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ON GENERALIZED STRONG NÖRLUND SUMMABILITY FIELDS

1. Introduction

The convergence fields of generalized strong Nörlund summability have been investigated by Kumar [2], Sinha [6] and Schaper [5]. In this paper we discuss the relations between the convergence fields $0[N,p,\alpha]_\lambda$ and $0[N,p*r,\alpha]_\lambda$, $r \in 1$, $\lambda > 1$. The results are generalisations of known theorems of Kuttner and Thorpe [3] ($\alpha = 1$, $\lambda = 1$) and Schaper [5]($\lambda = 1$).

2. Preliminaries

Let $\$:= \{s \mid s : \mathbf{N}_0 \to \mathbf{C}\}$ be the set of complex sequences. If $s,t \in \$$, then $s*t \in \$$ is defined by

$$s*t_n := (s*t)_n := \sum_{\nu=0}^{n} s_{n-\nu}t_\nu, \qquad n \geq 0.$$

We define $\frac{s}{t} \in \$$ by

$$\frac{s}{t}n := \begin{cases} \frac{s_n}{t_n} & \text{if} \quad t_n \neq 0 \\ 0 & \text{if} \quad t_n = 0. \end{cases}$$

We write $s \leq t$ if $s_n, t_n \in \mathbf{R}$ and $s_n \leq t_n$ for all n.

If $s \in \$$ we define as usual $\Delta s \in \$$ by

$$\Delta s_0 := s_0, \quad \Delta s_n := s_n - s_{n-1}$$

and $|s| \in \$$ by $|s|_n := |s_n|$.

We often use the following abbreviations

$$1 := \{1,1,1,\dots\},$$
$$e := \{1,0,0,\dots\}.$$

It can be easily verified that

$$\Delta s * t := \Delta(s * t) = (\Delta s) * t = s * (\Delta t),$$
$$\Delta s * 1 = s, \quad s * e = s.$$

We consider now the following sets of sequences:

$$\Omega := \{s \in \$ \mid s_n \neq 0 \quad \text{for almost all} \quad n\},$$

$$\ell := \{s \in \$ \mid \sum_{n=0}^{\infty} |s_n| < \infty\},$$

$$o(t) := \{s \in \$ \mid \lim_{n\to\infty} \frac{s}{t} n = 0\} \quad \text{for} \quad t \in \$,$$

$$O(t) := \{s \in \$ \mid |s| < K|t|\} \quad \text{for} \quad t \in \$.$$

Let α, $p \in \$$ be such that $p_0 \neq 0$, $\alpha_n \neq 0$ for $n \geq 0$ and $p * \alpha \in \Omega$, then $\frac{p*\alpha s}{p*\alpha}$ is the generalized Nörlund mean of $s \in \$$. The sequence s is said to be generalized Nörlund summable to zero if $p*\alpha s \in o(p*\alpha)$; [1], [5]. Then we write

$$o(N,p,\alpha) := \{s \in \$ \mid p*\alpha s \in o(p*\alpha)\}.$$

If furthermore $\Delta p * \alpha \in \Omega$ we define generalized strong Nörlund summability. The sequence s is said to be generalized strongly summable (N,p,α) with index $\lambda (\lambda > 0)$ to σ, if

$$\left\{\sum_{\nu=0}^{n} |\Delta p*\alpha_\nu| \cdot \left|\frac{\Delta p*\alpha s}{\Delta p*\alpha}\nu - \sigma\right|^\lambda\right\} \in o(p*\alpha)$$

and this is denoted by $s \to \sigma\ [N,p,\alpha]_\lambda$; [2], [6]. Since the sequence on the left side is monotonic, this definition is of use only if $|p*\alpha|_n \to \infty$ as $n \to \infty$.

We have $s \to \sigma [N,p,\alpha]_\lambda$ if and only if $\{s_n - \sigma\} \to 0$ $[N,p,\alpha]_\lambda$, hence we only consider the limits $0\ [N,p,\alpha]_\lambda$.

