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Wanda Ślósarska

ON CONFORMAL KILLING TENSORS OF TYPE 2/2 IN THE RIEMANNIAN SPACE

1. Introduction

Let there be given an n-dimensional Riemannian space M with the nonsingular metric form $ds^2 = g_{ij}dx^i dx^j$. By R_{ijkl}, R_{ij} and R we shall understand the covariant curvature tensor, the Ricci tensor and the scalar curvature of the considered space, respectively.

By C_{ijkl} we shall denote the tensor of the conformal curvature of the space M. Thus we have by definition

$$(1) \quad C_{ijkl} = R_{ijkl} + \frac{1}{n-2}(R_{ik}g_{jl} - R_{jk}g_{il} + R_{jl}g_{ik} - R_{il}g_{jk}) + \frac{R}{(n-1)(n-2)}(g_{jk}g_{il} - g_{ik}g_{jl}).$$

It is known that the Riemannian space M is a conformally-Euclidean space if and only if the tensor C_{ijkl} vanishes identically.

2. The conformal Killing tensor of type 2/2

Consider the tensor V_{ijcd} which satisfies the following identities

$$(2) \qquad V_{ij[cd]} = V_{ijcd}.$$

In order to introduce the notion of conformal Killing tensor of type 2/2 we state the following definition.

D e f i n i t i o n 1. The tensor V_{ijcd} of class C^2 satisfying the condition $V_{ij[cd]} = V_{ijcd}$ is said to be a conformal Killing tensor of type 2/2 if there exists in the space M a tensor f_{ijc} such that

$$V_{ijcd;b} + V_{ijbd;c} = 2f_{ijd}g_{bc} - f_{ijb}g_{cd} - f_{ijc}g_{bd}. \tag{3}$$

The tensor f_{ijc} occurring in definition 1 will be called the tensor associated with the conformal Killing tensor of type 2/2.

It is easy to see that the above introduced notion of conformal Killing tensor of type 2/2 is an attempt to extend to tensors of type (4,0) satisfying the condition (2) the notions of conformal vector and Killing tensor introduced in papers [1] and [2].

It is worth while noting that if we set in the definition $f_{ijc} = 0$ and $V_{[ij]cd} = V_{ijcd}$ we get the definition of Killing tensors of type 2/2 introduced in paper [4]; if we set $f_{ijc} = 0$ we get the definition of the Killing tensors of type 1/3 from paper [3].

We are now going to prove some fundamental lemmas about the properties of conformal Killing tensors of type 2/2 which will be useful in the proof of the main theorem of the present paper.

L e m m a 1. For any conformal Killing tensor of type 2/2 of class C^2 the following identity holds

$$V_{ijcd;b} + V_{ijdb;c} + V_{ijbc;d} = \tag{4}$$

$$= 3(V_{ijcd;b} + f_{ijc}g_{bd} - f_{ijd}g_{bc}).$$

P r o o f . The definition implies the following identities

$$(5)\quad V_{ijbd;c} + V_{ijcd;b} = 2f_{ijd}g_{bc} - f_{ijc}g_{db} - f_{ijb}g_{dc}$$

and

$$(6)\quad V_{ijbc;d} + V_{ijdc;b} = 2f_{ijc}g_{bd} - f_{ijd}g_{bc} - f_{ijb}g_{dc}.$$

Subtracting the equalities (5) from equality (6) and adding $V_{ijdc;b}$ to both sides we get the required identity.

