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AN APPLICATION OF QUASI-DIFFUSION PROCESSES TO A CHARACTERIZATION OF GAUSSIAN PROCESSES

Let X_t where $t \in (0,T)$ be a stochastic process with values in R^1, the covariance function $k_{ij} = K(t_i,t_j) = E(X_{t_i}X_{t_j})$ and positive definite matrix $[k_{ij}]_{ij=1}^n$ for $t_i \neq t_j$.

Let $t_1 < \ldots < t_{n+1}$, $\mathbf{t}_n = (t_1,\ldots,t_n)$, $\Delta t_n = t_{n+1}-t_n$, $\mathbf{x}_n = (x_1,\ldots,x_n) \in R^n$, $X_n = (X_{t_1},\ldots,X_{t_n})$,

$\mu_n = \mu_n(t_n,X_{n-1}) = E(X_{t_n}|X_{n-1})$, $\mu_1 = E(X_t) = 0$,

$\sigma_n^2 = \sigma_n^2(t_n,X_{n-1}) = E\left[(X_{t_n}-\mu_n)^2|X_{n-1}\right]$, $\sigma_1^2 = E(X_t^2)$,

$K^{(n)} = \det[k_{ij}]_{ij=1}^n$, $K_{ij}^{(n)}$ - the cofactor of k_{ij}.

It is well known that for gaussian processes we have

$$\mu_n = -\frac{1}{K^{(n-1)}}\sum_{i=1}^{n-1} K_{ni}^{(n)}\, x_i, \tag{1}$$

$$\sigma_n^2 = \frac{K^{(n)}}{K^{(n-1)}}. \tag{2}$$

In this paper we consider the following problem: Do (1), (2) characterize gaussian processes? In other words what is the class of processes for which it follows from (1), (2) that X_t is a gaussian process. The main assumption in our consideration will be $K \in C^2$.

In the paper [1] the analogical problem of characterization in the case $K \notin C^1$ was considered.

The case $K \in C^1$ and simultaneously $K \notin C^2$ is not solved yet. None of the proposed methods can be applied to this case.

We shall use the following denotations for probabilities and conditional probabilities

$$P^{(n)}(t_n, X_n; t_{n+1}, \ldots, t_{n+k}, A_1, \ldots, A_k) =$$

$$= \begin{cases} P(X_{t_{n+1}} \in A_1, \ldots, X_{t_{n+k}} \in A_k | X_n) & \text{for } n > 0 \\ P(X_{t_{n+1}} \in A_1, \ldots, X_{t_{n+k}} \in A_k) & \text{for } n = 0 \end{cases}$$

and the following denotations for integrals

$$\frac{1}{(\Delta t_n)^r} \int_A (x - x_n)^i \, P^{(n)}(t_n, x_n; t_{n+1}, dx) = Q_{ir}(t_{n+1}, x_n, A),$$

$$\frac{1}{(\Delta t_n)^r} \int_B P^{(n-1)}(t_{n-1}, x_{n-1}; t_n, t_{n+1}, dx, dy) = Q_r(t_{n+1}, x_{n-1}, B)$$

$$i = 0,1,2; \; r = 1,2.$$

We say that X_t is a quasi-diffusion process if for $n \geq 1$ and all ε, t_n, x_n we have

$$(3) \qquad \lim_{\Delta t_n \to 0} Q_{01}(t_{n+1}, x_n, V_\varepsilon(x_r)) = 0$$

and there exist limits

$$(4) \qquad \lim_{\Delta t_n \to 0} Q_{i1}(t_{n+1}, x_n, U_\varepsilon(x_n)) = a_{i1}(t_n, x_n), \qquad i = 1,2,$$

where

$$U_\varepsilon(x_n) = \{x : |x-x_n| < \varepsilon\}, \quad V_\varepsilon(x_n) = R^1 - U_\varepsilon(x_n).$$

Here $\Delta t_n \to 0$ always denotes $\Delta t_n \to 0^+$.