If $t, \Delta t \in \Omega$ and $\lambda > 0$ we define the set

$$s_\lambda(t) := \left\{ s \in \$ \,\middle|\, \left\{ \sum_{\nu=0}^{n} |\Delta t_\nu| \cdot \left| \frac{\Delta s}{\Delta t}\nu \right|^\lambda \right\} \in o(t) \right\} .$$

So we have $0[N,p,\alpha]_\lambda := \{ s \in \$ \mid p*\alpha s \in s_\lambda(p*\alpha)\}$. Thus the structure of the convergence field $0[N,p,\alpha]_\lambda$ is determined by the structure of the set $s_\lambda(p*\alpha)$.

For the remainder of this paper, we shall assume that if (N,p,α) is a generalized Nörlund method, then $p_0 \neq 0$, $\alpha_n \neq 0$ for $n \geq 0$ and $p*\alpha \in \Omega$. Considering strong summability we furthermore shall assume $\Delta p*\alpha \in \Omega$, $\lim_{n\to\infty} |p*\alpha|_n = \infty$ and $\lambda > 1$ unless mention is made to the contrary.

If $p \in \$$ we write $P(z) = \sum_{\nu=0}^{\infty} p_\nu z^\nu$ and denote the radius of convergence of this power series by $\varrho(p)$. We use similar notations with other letters in place of p.

If $p_0 \neq 0$ there exists a $k \in \$$ such that $p*k = e$. If furthermore $\varrho(p) > 0$, then also $\varrho(k) > 0$ and for $|z| < \varrho(k)$

(1) $$K(z) = \frac{1}{P(z)} = \sum k_\nu z^\nu$$

holds.

We often write $\sum$ insted of $\sum_{\nu=0}^{\infty}$.

3. Representation theorem

We first give a representation theorem which is an easy consequence of the definition of $[N,p,\alpha]_\lambda$ summability.

T h e o r e m 1. Suppose that $s_\lambda(p*\alpha) \subseteq \{x \in \$ \mid \varrho(x) \geq 1\}$ for $\lambda > 0$. Then $s \in O[N,p,\alpha]_\lambda$ if and only if there exists a $g \in s_\lambda(p*\alpha)$ such that

$$AS(z) := \sum \alpha_\nu s_\nu z^\nu = \frac{\sum g_\nu z^\nu}{\sum p_\nu z^\nu} = \frac{G(z)}{P(z)}$$

for $|z| < \min(1, \varrho(k))$.

Lemma 5 will give conditions that guarantee the inclusion $s_\lambda(p*\alpha) \subseteq \{x \mid (x) \geq 1\}$.

4. Inclusion theorem

We first list some conditions on α, p, $r \in \$$. So we have

(2) $$\lim_{n\to\infty} \frac{p*\alpha_{n-1}}{p*\alpha_n} = 1.$$

There is an M_1 such that for all n, μ with $n \geq \mu \geq 0$

(3) $$\left|\frac{p*\alpha_{n-\mu}}{p*\alpha_n}\right| < M_1.$$

(4) $$\lim_{n\to\infty} \frac{\Delta p*\alpha_{n-1}}{\Delta p*\alpha_n} = 1.$$

There is an M_2 such that for all n, μ with $n \geq \mu \geq 0$

(5) $$\left|\frac{\Delta p*\alpha_{n-}}{\Delta p*\alpha_n}\right| < M_2.$$

(6) Let $r, u, v \in \ell$ be such that

$$R(z) = U(z)\cdot V(z)\cdot \prod_{i=1}^{k} \left(1 - \frac{z}{a_i}\right)^{\varrho_i} \quad \text{for } |z| \leq 1,$$

with $U(z) \neq 0$, $V(\xi) \neq 0$ for $|\xi| < 1$ and $\xi = 1$, $\varrho_i \geq 0$, $0 < |a_i| < 1$.

Let $w \in \$$ be such that $v*w = e$.

We use the conditions (2) and (3) instead of the conditions of the regularity of (N,p,α). The conditions (2) and (3) are independent of the conditions of regularity of (N,p,α) since there are regular (N,p,α) methods which do not satisfy (2) and (3) and there are also (N,p,α) methods which satisfy (2) and (3) but are not regular [5].

The conditions (2) and (3) are the exact conditions for $o(p*\alpha) = o(p*r*\alpha)$ and (2)-(5) guarantee $s_\lambda(p*\alpha) = s_\lambda(p*r*\alpha)$ (see Lemma C and Lemma 2).