L e m m a 2. Any conformal Killing tensor of type 2/2 of class C^2 satisfies the identities

$$(7)\qquad V_{ej}{}^{ba}R^{e}{}_{iba} + V_{ie}{}^{ba}R^{e}{}_{jba} = -2(n-1)f_{ija;}{}^{a},$$

where

$$f_{ija;}{}^{a} = g^{ca}f_{ijc;a}.$$

P r o o f . Transvecting the tensor g^{bc} onto (3) we get

$$(8)\qquad V_{ijcd;}{}^{c} = (n-1)f_{ijd},$$

where

$$V_{ijcd;}{}^{c} = g^{bc}V_{ijcd;b}.$$

Taking the covariant derivative in equality (8) and transvecting the tensor g^{dc} onto both sides we obtain

$$(9)\qquad V_{ij}{}^{cd}{}_{;c;d} = (n-1)f_{ijd;}{}^{d}.$$

On the other hand, from the Ricci identity we get the following equality

$$(10)\qquad 2V_{ij}{}^{ab}{}_{;a;b} = -V_{ej}{}^{ab}R^{e}{}_{iab} - V_{ie}{}^{ab}R^{e}{}_{jab}.$$

Finally (9) and (10) imply (7).

L e m m a 3. Any conformal Killing tensor of type 2/2 satisfies the identities

$$(11)\quad \Big\{3(\delta_i^\alpha\delta_j^\beta(\delta_d^\tau T^\sigma{}_{abc} + \delta_c^\tau T^\sigma{}_{bad} + \delta_b^\tau T^\sigma{}_{cda} + \delta_a^\tau T^\sigma{}_{dcb}) + \delta_j^\beta(\delta_d^\sigma W^\alpha{}_{iabc}{}^\tau + \delta_c^\sigma W^\alpha{}_{ibda}{}^\tau + \delta_d^\sigma W^\alpha{}_{icda}{}^\tau + \delta_a^\sigma W^\alpha{}_{idcb}{}^\tau) + \delta_i^\alpha(\delta_d^\sigma W^\alpha{}_{jabc}{}^\tau + \delta_c^\sigma W^\alpha{}_{jbda}{}^\tau + \delta_b^\sigma W^\alpha{}_{jcda}{}^\tau + \delta_a^\sigma W^\beta{}_{jdcb}{}^\tau) + \frac{2}{n-1}(g_{ca}g_{bd}-g_{ab}g_{cd})(\delta_j^\beta R^\alpha{}_i{}^{\tau\sigma} + \delta_i^\alpha R^\beta{}_j{}^{\tau\sigma})\Big\} V_{\alpha\beta\tau\sigma} = 0,$$

where

$$(12)\quad T^\sigma{}_{abc} = (n-2)\, R^\sigma{}_{abc} + R^\sigma_b g_{ca} - R^\sigma_c g_{ab}$$

and

$$(13)\quad W^\alpha{}_{iabc}{}^\tau = 3(R^\alpha{}_{ib}{}^\tau g_{ca} - R^\alpha{}_{jc}{}^\tau g_{ab}) + (n-2)\Big[\delta_a^\tau R^\alpha{}_{icb} + \frac{1}{2}(\delta_c^\tau R^\alpha{}_{iab} - \delta_b^\tau R^\alpha{}_{iac})\Big].$$

P r o o f . Taking the covariant derivative in identity (3) we get

$$(14)\quad V_{ijcd;b;a} + V_{ijbd;c;a} = 2f_{ijd;a}g_{bc} - f_{ijb;a}g_{cd} - f_{ijc;a}g_{bd}.$$

Permuting cyclically the indices a, b, c we obtain

$$(15)\quad V_{ijad;c;b} + V_{ijcd;a;b} = 2f_{ijd;b}g_{ca} + - f_{ijc;b}g_{ad} - f_{ija;b}g_{cd}$$

and

(16) $$V_{ijbd;a;c} + V_{ijad;b;c} = 2f_{ijd;c}g_{ab} + - f_{ija;c}g_{bd} - f_{ijb;c}g_{ad}.$$

Adding now identities (14) and (15) side by side and subtracting the identity (16) from the sum and making use of the Ricci identity and of the identity $R^e{}_{[dac]} = 0$ we obtain:

