q_u) \circ (\Gamma'(u)(s)),$$

where q_u are the components of the universal cone from Γ to $X^*/\sim$. In particular, for $g \in \mathbb{T}([\underline{m}],[\underline{1}])$ we have by 4.2 1) that

$$G_X(g) \cdot ((g_i,w)/\sim \mid i \in \underline{m}) = (g \circ \langle g_i : i \in \underline{m}\rangle, w)/\sim,$$

hence using 4.2 2) we obtain

$$(\beta) \quad G_X(g) \cdot ((f_i,u_i)/\sim \mid i \in \underline{m}) = (g \circ \langle f_i' : i \in \underline{m}\rangle, u)/\sim,$$

where u and f'_i are defined as in 4.2 2) for the family

$$((f_i,u_i)/\sim \mid i \in \underline{m}).$$

Now, using 4.3 (β) we have that $G_X(pr_i^{\underline{m}}) = Set(q_i^{\underline{m}}, A_X)$ and hence (A_X,G_X) is a $\underline{T}$-algebra.

4.4. Now we shall show that for $\eta_X : X \to A_X$ given by $x \mapsto (id_{[\underline{1}]}, q_x^X : \underline{1} \to X)/\sim$, and for each $\underline{T}$-algebra $\alpha = (A,G)$, and for each function $f:X \to A$ there exists a unique arrow $\varphi : \alpha_X \to \alpha$ in $Alg(\underline{T})$ such that the condition 4.1 (α) holds.

For each function $w:\underline{m} \to X$, let $/w/ : A_{\underline{m}} \to A$ be the function defined by $/w/(g) = (G(g)(f \circ w))(1)$ for each $g \in A_{\underline{m}} = T([\underline{m}],[\underline{1}])$. Since A_X is a colimit object of the diagram Γ and for each arrow (w,t,w') of $E{\downarrow}X$ we have that $/w'/ \circ T(h(t),[\underline{1}]) = /w/$ (because $G(I(t)) = Set(t,A)$ and $w' \circ t = w$), there exists a unique arrow $h:A_X \to A$ such that for each object w of $E{\downarrow}X$ the following holds

$$h \circ q_w = /w/,$$

or equivalently

$$(\gamma) \qquad h((g,w)/\sim) = (G(g)(f \circ w))(1).$$

Using 3.4, 4.3 (β) and 4.4 (γ) we see that $\varphi = (\alpha_X, h, \alpha)$ is the unique arrow in $Alg(\underline{T})$ such that the condition 4.1 (α) holds.

4.5. Corollary. The forgetful functor $U:Alg(\underline{T}) \to Set$ has a left adjoint F such that $F(X) = \alpha_X$; the components of the unit η of this adjunction are the functions η_X defined as in 4.4.

5. The construction of a left adjoint to an algebraic functor

5.1. Let $\underline{T} = (T,I)$, $\underline{T}' = (T',I')$ be two algebraic theories and let $J: T' \to T$ be a covariant functor such that $J \circ I' = I$.

The functor J induces the functor $\mathcal{J}:\mathrm{Alg}(\underline{T}) \longrightarrow \mathrm{Alg}(\underline{T}')$ defined as follows:

$\mathcal{J}(\alpha) = (A, G\circ J)$ for each $\underline{T}$-algebra $\alpha = (A,G)$,

$\mathcal{J}(\varphi) = (\mathcal{J}(\alpha), U(\varphi), \mathcal{J}(\beta))$ for each arrow $\varphi:\alpha \longrightarrow \beta$ in $\mathrm{Alg}(\underline{T})$, where $U:\mathrm{Alg}(\underline{T}) \longrightarrow \mathrm{Set}$ is the forgetful functor (cf.3.3). This functor $\mathcal{J}$ is called an algebraic functor induced by J.