Theorem 2. Let $r \in \ell$ be such that $R(0) \neq 0$, $R(1) \neq 0$. Suppose that the conditions (2)-(5) hold and $\varrho(p) > 0$. Then

$$o[N,p,\alpha]_\lambda \subseteq o[N,p*r,\alpha]_\lambda .$$

5. Structure theorems

Theorem 3. Suppose that the conditions (2)-(6) hold with $\varrho_i > 0$, $v = w = e$ and $\varrho(p) > 0$. If we define $g := p*r*\alpha s$, where $s \in o[N,p\ r,\alpha]_\lambda$, then $\varrho(g) \geq 1$. Let G satisfy

$$G(z) = {}_1G(z) \cdot \prod_{i=1}^{k} \left(1 - \frac{z}{a_i}\right)^{\gamma_i} \tag{7}$$

for $|z| < 1$ with ${}_1G(a_i) \neq 0$, $\gamma_i \geq \varrho_i$, i.e. the root a_i of R with multiplicity ϱ_i is a root of G with multiplicity γ_i, $\gamma_i \geq \varrho_i$. Then $s \in o[N,p,\alpha]_\lambda$.

Theorem 4. Suppose that (2)-(6) hold with $\varrho_i > 0$, $v = w = e$ and $p \in s_\lambda(p*\alpha)$, then $\varrho(1) \geq 1$. Let P satisfy

$$P(z) = {}_1P(z) \cdot \prod_{i=1}^{k} \left(1 - \frac{z}{a_i}\right)^{\pi_i} \tag{8}$$

for $|z| < 1$ with ${}_1P(a_i) \neq 0$, $\pi_i \geq 0$, i.e. the root a_i of R is a root of P with multiplicity π_i. If

$$D := \left\{ d \in \$ \,\middle|\, \alpha_n d_n = \sum_{i=1}^{k} a_i^{-n} \sum_{j=\pi_i+1}^{\pi_i+\rho_i} c_{ij}\binom{n+j-1}{n},\ c_{ij} \in C \right\},$$

then $O[N,p*r,\alpha]_\lambda = \{s \mid s = t + d,\ t \in O[N,p,\alpha]_\lambda,\ d \in D\}$.

If R has no zero, then $D = \emptyset$ and we get the following corollary.

C o r o l l a r y 1. If the conditions (2)-(6) hold with $R(z) \neq 0$ for $|z| \leq 1$ and let $p \in s_\lambda(p*\alpha)$, then $O[N,p*r,\alpha]_\lambda = O[N,p,\alpha]_\lambda$.

T h e o r e m 5. If the conditions (2)-(6) hold with $\rho_i = 0$ and $\rho(p) > 0$, then

$$O[N,p*r,\alpha]_\lambda = \{s \mid \alpha s = w*\alpha t,\ t \in O[N,p,\alpha]_\lambda\}.$$

T h e o r e m 6. If the conditions (2)-(6) hold and p satisfies the conditions of Theorem 4, then

$$O[N,p*r,\alpha]_\lambda = \{s \mid \alpha s = (w*\alpha t) + (w*\alpha d),\ t \in O[N,p,\alpha],\ d \in D\} =$$
$$= \{s \mid \alpha s = (w*\alpha \tau) + \delta,\ \tau \in O[N,p,\alpha],\ \delta \in D\}.$$

If $\lambda = 1$ Theorems 1-6 hold under the conditions (2) and (3) instead of (2)-(6); [5]. In [5] we also proved analogous theorems in the case of ordinary and absolute generalized Nörlund summability.

The case $\lambda = 1$ and $\alpha = 1$ was treated by Kuttner and Thorpe [3] under the condition of regularity of (N,p,1) which is stronger than (2) and (3); ([5], p.36).

Naturally in special cases conditions (2)-(6) are not independent. Let us denote $m := p*\alpha$ and suppose $\Delta m_n \neq 0$, $n \geq 0$. Then we have

$$\frac{m_{n-1}}{m_n} = \sum_{\nu=0}^{n} \frac{\Delta m_\nu}{m_\nu} \frac{\Delta m_{\nu-1}}{\Delta m_\nu}.$$

Hence $\lim \frac{\Delta m_{n-1}}{\Delta m_n} = 1$ implies $\lim \frac{m_{n-1}}{m_n} = 1$ if $(N,1,\Delta m)$ is regular. Since $\lim |m_n| = \infty$, this means $|\Delta m| * 1 \in O(m)$ (by the well known Toeplitz-theorem [4], p.11).