(17) $$\begin{aligned}&2V_{ijcd;b;a} - 2V_{ijed}R^e{}_{acb} - V_{ejcd}R^e{}_{iab} + \\ &- V_{ejbd}R^e{}_{ica} - V_{ejad}R^e{}_{icb} - V_{iecd}R^e{}_{jab} + \\ &- V_{iebd}R^e{}_{jca} - V_{iead}R^e{}_{jcb} - V_{ijce}R^e{}_{dab} + \\ &- V_{ijbe}R^e{}_{dca} - V_{ijae}R^e{}_{dcb} = 2(f_{ijd;a}g_{bc} + \\ &+ f_{ijd;b}g_{ca} - f_{ijd;c}g_{ab}) - (f_{ijb;a} + f_{ija;b})g_{cd} + \\ &+ (f_{ija;c} - f_{ijc;a})g_{bd} + (f_{ijb;c} - f_{ijc;b})g_{ad}.\end{aligned}$$

Permutting cyclically the indices b, c, d we get from (17) the following two identities

(18) $$\begin{aligned}&2V_{ijdb;c;a} - 2V_{ijeb}R^e{}_{adc} - V_{ejdb}R^e{}_{iac} + \\ &- V_{ejcb}R^e{}_{ida} - V_{ejab}R^e{}_{idc} - V_{iedb}R^e{}_{jac} + \\ &- V_{iecb}R^e{}_{jda} - V_{ieab}R^e{}_{jdc} - V_{ijde}R^e{}_{bac} + \\ &- V_{ijce}R^e{}_{bda} - V_{ijae}R^e{}_{bdc} = 2(f_{ijb;a}g_{cd} + \\ &+ f_{ijb;c}g_{da} - f_{ijb;d}g_{ac}) - (f_{ijc;a} + f_{ija;c})g_{bd} + \\ &+ (f_{ija;d} - f_{ijd;a})g_{cb} + (f_{ijc;d} - f_{ijd;c})g_{ab}\end{aligned}$$

and

$$(19)\quad 2V_{ijbc;d;a} - 2V_{ijec}R^{e}{}_{abd} - V_{ejbc}R^{e}{}_{iad} + \\ - V_{ejdc}R^{e}{}_{iba} - V_{ejac}R^{e}{}_{ibd} - V_{iebc}R^{e}{}_{jad} + \\ - V_{iedc}R^{e}{}_{jba} - V_{ieac}R^{e}{}_{jbd} - V_{ijbe}R^{e}{}_{cad} + \\ - V_{ijde}R^{e}{}_{cba} - V_{ijae}R^{e}{}_{cbd} = 2(f_{ijc;a}g_{db} + \\ + f_{ijc;d}g_{ba} - f_{ijc;b}g_{ad}) + (f_{ijd;a} + f_{ija;d})g_{bc} + \\ + (f_{ija;b} - f_{ijb;a})g_{cd} + (f_{ijd;b} - f_{ijb;d})g_{ac}.$$

Adding equalities (17), (18) and (19) side by side and making use of identity (4) and identity $R^{e}{}_{[ijk]} = 0$ we have

$$(20)\quad 6V_{ijcd;b;a} + 3(V_{ijde}R^{e}{}_{acb} + V_{ijbe}R^{e}{}_{adc} + V_{ijce}R^{e}{}_{abd}) + \\ + 2(V_{ejdc}R^{e}{}_{iab} + V_{ejdb}R^{e}{}_{ica} + V_{ejbc}R^{e}{}_{ida}) + \\ + 2(V_{iedc}R^{e}{}_{jab} + V_{iedb}R^{e}{}_{jca} + V_{iebc}R^{e}{}_{jda}) + \\ + V_{ejda}R^{e}{}_{icb} + V_{ejba}R^{e}{}_{idc} + V_{ejca}R^{e}{}_{ibd} + \\ + V_{ieda}R^{e}{}_{jcb} + V_{ieba}R^{e}{}_{jdc} + V_{ieca}R^{e}{}_{jbd} = \\ = 3\Big[(f_{ijb;c} - f_{ijc;b})g_{da} + (f_{ijd;b} - f_{ijb;d})g_{ca} + \\ + (f_{ijc;d} - f_{ijd;c})g_{ba} + 2f_{ijd;a}g_{cd} - 2f_{ijc;a}g_{bd}\Big].$$