Introducing the quasi-diffusion process we do not assume that there exist conditional moments of the second order, we operate only the truncated conditional moments. In the present paper, for another reason, we assume the existence of the covariance function. Thus consequently we assume that (4) is satisfied also for $\varepsilon = +\infty$. We shall employ the following conditions

(3')
$$\lim_{\Delta t_n \to 0} Q_r^*(t_{n+1}, x_{n-1}, V_\varepsilon) = 0,$$

where

$$V_\varepsilon = \{(x,y) : |x-y| > \varepsilon\}$$

(5)
$$\lim_{\Delta t_n \to 0} \int_{(a,b)} Q_{ir}(t_{n+1}, x_n, R^1) P^{(n-1)}(t_{n-1}, x_{n-1}; t_n, dx_n) =$$

$$= \int_{(a,b)} \lim_{\Delta t_n \to 0} Q_{ir}(t_{n+1}, x_n, R^1) P^{(n-1)}(t_{n-1}, x_{n-1}; t_n, dx_n)$$

$i, r = 1,2$; (a,b) - an optional interval.

We shall assume that there exists a density $f^{(n)}$ of the measure $P^{(n)}$ for $n \geq 0$.

In [1] the following theorem has been proved

T h e o r e m 1. Let (3'), (4), (5) be satisfied for $r = 1$. Then the following equations hold

$$(6)\qquad \frac{\partial}{\partial t_n} f^{(n-1)}(t_{n-1},x_{n-1};t_n,x_n) + \frac{\partial}{\partial x_n}\left[a_{11}(t_n,x_n) f^{(n-1)}(t_{n-1},x_{n-1};t_n,x_n)\right] = \frac{1}{2}\frac{\partial^2}{\partial x_n^2}\left[a_{21}(t_n,x_n) f^{(n-1)}(t_{n-1},x_{n-1};t_n,x_n)\right], \quad n \geq 1$$

under the assumptions that all the derivatives in (6) are continuous.

If (1), (2) hold the following limits exist:

$$(7)\qquad \begin{cases} \lim\limits_{\Delta t_n \to 0} \frac{1}{\Delta t_n}\left[K(t_i,t_{n+1}) - K(t_i,t_n)\right] = k'_{in}, & i=1,\dots,n \\ \lim\limits_{\Delta t_n \to 0} \frac{1}{\Delta t_n}\left[K(t_{n+1},t_{n+1}) - K(t_n,t_{n+1})\right] = k'_n \end{cases}$$

and

$$(4\ )\qquad \lim_{\Delta t_n \to 0} Q_{21}(t_{n+1},x_n,V_\varepsilon(x_n)) = 0.$$

We then have

$$(8)\qquad a_{11}(t_n,x_n) = \lim_{\Delta t_n \to 0} \frac{1}{\Delta t_n}(\mu_{n+1}-x_n) = \frac{\partial}{\partial t_{n+1}} \mu_{n+1}\Big|_{t_{n+1}=t_n} = -\frac{1}{K^{(n)}} \sum_{i=1}^{n} K'^{(n+1)}_{i,n+1}\, x_i$$

$$(9)\qquad a_{21}(t_n,x_n) = \lim_{\Delta t_n \to 0} \frac{1}{\Delta t_n}\left[k_{n+1,n+1} - 2k_{n,n+1} + k_{nn}\right] = k'_n - k'_{nn},$$

where

$$K'^{(n+1)} = \begin{vmatrix} K^{(n)}, & k'_{nn} \\ k_{nn}, & 1 \end{vmatrix}$$

$$k_{nn} = (k_{n1},\dots,k_{nn}), \quad k'_{nn} = (k'_{1n},\dots,k'_{nn})$$

and $K'^{(n+1)}_{ij}$ - denotes the cofactor of the corresponding element.

In the paper [1] it was proved that, if there exists a constant $\delta > 0$ such that

(10) $$a_{21} > \delta > 0$$

and some regularity conditions hold then the unique fundamental solution of (6) is a gaussian density.

In this paper we shall consider the case inverse to (10), namely the case

(11) $$a_{21}(t_n,x_n) = 0.$$

The case (11) is characterized by the following two properties

P r o p e r t y 1. If $K \in C^1$, then (2) implies (11).