If m is a real sequence, then $\lim \frac{\Delta m_{n-1}}{\Delta m_n} = 1$ implies the existence of a number N such that all Δm_n with $n \geq N$ have the same sign. Hence by

$$\sum_{\nu=0}^{n} |\Delta m_\nu| = \sum_{\nu=0}^{N-1} |\Delta m_\nu| + \left|\sum_{\nu=N}^{n} \Delta m_\nu\right|$$

and $\lim |m_n| = \infty$ we get $|\Delta m| * 1 \in O(m)$.

On the other hand $|\Delta m| * 1 \in O(m)$ implies $\left|\frac{m_{n-\mu}}{m_n}\right| < M_3$ for all n, $\mu \geq 0$, since

$$|m_{n-\mu}| = |(\Delta m * 1)_{n-\mu}| \leq (|\Delta m| * 1)_{n-\mu} \leqslant$$
$$\leqslant (|\Delta m| * 1)_n \leq M_3 |m_n|.$$

Since also $\frac{m_{n-\mu}}{m_n} = \sum_{\nu=0}^{n} \frac{\Delta m_\nu}{m_n} \frac{\Delta m_{\nu-\mu}}{\Delta m_\nu}$

by $\left|\frac{\Delta m_{n-\mu}}{\Delta m_n}\right| < M_4$ and $|\Delta m| * 1 \in O(m)$, so we get $\left|\frac{m_{n-\mu}}{m_n}\right| < M_5$ (uniformly in n, μ).

We remark that if $p \in s_\lambda(p*\alpha)$ is not assumed, the conclusion of Theorem 4 may be false even in simple cases. But we cannot use the same counterexample as Kuttner and Thorpe ([3], p.393), since in their example $p_{2n+1} = 0$, $n \geq 0$. If we take $\alpha = 1$ and $P(z) := (1-z)^{-1} + (1-z^2)^{-1}$, then $p_{2n} = 2$, $p_{2n+1} = 1$, $n \geq 0$. It follows by

$\sum_{\nu=0}^{n} |\Delta p * 1|_\nu^{\lambda-1} \cdot |\alpha p|_\nu^{\lambda} \geq n$ that $p \notin s_\lambda(p*1)$.

Let $R(z) := 1 + 2z$ and $S(z) := (1 + 2z)^{-1}$. If we denote $q := p*r$ and $w := \Delta c*r*s$ then an easy computation gives $q_0 = 2$, $q_{2n-1} = 5$, $q_{2n} = 4$, $w_0 = 2$, $w_n = (-1)^n$, $n \geq 1$. Since $c_n := \sum_{\nu=0}^{n} |\Delta q*1|_\nu^{\lambda-1} \cdot |w|_\nu^\lambda \geq \sum_{\nu=0}^{n} 2^{\lambda-1} \geq n$, $\lambda > 1$, and $5\cdot(n+1) \geq (q*1)_n$, we get $c \notin o(q*1)$, that means $s \notin O[N,p*r,1]_\lambda$.

6. Proofs of the theorems

Theorems 1 - 6 are special cases of the general representation theorems proved in [5]. These theorems are of the same structure as Theorems 1 - 6 but they deal with some "abstract" sets $e(m,p,\alpha)$ and $e(p*\alpha)$ instead of $O[N,p,\alpha]_\lambda$ and $s_\lambda(p*\alpha)$. First we state these general theorems and we prove some suitable lemmas on $s_\lambda(p*\alpha)$.

Let $e(m)$ be a set of sequences such that

$$e(m) \subseteq \{x \in \$ \mid \rho(x) \geq 1\} \quad \text{for } m \in \$.$$

(e.g. $e(m) = o(m)$ or $e(m) = s_\lambda(m)$ under certain conditions on m).

We define the set

$$e(m,p,\alpha) := \{s \in \$ \mid p*\alpha s \in e(m)\} \tag{10}$$

where $m,p,\alpha \in \$$ with $\rho(p) > 0$, $p_0 \neq 0$, $\alpha_n \neq 0$ for $n \geq 0$. (Later we will have $m = p*\alpha$ and if e.g. $e(m) = o(p*\alpha)$, then $e(m,p,\alpha) = o(N,p,\alpha)$).