Multiplying now the identities (17) by 3 and subtracting them from identity (20) we get

$$(21)\quad 3(V_{ijde}R^{e}{}_{abc} + V_{ijce}R^{e}{}_{bad} + V_{ijbe}R^{e}{}_{cda} + \\ + V_{ijae}R^{e}{}_{dcb}) + 2(V_{ejad}R^{e}{}_{icb} + V_{ejbc}R^{e}{}_{ida}) + \\ + 2(V_{iead}R^{e}{}_{jcb} + V_{iebc}R^{e}{}_{jda}) + V_{ejcd}R^{e}{}_{iab} +$$

$$+ V_{ejba}R^{e}{}_{idc} + V_{ejbd}R^{e}{}_{ica} + V_{ejca}R^{e}{}_{ibd} +$$

$$+ V_{iecd}R^{e}{}_{jab} + V_{ieba}R^{e}{}_{jdc} + V_{iebd}R^{e}{}_{jca} +$$

$$+ V_{ieca}R^{e}{}_{ibd} = 3(\sigma_{ijdc}g_{ab} - \sigma_{ijdb}g_{ca} + \sigma_{ijba}g_{cd} +$$

$$- \sigma_{ijac}g_{bd}),$$

where

$$\sigma_{ijdc} = f_{ijd;c} + f_{ijc;d}.$$

Transvecting the tensor g^{ab} onto both sides of the last equalities and taking into account the identity (7) and the following one, easy to check,

$$R_{bdce}u_{ij}{}^{de} + R_{cdbe}u_{ij}{}^{de} = 0$$

we get

$$V_{ijde}R^{e}{}_{c} + V_{ijce}R^{e}{}_{d} + V_{ejbd}R^{e}{}_{ic}{}^{b} +$$

$$+ V_{ejbc}R^{e}{}_{id}{}^{b} + V_{iebd}R^{e}{}_{jc}{}^{b} + V_{iebc}R^{e}{}_{jd}{}^{b} =$$

$$= (n-2)\,\sigma_{ijdc} - \frac{g_{cd}}{n-1}(V_{ej}{}^{ba}R^{e}{}_{iba} + V_{ie}{}^{ba}R^{e}{}_{jba}),$$

whence

$$(22)\quad \sigma_{ijdc} = \frac{1}{n-2}(V_{ijde}R^{e}{}_{c} + V_{ijce}R^{e}{}_{d} +$$

$$+ V_{ejbd}R^{e}{}_{ic}{}^{b} + V_{ejbc}R^{e}{}_{id}{}^{b} + V_{iebd}R^{e}{}_{jc}{}^{b} +$$

$$+ V_{iebc}R^{e}{}_{jd}{}^{b} + \frac{g_{cd}}{n-1}(V_{ej}{}^{ba}R^{e}{}_{iba} + V_{ie}{}^{ba}R^{e}{}_{jba}).$$