P r o o f. If $K \in C^1$ then the limits (7) exist and $k'_{nn} = k'_n$, thus (11) holds. Q.E.D.

From Property 1 it follows that

P r o p e r t y 2. If $K \in C^1$, X_t is a gaussian process then (11) holds.

Let X'_t denote the mean-square derivative of X_t.

P r o p e r t y 3. If there exists X'_t and

$$(5') \qquad \lim_{\Delta t_n \to 0} \int_{R^n} Q_{21}(t_{n+1}, x_n, R^1) P^{(o)}(t_n, dx_n) =$$

$$= \int_{R^n} \lim_{\Delta t_n \to 0} Q_{21}(t_{n+1}, x_n, R^1) P^{(o)}(t_n, dx_n)$$

then (11) holds for almost all x_n (mod $P^{(o)}(x_n, .)$).

P r o o f . By assumption the following limit exists

$$\lim_{h,k \to 0} \frac{1}{hk} \left[K(t+h,t+k) - K(t,t+h) - K(t+k,t) + K(t,t)\right].$$

Hence there exist limits

$$\lim_{h \to 0} \frac{1}{h^2} E(X_{t+h} - X_t)^2$$

and consequently we have

$$0 = \lim_{\Delta t_n \to 0} \frac{1}{\Delta t_n} E(X_{t_{n+1}} - X_{t_n})^2 =$$

$$= \lim_{\Delta t_n \to 0} \int_{R^{n-1}} Q_2^*(t_{n+1}, x_{n-1}, R^2) P^{(o)}(t_{n-1}, dx_{n-1}) =$$

$$= \lim_{\Delta t_n \to 0} \int_{R^n} Q_2(t_{n+1}, x_n, R^1) P^{(o)}(t_n, dx_n) =$$

$$= \int_{R^n} \lim_{\Delta t_n \to 0} Q_2(t_{n+1}, x_n, R^1) P^{(o)}(t_n, dx_n).$$

Thus (11) holds almost everywhere (mod $P^{(o)}(t_n, .)$).

We see that (11) holds in a large class of processes. If (11) holds, then (6) is a system of partial differential equations of the first order. In general for a_{11} given by (8) the solution in the class of density function is not unique. Thus we introduce additionally new conditions and new methods. Namely we shall base our considerations on conditions (3')-(5) not only for $r = 1$ but also for $r = 2$ and the fact that $K \in C^2$. In the case $K \in C^2$ we introduce the following notation

$$k''_{in} = \frac{\partial^2}{\partial t^2_{n+1}} k_{i,n+1}\Big|_{t_{n+1}=t_n}, \quad k''_{in} = (k''_{1n},\dots,k''_{in}), \quad 1 \leqslant i < n;$$

$$\tilde{k}''_{nn} = \frac{\partial}{\partial t_n} k'_{nn},$$

$$\hat{k}''_{nn} = \lim_{\Delta t_n \to 0} \frac{1}{\Delta t_n} (k_{n+1,n+1} - 2k_{n,n+1} + k_{nn}),$$

$$K''(n) = \begin{vmatrix} K^{(n-1)}, & k''_{n-1,n} \\ k_{n,n-1}, & 1 \end{vmatrix}.$$

If $K \in C^2$, then from Taylor's formula it follows that

$$k_{n,n+1} = k_{nn} + \Delta t_n k'_{nn} + \frac{1}{2} (\Delta t_n)^2 k''_{nn} + o(\Delta t_n)^2.$$

Hence for μ_{n+1} and a_{11} given respectively by (1), (8) we have

(12) $$\lim_{\Delta t_n \to 0} \frac{1}{(\Delta t_n)^2} (\mu_{n+1} - x_n - \Delta t_n a_{11}(t_n, x_n)) =$$

$$= \frac{1}{2K^{(n)}} \begin{vmatrix} K^{(n)}, & k''_{nn} \\ x_n, & 0 \end{vmatrix} = \frac{\partial^2}{2\partial t^2_{n+1}} \mu_{n+1}\Big|_{t_{n+1}=t_n} = \frac{1}{2} \mu''_n$$