The following theorems are proved in [5].

T h e o r e m A. If the conditions (1) and (10) hold, then $s \in e(m,p,\alpha)$ if and only if there exists a $g \in e(m)$ such that

$$\alpha S(z) := \sum \alpha_\nu s_\nu z^\nu = \frac{\sum g_\nu z^\nu}{\sum \rho_\nu z^\nu} = \frac{G(z)}{P(z)}$$

for $|z| < \min\{1,\rho(k)\}$.

T h e o r e m B. If condition (10) holds, $r \in \ell$ with $R(0) \neq 0$ and

$$\{x \mid x = b*r,\ b \in e(m)\} \subseteq e(m*r)$$

hold, then $e(m,p,\alpha) \subset e(m*r,p*r,\alpha)$.

For the next theorems we need the following conditions:

(11) $e(m)$ is a complex vector space.

(12) $e(m) = e(m*r)$ for any $r \in \ell$ with $R(0) \neq 0$, $R(1) \neq 0$.

(13) $e(m) \supseteq \{x \mid x = b*r,\ b \in e(m),\ r \in \ell\}$.

(14) $e(m) \supseteq \{x \mid x_n = \sum b_{n+\nu} h_\nu,\ b \in e(m),\ \rho(h) > 1\}$.

T h e o r e m C. Suppose that the conditions (6), (10)-(14) hold with $\rho_1 > 0$, $v = w = e$. If $g := p*r*\alpha s$, where $s \in e(m*r,p*r,\alpha)$, then $\rho(g) \geq 1$. If G satisfies (7), then $s \in e(m,p,\alpha)$.

T h e o r e m D. If the conditions (6), (10)-(14) hold with $\rho_1 > 0$, $v = w = e$ and $p \in e(m)$, then $\rho(p) \geq 1$. If P satisfies (8) and D is as in (9), then

$$e(m*r,p*r,\alpha) = \{s \mid \alpha s = t + d,\ t \in e(m,p,\alpha),\ d \in D\}.$$

T h e o r e m E. If the conditions (6), (10), (12), (13) hold with $\rho_1 = 0$, then

$$e(m*r,p*r,\alpha) = \{s \mid \alpha s = w*\alpha t,\ t \in e(m,p,\alpha)\}.$$

T h e o r e m F. If the conditions (6), (10) - (15) hold and p satisfies the conditions of Theorem D, then

$$e(m*r,p*r,\alpha) = \{s \mid \alpha s = (w*\alpha t) + (w*\alpha d),\ t \in e(m,p,\alpha),\ d \in D\} =$$
$$= \{s \mid \alpha s = (w*\alpha\tau) + \delta,\ \tau \in e(m,p,\alpha),\ \delta \in D\}.$$

Now we want to prove some lemmas on $s_\lambda(m)$. We need three known lemmas [5] and the following conditions.

(15) $$\lim_{n\to\infty} \frac{m_{n-1}}{m_n} = 1.$$

There is an M_6 such that for all n, μ with $n \geq \mu \geq 0$

(16) $$\left|\frac{m_{n-1}}{m_n}\right| < M_6.$$

(17) $$\lim_{n\to\infty} \frac{\Delta m_{n-1}}{\Delta m_n} = 1.$$

There is an M_7 such that for all n, μ with $n \geq \mu \geq 0$

(18) $$\left|\frac{\Delta m_{n-\mu}}{\Delta m_n}\right| < M_7.$$

L e m m a A. Let $r \in \ell$. Then $\lim_{n\to\infty} \frac{m*r}{m}n = R(1)$ if and only if (15) and (16) hold.

L e m m a B. If (15) hold and $\eta > 0$ then there exists an M_8 such that for all n, μ

$$\left|\frac{m_{n+\mu}}{m_n}\right| < M_8 \ (1+\eta)^{\mu}.$$

L e m m a C. Let $r \in \ell$ with $R(1) \neq 0$. Then $o(m) = o(m*r)$ if and only if (15) and (16) hold.

L e m m a 1. The set $s_\lambda(m)$ where $\lambda > 1$ is a complex vector space.

The proof follows by Minkowski's inequality.