Formulas (21) and (22) imply the identities

$$
\begin{aligned}
(23)\quad & 3\Big[V_{ijde}T^{e}{}_{abc} + V_{ijce}T^{e}{}_{bad} + V_{ijbe}T^{e}{}_{cda} + \\
& + V_{ijae}T^{e}{}_{dcb} + V_{ejkd}(R^{e}{}_{ib}{}^{k}g_{ca} - R^{e}{}_{ic}{}^{k}g_{ab}) + \\
& + V_{ejkc}(R^{e}{}_{ia}{}^{k}g_{bd} - R^{e}{}_{id}{}^{k}g_{ba}) + V_{ejkb}(R^{e}{}_{id}{}^{k}g_{ca} + \\
& - R^{e}{}_{ia}{}^{k}g_{cd}) + V_{ejka}(R^{e}{}_{ic}{}^{k}g_{bd} - R^{e}{}_{ib}{}^{k}g_{cd}) + \\
& + V_{iekd}(R^{e}{}_{jb}{}^{k}g_{ca} - R^{e}{}_{jc}{}^{k}g_{ab}) + V_{iekc}(R^{e}{}_{ja}{}^{k}g_{bd} - R^{e}{}_{jd}{}^{k}g_{ba}) + \\
& + V_{iekb}(R^{e}{}_{jd}{}^{k}g_{ca} - R^{e}{}_{ja}{}^{k}g_{cd}) + V_{ieka}(R^{e}{}_{jc}{}^{k}g_{bd} - R^{e}{}_{jb}{}^{k}g_{cd})\Big] + \\
& + (n-2)\Big[2(V_{ejad}R^{e}{}_{icb} + V_{ejbc}R^{e}{}_{ida}) + 2(V_{iead}R^{e}{}_{jcb} + \\
& + V_{iebc}R^{e}{}_{jda}) + V_{ejcd}R^{e}{}_{iab} + V_{ejba}R^{e}{}_{idc} + V_{ejbd}R^{e}{}_{ica} + \\
& + V_{ejca}R^{e}{}_{ibd} + V_{iecd}R^{e}{}_{jab} + V_{jeba}R^{e}{}_{jdc} + V_{iebd}R^{e}{}_{jca}\Big] + \\
& + \frac{2}{n-1}(g_{ca}g_{db} - g_{ab}g_{cd})(V_{ejkl}R^{e}{}_{i}{}^{kl} + V_{iekl}R^{e}{}_{j}{}^{kl}) = 0
\end{aligned}
$$

and

$$
\begin{aligned}
(24)\quad & 3(V_{ijde}T^{e}{}_{abc} + V_{ijce}T^{e}{}_{bad} + V_{ijbe}T^{e}{}_{cda} + \\
& + V_{ijae}T^{e}{}_{dcb}) + V_{ejkd}W^{e}{}_{iabc}{}^{k} + V_{ejkc}W^{e}{}_{ibda}{}^{k} + \\
& + V_{ejkb}W^{e}{}_{icda}{}^{k} + V_{ejka}W^{e}{}_{idcb}{}^{k} + V_{iekd}W^{e}{}_{jabc}{}^{k} + \\
& + V_{iekc}W^{e}{}_{jbda}{}^{k} + V_{iekb}W^{e}{}_{jcda}{}^{k} + V_{ieka}W^{e}{}_{jdcb}{}^{k} + \\
& + \frac{2}{n-1}(g_{ca}g_{db} - g_{ac}g_{cd})(V_{ejkl}R^{e}{}_{i}{}^{kl} + V_{iekl}R^{e}{}_{j}{}^{kl}) = 0,
\end{aligned}
$$

where $T^{e}{}_{abc}$ is given by formula (12), $W^{e}{}_{iabc}{}^{k}$ by formula (13). From (24) the relation (11) follows.

T h e o r e m 1. If at any point of the Riemannian space M there exists a conformal Killing tensor of type 2/2, then the space M is a conformally-Euclidean space.

P r o o f . Let the assumptions of the theorem hold. Then it follows from lemma 3 and condition (2) that the symmetric part (with respect to γ and δ) of the coefffioient of $V_{\alpha\beta\gamma\delta}$ vanishes, that is