For $\mu_{n+1}, \mu'_n, \mu''_n, \sigma^2_{n+1}$ given respectively by (1), (8), (12), (2) we denote

$$(1') \qquad \mu_{n+1} = x_n b_{n+1} + c_{n+1} = x_n b_{n+1}(t_{n+1}) + c_{n+1}(x_{n-1}, t_{n+1})$$

$$(8') \qquad a_{11}(t_n, x_n) = \mu'_n = x_n b'_n + c'_n = x_n b'_n(t_n) + c'_n(x_{n-1}, t_n)$$

$$(12') \qquad \mu''_n = x_n b''_n + c''_n = x_n b''_n(t_n) + c''_n(x_{n-1}, t_n)$$

$$(13) \qquad d^2_n = \lim_{\Delta t_n \to 0} \frac{1}{(\Delta t_n)^2} \sigma^2_{n+1},$$

where evidently

$$b'_n = \frac{\partial}{\partial t_{n+1}} b_{n+1} \Big|_{t_{n+1}=t_n}, \quad b''_n = \frac{\partial^2}{\partial t^2_{n+1}} b_{n+1} \Big|_{t_{n+1}=t_n}$$

and analogically for c'_n, c''_n.

In our considerations we shall use the following property proved in [1] and connected with the function defined by (1), (2), (8').

P r o p e r t y 4. If (1), (2), (7), (11) hold then we have

$$(14) \qquad \frac{\partial}{\partial t_n} \sigma^2_n = 2\sigma^2_n b'_n, \quad \tilde{\mu}_n = \frac{\partial}{\partial t_n} \mu_n = c'_n + \mu_n b'_n.$$

We are going to continue the investigation of the problem of characterization by considering the following cases and differential equations joined with them

I. $a_{21} > 0$, parabolic equations (the case considered in [1])

II. $a_{21} = 0$, $a_{22} = 0$, a trivial case,

III. $a_{21} = 0$, $a_{22} > 0$, $a_{11} \neq 0 \pmod{P^{(0)}(t_n, \cdot)}$, partial differential equations of the first order (6),

IV. $a_{21} = 0$, $a_{22} > 0$, $a_{11} = 0$ (mod $P^{(0)}(t_n,\cdot)$), ordinary differential equations of the first and of the second order, which will be given.

Ad.II. The fact that this case is trivial is explained by the following property:

P r o p e r t y 5. Let $K \in C^2$, and let (1), (2) hold. If $a_{22}(t_n,x_n) = 0$, then the distribution of $Y_n = (X_n, X'_{t_n})$ is improper (of the singular type).

P r o o f . It is evident that $k'_{in} = E(X_{t_i} X'_{t_n})$, $\hat{k}''_{nn} = E(X'_{t_n})^2$ and

$$a_{22}(t_n,x_n) - a_{11}^2(t_n,x_n) = d_n^2 = \frac{1}{K^{(n)}} \begin{vmatrix} K^{(n)}, & k'_{in} \\ k'_{in}, & \hat{k}''_{nn} \end{vmatrix}$$

and that $K^{(n)} d_n^2$ is the covariance matrix of Y_n. Thus $a_{22} = 0$ implies $d_n^2 = 0$, in other words the distribution of Y_n is improper. Q.E.D.

In the begining of the present paper it was assumed that $K^{(n)} > 0$ i.e. the distribution of X_n is proper. Thus the distribution of Y_n could be improper only in the very trivial cases.

Ad.III. Now we are going to prove the following theorem.

T h e o r e m 2. If (1), (2) hold, $K \in C^2$, $a_{11} \neq 0$ mod $P^{(0)}(t_n,\cdot)$), then the unique explicit solution $f^{(n-1)}$ of (6) is a gaussian density, and the family $\{f^{(n-1)}\}_{n \in N}$ satisfies the consistency conditions.

First we are going to prove three auxiliary lemmas, Lemma 1 has the analogical character to Property 4. This lemma expresses some relations between the parameters $b'_n, c'_n, b''_n, c''_n, d_n^2$ connected with the function $f^{(n)}$ with the parameters μ_n, σ_n^2 of the function $f^{(n-1)}$.