5.2. Let J and $\mathcal{J}$ be the functors as in 5.1 and let $\mathfrak{x} = (X,H)$ be a $\underline{T}'$-algebra. We shall construct a $\underline{T}$-algebra $\hat{\mathfrak{x}}$ and an arrow $\hat{\eta}_{\mathfrak{x}} : \mathfrak{x} \longrightarrow \mathcal{J}(\hat{\mathfrak{x}})$ in $\mathrm{Alg}(\underline{T}')$ satisfying the following condition:

1) for each arrow $\varphi : \mathfrak{x} \longrightarrow \mathcal{J}(\beta)$ in $\mathrm{Alg}(\underline{T}')$ there is a unique arrow $\psi : \hat{\mathfrak{x}} \longrightarrow \beta$ in $\mathrm{Alg}(\underline{T})$ such that

i) $\mathcal{J}(\psi)\circ\hat{\eta}_{\mathfrak{x}} = \varphi$.

Let $F(X) = \alpha_X = (A_X, G_X)$ be the free $\underline{T}$-algebra generated by the underlying set X of the $\underline{T}'$-algebra $\mathfrak{x} = (X,H)$. We shall say that a $\underline{T}$-congruence Q on $F(X) = \alpha_X$ is $\mathfrak{x}$-regular if the following condition holds:

2) if $H(f)(u) = H(g)(v)$, then $(J(f),u)/\sim Q (J(g),v)/\sim$, where $\sim$ is the equivalence relation defined in 4.2. Let $Q_{\mathfrak{x}}$ be the smallest $\mathfrak{x}$-regular $\underline{T}$-congruence on F(X). We shall show that the condition 5.2 1) holds for $\hat{\mathfrak{x}} = F(X)/Q_{\mathfrak{x}}$ and for $\hat{\eta}_{\mathfrak{x}} = (\mathfrak{x}, U(\iota_{Q_{\mathfrak{x}}})\circ\eta_X, \mathcal{J}(F(X)/Q_{\mathfrak{x}}))$, where $\iota_{Q_{\mathfrak{x}}}$ is defined for the $\underline{T}$-congruence $Q_{\mathfrak{x}}$ as in 3.6 and $\eta_X : X \longrightarrow U(F(X))$ is defined as in 4.4. Let $\varphi : \mathfrak{x} \longrightarrow \mathcal{J}(\beta)$ be an arrow in $\mathrm{Alg}(\underline{T}')$ and let $\xi : F(X) \longrightarrow \beta$ be the unique arrow in $\mathrm{Alg}(\underline{T})$ such that

3) $U(\xi)\circ\eta_X = U'(\varphi)$,

where $U':\mathrm{Alg}(\underline{T}') \longrightarrow \mathrm{Set}$ is the forgetful functor. Since the $\underline{T}$-congruence Q_ξ on F(X) induced by the arrow ξ (cf. 3.6) is $\mathfrak{x}$-regular, $Q_{\mathfrak{x}} \subseteq Q_\xi$ and hence $\sigma = (F(X)/Q_{\mathfrak{x}}, s, F(X)/Q_\xi)$ is an arrow in $\mathrm{Alg}(\underline{T})$, where the function $s:U(F(X)/Q_{\mathfrak{x}}) \longrightarrow U(F(X)/Q_\xi)$ is given by $s(c/Q_{\mathfrak{x}}) = c/Q_\xi$. Moreover, $\nu_\xi\circ\sigma\circ\iota_{Q_{\mathfrak{x}}} = \xi$ (for the definition of ν_ξ see 3.6) and hence by 5.2 3) $\psi = \nu_\xi\circ\sigma$ is an arrow in $\mathrm{Alg}(\underline{T})$ such that the condition 5.2 1) i) holds. We shall show that $\psi = \nu_\xi\circ\sigma$ is the unique arrow in $\mathrm{Alg}(\underline{T})$ satisfying 5.2. 1) i), i.e. if

$\mathcal{J}(\psi')\circ\hat{\eta}_{\mathfrak{x}}=\varphi$, then $\psi'=\nu_{\xi}\circ\sigma$. In fact, since $F(X)$ is a free $\underline{T}$-algebra generated by the set X,

$$\psi'\circ\iota_{Q_{\mathfrak{x}}}=\xi=\nu_{\xi}\circ\sigma\circ\iota_{Q_{\mathfrak{x}}}$$

hence using the fact $U(\iota_{Q_{\mathfrak{x}}})$ is a surjection we deduce that $\psi'=\nu_{\xi}\circ\sigma$.