$$
\begin{aligned}
(25)\quad & 3\delta^{\alpha}_{i}\delta^{\beta}_{j}(\delta^{\tau}_{d}T^{\sigma}{}_{abc} + \delta^{\tau}_{c}T^{\sigma}{}_{bad} + \delta^{\tau}_{b}T^{\sigma}{}_{cda} + \\
& + \delta^{\tau}_{a}T^{\sigma}{}_{dcb}) + \delta^{\beta}_{j}(\delta^{\sigma}_{d}W^{\alpha}{}_{iabc}{}^{\tau} + \delta^{\tau}_{c}W^{\alpha}{}_{ibda}{}^{\tau} + \\
& + \delta^{\sigma}_{b}W^{\alpha}{}_{icda}{}^{\tau} + \delta^{\sigma}_{a}W^{\alpha}{}_{idcb}{}^{\tau}) + \delta^{\alpha}_{i}(\delta^{\sigma}_{d}W^{\beta}{}_{jabc}{}^{\tau} + \\
& + \delta^{\sigma}_{c}W^{\beta}{}_{jbda}{}^{\tau} + \delta^{\sigma}_{b}W^{\beta}{}_{jcda}{}^{\tau} + \delta^{\sigma}_{a}W^{\beta}{}_{jdcb}{}^{\tau}) + \\
& + \frac{2}{n-1}(g_{ca}g_{db} - g_{ab}g_{cd})(\delta^{\beta}_{j}R^{\alpha}{}_{i}{}^{\tau\sigma} + \delta^{\alpha}_{i}R^{\beta}{}_{j}{}^{\tau\sigma}) = \\
& = 3\delta^{\alpha}_{i}\delta^{\beta}_{j}(\delta^{\sigma}_{d}T^{\tau}{}_{abc} + \delta^{\sigma}_{c}T^{\tau}{}_{bad} + \delta^{\sigma}_{c}T^{\tau}{}_{cda} + \\
& + \delta^{\sigma}_{a}T^{\tau}{}_{dcb}) + \delta^{\beta}_{j}(\delta^{\tau}_{d}W^{\alpha}{}_{iabc}{}^{\sigma} + \delta^{\tau}_{c}W^{\alpha}{}_{ibda}{}^{\sigma} + \\
& + \delta^{\tau}_{b}W^{\alpha}{}_{icda}{}^{\sigma} + \delta^{\tau}_{a}W^{\alpha}{}_{idcb}{}^{\sigma}) + \delta^{\alpha}_{i}(\delta^{\tau}_{d}W^{\beta}{}_{jabc}{}^{\sigma} + \\
& + \delta^{\tau}_{c}W^{\beta}{}_{jbda}{}^{\sigma} + \delta^{\tau}_{b}W^{\beta}{}_{jcda}{}^{\sigma} + \delta^{\tau}_{a}W^{\beta}{}_{jdcb}{}^{\sigma}) + \\
& + \frac{2}{n-1}(g_{ca}g_{bd} - g_{ab}g_{cd})(\delta^{\beta}_{j}R^{\alpha}{}_{i}{}^{\sigma\tau} + \delta^{\alpha}_{i}R^{\beta}{}_{j}{}^{\sigma\tau}),
\end{aligned}
$$

where $T^{\sigma}{}_{abc}$ and $W^{\beta}{}_{jbda}{}^{\sigma}$ are given by formulas (12) and (13), respectively. Contracting in (25) first the indices α, i, then the indices β, j, we get the identities

$$
\begin{aligned}
(26)\quad & \delta^{\tau}_{d}T^{\sigma}{}_{abc} + \delta^{\tau}_{c}T^{\sigma}{}_{bad} + \delta^{\tau}_{b}T^{\sigma}{}_{cda} + \\
& + \delta^{\tau}_{a}T^{\sigma}{}_{dcb} = \delta^{\sigma}_{d}T^{\tau}{}_{abc} + \delta^{\sigma}_{c}T^{\tau}{}_{bad} + \\
& + \delta^{\tau}_{b}T^{\tau}{}_{cda} + \delta^{\sigma}_{a}T^{\tau}{}_{dcb}.
\end{aligned}
$$

From (26) we draw, contracting the indices τ, d,

$$
(27)\quad T^{\sigma}_{abc} = \delta^{\sigma}_{c}(R_{ab} - \frac{R}{n-1}g_{ab}) + (-R_{ac} + \frac{R}{n-1}g_{ac})\delta^{\sigma}_{b}
$$