L e m m a 1. If the assumptions of Theorem 2 are satisfied, then we have

(15) $$(\tilde{b}''_n + (b'_n)^2 - b''_n)\,\sigma_n^2 = d_n^2$$

(16) $$(c''_n - b'_n c'_n - \tilde{c}''_n)\,\sigma_n^2 = d_n^2 \mu_n,$$

where

$$\tilde{b}''_n = \frac{\partial}{\partial t_n} b'_n, \quad \tilde{c}''_n = \frac{\partial}{\partial t_n} c'_n.$$

P r o o f . From the properties of determinants and from the fact that $K \in C^2$ it follows that

(17) $$((b'_n)^2 + \tilde{b}''_n)\,(K^{(n)})^2 =$$

$$= \left(\begin{vmatrix} K^{(n-1)}, & k'_{n-1,n} \\ k'_{n-1,n}, & \tilde{k}''_{nn} \end{vmatrix} + \begin{vmatrix} K^{(n-1)}, & k''_{n-1,n} \\ k_{n-1,n}, & 0 \end{vmatrix} \right) K^{(n)} +$$

$$- \begin{vmatrix} K^{(n-1)}, & k'_{n-1,n} \\ k_{n-1,n}, & k'_{nn} \end{vmatrix}^2 .$$

In virtue of (12), (12') we have

(18) $$b''_n K^{(n)} = \begin{vmatrix} K^{(n-1)}, & k''_{n-1,n} \\ k_{n-1,n}, & k''_{nn} \end{vmatrix} .$$

It is evident that

(19) $$d_n^2 \left[K^{(n)}\right]^2 = \begin{vmatrix} K^{(n)}, & k'_{nn} \\ k'_{nn}, & \hat{k}''_{nn} \end{vmatrix} K^{(n-1)}\,\sigma_n^2 .$$

Now let us denote

$$D = \begin{vmatrix} K^{(n)}, & k'_{nn} \\ k'_{nn}, & \hat{k}''_{nn} \end{vmatrix}.$$

Taking into account the known relation for the symmetric determinant

$$DD_{ij,n+1,n+1} - D_{i,n+1}D_{n+1,j} - D_{ij}D_{n+1,n+1} = 0,$$

the formulas (17) - (19) and the fact that $K \in C^2$ we get (15).

Analogically as (15) it can be shown that (16) holds.

L e m m a 2. If the assumptions of Theorem 2 are satisfied, then for fixed n the general explicit solution of (6) is

$$f^{(n-1)}(t_{n-1},x_{n-1};t_n,x_n) = 1/\sigma_n \, \varphi_n \left((x_n-\mu_n)/\sigma_n\right), \tag{20}$$

where φ_n is an optional function belonging to C^1.

P r o o f . Let us denote $t_n = t$, $x_n = x$

$$f^{(n-1)}(t_{n-1},x_{n-1};t_n,x_n) = g_n(t_n,x_n) = g_n(t,x). \tag{21}$$

Taking into account (8') we can write equation (6) in the following form

$$\frac{\partial g_n}{\partial t} + (xb'_n+c'_n)\frac{\partial g_n}{\partial x} + b'_n g_n = 0.$$

Using the standard methods for partial differential equations of the first order we solve equations [2]

$$\frac{dx}{xb'_n + c'_n} = dt = -\frac{dg_n}{g_n b'_n}.$$

In virtue of (14) the general solution of

$$b_n' dt = - dg_n/g_n$$

can be written in the following form

$$1/C_1 \; g_n(t,x) = \exp\left(-\int b_n' \, dt\right) = 1/\sigma_n.$$

Analogically in virtue of (14) the general solution of

$$\frac{dx}{xb_n' + c_n'} = dt$$

can be written in the following form

$$x = \left[\exp \int b_n' dt\right] \times \left[\int c_n' \exp\left(-\int b_n dt\right) dt + C_2\right] = (\mu_n/\sigma_n + C_2)\,\sigma_n$$

The general solution of (6) is then $\psi(C_1, C_2) = 0$, where ψ is an optional function belonging to C^1.