L e m m a 2. If $r \in \ell$ with $R(1) \neq 0$ and (15) - (18) hold, then for $\lambda > 0$ $s_\lambda(m) = s_\lambda(m*r)$.

R e m a r k . By an easy consideration we get $s_\lambda(m) = s_\lambda(\tilde{m})$ if $m_n = \tilde{m}_n$ for $n \geq N_0$. Therefore it is no loss of generality to assume $m_n \neq 0$, $m*r_n \neq 0$, $\Delta m_n \neq 0$, $\Delta m*r_n \neq 0$ for all n.

P r o o f of Lemma 2: If $x \in s_\lambda(m)$, then

$$|\Delta m| \cdot \left|\frac{\Delta x}{\Delta m}\right|^{\lambda} = |\Delta m*r| \cdot \left|\frac{\Delta x}{\Delta m*r}\right|^{\lambda} \cdot \left|\frac{\Delta m*r}{\Delta m}\right|^{\lambda}.$$

Since $\lim \frac{\Delta m * r}{\Delta m} n = R(1) \neq 0$ by Lemma A and $o(m) = o(m * \dot{r})$ by Lemma C we get. $x \in s_\lambda(m*r)$. There other direction is similar. Q.E.D.

Lemma 3. Let $\lambda > 1$ and $\lim |m_n| = \infty$. If (16) and (18) hold, then

$$s_\lambda(m) \supseteq \left\{ x \,|\, x = b*r,\ b \in s_\lambda(m),\ r \in \ell \right\}.$$

Proof. Let $b \in s_\lambda(m)$ and define $c := \frac{\Delta b}{\Delta m}$, $y := \frac{\Delta b*r}{\Delta m}$, then $y \cdot \Delta m = r*(c \cdot \Delta m)$. Thus using Hölder's inequality and (18) we obtain

$$|y \cdot \Delta m|^\lambda \le (|r| * |c \cdot \Delta m|)^\lambda \le$$

$$\le (|r| * |\Delta m|)^{\lambda-1} \cdot (|r| * |c^\lambda \cdot \Delta m|) \le$$

$$\le \frac{M_8}{|\Delta m|^{\lambda-1}} (|r| * 1)^{\lambda-1} \cdot (|r| * |c^\lambda \cdot \Delta m|).$$

This means $|\Delta m| \cdot |y|^\lambda \le M_{10} \cdot (|r| * |\Delta m| \cdot |c|^\lambda)$. Hence

$$\sum_{\nu=0}^{n} |\Delta m_\nu| \cdot |y|_\nu^\lambda \le M_{10} \cdot \sum_{\nu=0}^{n} \sum_{\mu=0}^{\nu} |r_{\nu-\mu} \cdot \Delta m_\mu| \cdot |c|_\mu^\lambda \le$$

$$\le M_{10} \cdot \sum_{\mu=0}^{n} |r_{\nu-\mu}| \sum_{\nu=0}^{\mu} |\Delta m_\nu| \cdot |c|_\mu^\lambda .$$

Denoting $\varepsilon := \frac{|\Delta m| \cdot |c| * 1}{|m|}$ we get $\varepsilon \in o(1)$ by $b \in s_\lambda(m)$ and

$$\frac{1}{|m_n|} \cdot \sum_{\nu=0}^{n} |\Delta m_\nu| \cdot |y|_\nu^\lambda \le M_{10} \frac{1}{|m_n|} \cdot \sum_{\nu=0}^{n} |r_{n-\nu}| \cdot |m_\nu| \cdot \varepsilon_\nu .$$

The right side tends to zero if the limitation method defined by

$$a_{n\nu} := \begin{cases} \left|\dfrac{r_{n-\nu} m_\nu}{m_n}\right|, & \nu \le n, \\ 0 \quad , & \nu > n, \end{cases}$$

transforms sequences of o(1) into sequences of o(1). By Toeplitz' theorem this holds if and only if $|m_n| \to \infty$ and $|r| * |m| \in O(m)$. The last condition follows from $r \in \ell$ and (16). Hence $b*r \in s_\lambda(m)$. Q.E.D.

L e m m a 4. If (15) and (17) holds, then for $\lambda > 1$ $s_\lambda(m) \supseteq \left\{x \mid x_n = \sum b_{n+\nu} h_\nu,\ b \in s_\lambda(m),\ \varrho(h) > 1\right\}$.