Substituting into (12) the identities (27) we get

$$
C^{\sigma}{}_{abc} = 0.
$$

3. Sufficient condition for the existence of a conformal Killing tensor of type 2/2

Let V_{ijcd} be a conformal Killing tensor of type 2/2. Transvecting the tensor g^{ab} onto (17) and substituting (7) and the identity $R^{e}{}_{[abc]} = 0$ we get

$$(28)\qquad \begin{aligned} &V_{ijcd;}{}^{a}{}_{;a} - V_{ijde}R^{e}{}_{c} - V_{ijbe}R^{e}{}_{dc}{}^{b} + \\ &- (V_{ejbd}R^{e}{}_{ic}{}^{b} - V_{iebd}R^{e}{}_{jc}{}^{b}) + \\ &- \tfrac{1}{2}(n-1)g_{cd}(V_{ej}{}^{ba}R^{e}{}_{iba} + V_{ie}{}^{ba}R^{e}{}_{jba} = \\ &= -(n-3)f_{ijd;c} - f_{ijc;d}. \end{aligned}$$

Taking the antisymmetric part of (28) with respect to c,d we get

$$(29)\qquad \begin{aligned} &2V_{ijcd;}{}^{a}{}_{;a} - V_{ijde}R^{e}{}_{c} - V_{ijce}R^{e}{}_{d} + \\ &- R_{dc}{}^{be}V_{ijbe} = (n-4)(f_{ijc;d} - f_{ijd;c}). \end{aligned}$$

In this section we are going to show that the tensor V_{ijcd} satisfying (28) and (29) is a conformal Killing tensor of type 2/2 in the compact space M. For this purpose we shall establish an integral formula for V_{ijcd}. Let us define the tensor A_{ijbcd} by the formula

$$(30)\qquad \begin{aligned} A_{ijbcd} = {} & V_{ijcd;b} + V_{ijbd;c} - 2f_{ijd}g_{bc} + \\ & + f_{ijc}g_{bd} + f_{ijb}g_{cd}, \end{aligned}$$

where f_{ijd} is given by formula (8).

Differentiating (30), substituting (7) and transvecting g^{ab} we get

$$(31)\quad A_{ij}{}^{a}{}_{cd;a} = V_{ijcd;}{}^{a}{}_{;a} + (n-3)f_{ijd;c} + f_{ijc;d} + \\ - (V_{ejbd}R^{e}{}_{ic}{}^{b} + V_{iebd}R^{e}{}_{jc}{}^{b}) - V_{ijde}R^{e}{}_{c} + \\ - V_{ijbe}R^{b}{}_{cd}{}^{e} - \frac{1}{2(n-1)} g_{dc}(V_{ejba}R^{e}{}_{i}{}^{ba} + V_{ieba}R^{e}{}_{j}{}^{ba}).$$

T h e o r e m 2. In the compact orientable Riemannian space M for any tensor satisfying condition (2) the following integral formula holds:

$$(32)\quad \int_V \Big\{ V^{ijcd}\Big[V_{ijcd;}{}^{a}{}_{;a} - (V_{ejbd}R^{e}{}_{ic}{}^{b} + V_{iebd}R^{e}{}_{jc}{}^{b} + \\ - V_{ijde}R^{e}{}_{c} + V_{ijbe}R^{b}{}_{cd}{}^{e} - \frac{1}{2(n-1)} g_{dc}(V_{ejba}R^{e}{}_{i}{}^{ba} + \\ + V_{ieba}R^{e}{}_{j}{}^{ba} + (n-3)f_{ijd;c} + f_{ijc;d}\Big] + \\ + \frac{1}{2} A_{ijbcd}A^{ijbcd}\Big\} d\sigma = 0,$$

where $d\sigma$ denotes the volume element of M.