5.3. Corollary. The functor $\mathcal{J}:\mathrm{Alg}(\underline{T})\longrightarrow\mathrm{Alg}(\underline{T}')$ has a left adjoint $\mathcal{F}$ such that $\mathcal{F}(\mathfrak{x})=\hat{\mathfrak{x}}=F(U'(\mathfrak{x}))/Q_{\mathfrak{x}}$ for each $\underline{T}'$-algebra $\mathfrak{x}$; besides that the components of the unit $\hat{\eta}$ of this adjunction are the arrows $\hat{\eta}_{\mathfrak{x}}$ defined as in 5.2.

5.4. Corollary. Let $\varphi:\mathfrak{x}\longrightarrow\mathcal{J}(\mathfrak{d})$ be an arrow in $\mathrm{Alg}(\underline{T}')$. If $U'(\varphi)$ is an injection, then $U'(\hat{\eta}_{\mathfrak{x}})$ is an injection.

Proof. Since $U'(\varphi)$ is an injection, the following proposition is true:

(δ) if $(id_{[1]},q_x^X)/\sim Q_{\xi}\ (id_{[1]},q_y^X)/\sim$, then $x=y$,

where Q_{ξ} is the $\underline{T}$-congruence on $F(X)$ induced by the unique arrow ξ satisfying 5.2. 3). Since $Q_{\mathfrak{x}}\subseteq Q_{\xi}$, $U'(\hat{\eta}_{\mathfrak{x}})$ is an injection by the definition of $\hat{\eta}_{\mathfrak{x}}$ and by 5.4. (δ).

5.5. We shall present certain characterization of the $\underline{T}$-congruence $Q_{\mathfrak{x}}$ on $F(X)=\alpha_X=(A_X,G_X)$ defined for a $\underline{T}'$-algebra $\mathfrak{x}=(X,H)$ in 5.2.

Let a binary relation $R_{\mathfrak{x}}$ on the set $U(F(X))=A_X=X^*/\sim$ be defined as follows:

$c_1/\sim R_{\mathfrak{x}}\,c_2/\sim$ iff there exist $k,m,n\in\underline{N}$, $h\in T([\underline{k}],[\underline{1}])$, $v\in X^{\underline{m}}$, $u\in X^{\underline{n}}$, and there exist families $(f_i:[\underline{m}]\longrightarrow[\underline{1}]\,|\,i\in\underline{k})$, $(g_i:[\underline{n}]\longrightarrow[\underline{1}]\,|\,i\in\underline{k})$ of arrows of T' such that $H(f_i)(v)=H(g_i)(u)$ for each $i\in\underline{k}$ and $c_1\sim(h\circ\langle J(f_i):i\in\underline{k}\rangle,v)$, and $c_2\sim(h\circ\langle J(g_i):i\in\underline{k}\rangle,u)$.

Let now $R^1_{\mathfrak{x}}=R_{\mathfrak{x}}$, $R^{n+1}_{\mathfrak{x}}=R^n_{\mathfrak{x}}\odot R_{\mathfrak{x}}$ for each $n\geq 1$, where $\odot$ denotes the composition of relations.

We prove by induction on n the following two propositions:

1) if Q is $\mathfrak{x}$-regular $\mathbb{T}$-congruence on $F(X)$, then $R_{\mathfrak{x}}^{n} \subseteq Q$,

2) if $f \in \mathbb{T}([\underline{m}],[\underline{1}])$ and $c_i/\sim R^n c_i'/\sim$ for each $i \in \underline{m}$, then

$$G_X(f).(c_i/\sim | i \in \underline{m}) \; R_{\mathfrak{x}}^{n} \; G_X(f).(c_i'/\sim | i \in \underline{m}).$$

The relation $R_{\mathfrak{x}}$ satisfies also the following condition:

3) if $H(f)(u) = H(g)(v)$, then $(J(f),u)/\sim R_{\mathfrak{x}}(J(g),v)/\sim$.

Since the relation $R_{\mathfrak{x}}$ is reflexive and symmetric, we deduce from 5.5. 1), 2), 3) that

4) $\bigcup\limits_{n \geq 1} R_{\mathfrak{x}}^{n} = Q_{\mathfrak{x}}$.