P r o o f . Let $b \in s_\lambda(m)$ and define $c := \frac{\Delta b}{\Delta m}$, $x_n := \Delta b_{n+\nu} h_\nu$, $y := \frac{\Delta x}{\Delta m}$. Hence we have $y_n \cdot \Delta m_n = \sum h_\nu c_{n+\nu} \Delta m_{n+\nu}$. If $\eta > 0$ such that $(1+\eta) < \varrho(h)$ and using Hölder's inequality and Lemma B we get

$$\begin{aligned} |y_n \cdot \Delta m_n|^\lambda &\le \left(\sum |h_\nu c_{n+\nu} \Delta m_{n+\nu}|\right)^\lambda \le \\ &\le \left(\sum |h_\nu \Delta m_{n+\nu}|\right)^{\lambda-1} \cdot \left(\sum |h_\nu \Delta m_{n+\nu}| \cdot |c_{n+\nu}|^\lambda\right) \le \\ &\le |\Delta m_n|^{\lambda-1} \cdot M_8 \cdot \left(\sum |h_\nu| \cdot (1+\eta)^\lambda\right)^{\lambda-1} \cdot \left(\sum |h_\nu \Delta m_{n+\nu}| \cdot |c_{n+\nu}|^\lambda\right). \end{aligned}$$

Hence $|\Delta m_n| \cdot |y|_n^\lambda \le M_{11} \sum |h_\nu \Delta m_{n+\nu}| \cdot |c_{n+\nu}|^\lambda$.

Denoting

$$\varepsilon_n := \frac{1}{|m_n|} \cdot \sum_{\mu=0}^{n} |h_\nu \Delta m_{n+\nu}| \cdot |c_{n+\nu}|^\lambda$$

and

$$\hat{\varepsilon}_n := \max_{\nu \ge 0} \varepsilon_{n+\nu} \quad \text{for} \quad n \ge 0$$

we have $\varepsilon \in o(1)$ and $\hat{\varepsilon} \in o(1)$ since $b \in s_\lambda(m)$. Also we get

$$\sum_{\mu=0}^{n} |\Delta m_\mu| \cdot |y|_\mu^\lambda \leq M_{11} \sum |h_\nu| \sum_{\mu=0}^{n} |\Delta m_{\mu+\nu}| \cdot |c_{\mu+\nu}|^\lambda \leq$$

$$\leq M_{11} \sum |h_\nu| \sum_{\mu=0}^{n+\nu} |\Delta m_\mu| \cdot |c|_\mu^\lambda \leq$$

$$\leq M_{11} \sum |h_\nu| \cdot |\Delta m_{n+\nu}| \cdot \varepsilon_{n+\nu} \leq$$

$$\leq M_{11} \cdot \hat{\varepsilon}_n \cdot |\Delta m_n| \cdot M_8 \cdot \sum |h_\nu| \cdot (1 + \eta)^\nu$$

by Lemma B. Hence $x \in s_\lambda(m)$. Q.E.D.

L e m m a 5. If (15) and (17) hold, then for $\lambda > 0$

$$s_\lambda(m) \subseteq \{x \mid \varrho(x) \geq 1\}.$$

P r o o f . If $x \in s_\lambda(m)$, then

$$|\Delta m|^{\lambda-1} \cdot |\Delta x|^\lambda \leq |\Delta m|^{\lambda-1} \cdot |\Delta x|^\lambda * 1 \in o(m) \subseteq O(m).$$

Hence

$$|\Delta x|^\lambda \leq \frac{M_{12}|m|}{|\Delta m|^{\lambda-1}} := c.$$

By (15) and (17) $\lim \frac{c_{n-1}}{c_n} = 1$, hence $\varrho(c) = 1$ and $\varrho(\Delta x) \geq 1$, $\varrho(x) \geq 1$. Q.E.D.

Now the proofs of the Theorems are direct consequences of the Theorems A - F and our Lemmas. Theorem 1 follows from Theorem A. Lemma 5 gives some information on $s_\lambda(m) \subset \{x \mid \varrho(x) \geq 1\}$. The proof of Theorem 2 follows from Theorem B and Lemmas 2 and 3. The proofs of the Theorems 3 - 6 follow from the corresponding Theorems C - F and Lemmas 1 - 5.

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