P r o o f . Consider A_{ijbcd} given by formula (30) and introduce the scalar function $A_{ijbcd}A^{ijbcd}$

$$(33)\qquad A_{ijbcd}A^{ijbcd} = 2A_{ijbcd}V^{ijcd;b}.$$

From (33) we find $A_{ijbcd}V^{ijcd;b}$ and substitute it into formula

$$(34)\qquad g^{ab}(A_{ijbcd}V^{ijcd})_{;a} = \\ = g^{ab}(A_{ijbcd;a}V^{ijcd} + V^{ijcd}{}_{;a}A_{ijbcd}).$$

Substituting (31) and (33) into (34) and integrating we finally get the integral formula (32).

Taking into account the integral theorem of paper [3] we obtain the following theorem.

T h e o r e m 3. In a compact and orientable Riemannian space the tensor V_{ijcd} satisfying condition (2) is a conformal Killing tensor of type 2/2 if and only if the equations (28) and (29) hold.

E x a m p l e . Let E^{n+1} be an Euclidean space with the orthogonal coordinate system $\{y^{\lambda}\}$, $\lambda = 1,2,\dots,n+1$, and S^n the unit sphere with the local coordinate system $\{x^a\}$, $a = 1,\dots,n$. By $C_a^{\lambda} = \frac{\partial y^{\lambda}}{\partial x^a}$ we denote the intermediate tensor of these two spaces.

The curvature tensor of the embedded space S^n is expressed by the formula

$$(1')\qquad H^{\lambda}{}_{ba} = C^{\lambda}_{a;b} = \partial_b C_a^{\lambda} - C_c^{\lambda}\,'\Gamma^c{}_{ba} + C_a^{\mu}\Gamma^{\lambda}_{\mu\nu}C_b^{\nu},$$

where:

$\Gamma^{\lambda}_{\mu\nu}$ is the connection object of the surrounding space and $'\Gamma^c{}_{ba}$ the connection object of the embedded space.

Since S^n is an umbilical surface we may write

$$(2')\qquad H^{\lambda}{}_{ba} = g_{ab}N^{\lambda},$$

where N^{λ} is the unit normal vector to the sphere.

Let $\omega_{\mu\nu}$ be a skew-symmetric, covariantly constant field in E^{n+1}; then the tensor ω_{ab} on S^n is expressed by the formula

$$(3')\qquad \omega_{ab} = C_a^{\mu}C_b^{\nu}\omega_{\mu\nu}.$$

We denote by V_{cd} a covariantly constant field on S^n. Let us now consider the tensor

$$(4')\qquad U_{cdab} = V_{cd}\cdot\omega_{ab}.$$

It is easy to see that this tensor satisfies condition (2). Taking its covariant derivative we get

(5')
$$U_{cdba;e} = V_{cd;e}\omega_{ab} + (C^{\mu}_{a;e}C^{\nu}_{b}\omega_{\mu\nu} + C^{\mu}_{a}C^{\nu}_{b;e}\omega_{\mu\nu} + C^{\mu}_{a}C^{\nu}_{b}C^{\alpha}_{e}\omega_{\mu\nu;\alpha}).$$

Hence, making use of (1') and of the fact that the covariant derivatives are constant, we draw

(6')
$$U_{cdab;a} = V_{cd}(H^{\mu}{}_{ae}C^{\nu}{}_{b} + C^{\mu}_{a}H^{\nu}{}_{be})\omega_{\mu\nu}$$

or, which is equivalent,

(7')
$$U_{cdab;e} = (N^{\mu}g_{ae}C^{\nu}{}_{b} - N^{\nu}g_{be}C^{\mu}_{a})V_{cd}\omega_{\mu\nu}$$

We set $f_{cdb} = N^{\mu}C^{\nu}{}_{b}V_{cd}\omega_{\mu\nu}$; then

$$U_{cdab;e} = f_{cdb}g_{ae} - f_{cda}g_{be}$$

that is

$$U_{cdab;e} + U_{cdeb;a} = 2f_{cdb}g_{ae} - f_{cda}g_{be} - f_{cde}g_{ab}$$

which means that U_{cdab} is a conformal Killing tensor of type 2/2.

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INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW
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