The formula 5.5. 4) can serve as a characterization of $Q_{\mathfrak{x}}$.

6. The problem of extension of algebras

6.1. Let $\underline{\mathbb{T}} = (\mathbb{T},I)$, $\underline{\mathbb{T}}' = (\mathbb{T}',I')$ be two algebraic theories and let $J:\mathbb{T}' \to \mathbb{T}$ be a contravariant functor such that $J \circ I' = I$. Let $\mathcal{J}:\mathrm{Alg}(\underline{\mathbb{T}}) \to \mathrm{Alg}(\underline{\mathbb{T}}')$ be the algebraic functor induced by J. We shall say that the problem of extension of a $\underline{\mathbb{T}}'$-algebra $\mathfrak{x} = (X,H)$ w.r.t. the algebraic functor $\mathcal{J}$ induced by J has a positive solution if there exist a $\underline{\mathbb{T}}$-algebra $\mathfrak{b}$ and an arrow $\varphi : \mathfrak{x} \to \mathcal{J}(\mathfrak{b})$ in $\mathrm{Alg}(\mathbb{T}')$ with $U'(\varphi)$ being an injection.

The following proposition is an immediate consequence of 5.4.

6.2. P r o p o s i t i o n . The problem of extension of a $\underline{\mathbb{T}}'$-algebra $\mathfrak{x}$ w.r.t. the algebraic functor $\mathcal{J}$ induced by J has positive solution iff $U'(\hat{\eta}_{\mathfrak{x}})$ is an injection, where $\hat{\eta}_{\mathfrak{x}} : \mathfrak{x} \to \mathcal{J}(\mathcal{F}(\mathfrak{x}))$ is the $\mathfrak{x}$-th component of the unit $\hat{\eta}$ of the adjunction determined by functors $\mathcal{J}$, $\mathcal{F}$.

6.3. The proposition 6.2 and the characterization of $Q_{\mathfrak{x}}$ presented in 5.5 give rise to the following necessary conditions for the positive solution of the problem of extension of a $\underline{\mathbb{T}}$-algebra $\mathfrak{x} = (X,H)$ w.r.t. the algebraic functor $\mathcal{J}$ induced by J.

$(\mathcal{N}_0)$ if $J(f) = J(g)$, then $H(f)(v) = H(g)(v)$,
$(\mathcal{N}_1)$ if $(J(f),v)/\sim R_{\mathfrak{x}} (J(g),u)/\sim$, then $H(f)(v) = H(g)(u)$,
$(\mathcal{N}_n)$ if $(J(f),v)/\sim R^n_{\mathfrak{x}} (J(g),u)/\sim$, then $H(f)(v) = H(g)(u)$.
The condition $(\mathcal{N}_0)$ is equivalent to the following condition:
$(\mathcal{N}'_0)$ if $(J(f),v)\sim(J(g),u)$, then $H(f)(v) = H(g)(u)$.

Let us consider the following conditions:
$(\mathcal{J}_0)$ the functor J is full and $(\mathcal{N}_0)$ holds,
$(\mathcal{J}_1)$ $R_{\mathfrak{x}}$ is a transitive relation and $(\mathcal{N}_1)$ holds,
$(\mathcal{J}_n)$ $R^n_{\mathfrak{x}}$ is a transitive relation and $(\mathcal{N}_n)$ holds,
$(\mathcal{J}^\wedge)$ the condition $(\mathcal{N}_n)$ holds for $n = 2$ and if $(J(g),u)/\sim R^2_{\mathfrak{x}} (h,w)/\sim$, then $(J(g),u)/\sim R_{\mathfrak{x}}(h,w)/\sim$.

We deduce from 5.5. 4) and 6.2 that each of the conditions $(\mathcal{J}_0)$, $(\mathcal{J}_1)$, $(\mathcal{J}_n)$, $(\mathcal{J}^\wedge)$ is a sufficient condition for the positive solution of the problem of extension of $\mathfrak{x}$ w.r.t. $\mathcal{J}$ induced by J.

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