Finally the general explicit solution of (6) is given by

$$g_n(t,x) = 1/\sigma_n \; \varphi_n(C_2) = 1/\sigma_n \; \varphi_n((x-\mu_n)/\sigma_n),$$

where φ_n is an optional function belonging to C_1. Q.E.D.

L e m m a 3. If the assumptions of Theorem 2 hold, then g_n defined by (21) satisfies the following equation

$$(22) \qquad \sigma_n^2 \frac{\partial^2 g_n}{\partial x^2} + \frac{\partial}{\partial x}(x-\mu_n) g_n = 0.$$

P r o o f . Taking into account formula (1') and the fact that g_{n+1} is the density function of a distribution with the mean value μ_{n+1} and the variance σ_{n+1}^2 and substituting

$$1/\sigma_{n+1}(x-\mu_{n+1}) = 1/\sigma_{n+1}(x - x_n b_{n+1} - c_{n-1}) = u$$

we have

(23) $$1/\sigma_{n+1} \int_{R^1} \varphi_{n+1}\left(\frac{x-\mu_{n+1}}{\sigma_{n+1}}\right) dx_n = 1/b_{n+1} \int_{R^1} \varphi_{n+1}(u)du =$$

$$= 1/b_{n+1}$$

(24) $$1/\sigma_{n+1} \int_{R^1} (x_n-x)\varphi_{n+1}\left(\frac{x-\mu_{n+1}}{\sigma_{n+1}}\right) dx_n =$$

$$= \frac{1}{b^2_{n+1}} \int_{R^1} \left[x(1-b_{n+1}) - c_{n+1} - u\sigma_{n+1}\right] \varphi_{n+1}(u)du =$$

$$= -\frac{1}{b^2_{n+1}} \left[x(b_{n+1}-1) + c_{n+1}\right],$$

(25) $$1/\sigma_{n+1} \int_{R^1} (x_n-x)^2 \varphi_{n+1}\left(\frac{x- \ n+1}{\sigma_{n+1}}\right) dx_n =$$

$$= \frac{1}{b^3_{n+1}} \left[(x(b_{n+1}-1) + c_{n+1})^2 + \sigma^2_{n+1}\right].$$

It follows from (1), (1'), (7), (8), (8') that for fixed $x_{n-1}, x, t_n, \varepsilon$ for every positive $\varepsilon_1 < \varepsilon |b_{n+1}|$ there exists $\delta > 0$ such that for $t_{n+1}-t_n < \delta$

$$|x(b_{n+1}-1) + c_{n+1}| < \varepsilon_1$$

and consequently for $\varepsilon_2 = (\varepsilon |b_{n+1}| - \varepsilon_1)^1/\sigma_{n+1}$

$$\left\{u: |x(b_{n+1}-1) + c_{n+1}| > \varepsilon |b_n|\right\} \subset \left\{u: |u| > \varepsilon_2\right\}.$$

Therefore for such ε_2 we obtain

$$(26)\quad \lim_{\Delta t_n\to 0}\frac{1}{(\Delta t_n)^2\sigma_{n+1}}\int_{V_\varepsilon(x)}|x_n-x|^i\,\varphi_{n+1}\left(\frac{x-\mu_{n+1}}{\sigma_{n+1}}\right)dx_n\leqslant$$

$$\leqslant\lim_{\Delta t_n\to 0}\frac{1}{(\Delta t_n)^2b_{n+1}^3}\int_{|u|>\varepsilon_2}|x(b_{n+1}-1)+c_{n+1}+u\sigma_{n+1}|^i\varphi_{n+1}(u)du=0$$

Now using the generalized Chapman-Kolmogorov equation, expanding g into Taylor's formula and applying (23)-(26) we get

$$(27)\quad g_n(t_{n+1},x)=f^{(n-1)}(t_{n-1},x_{n-1};t_{n+1},x)=$$

$$=\int_{R^1}f^{(n-1)}(t_{n-1},x_{n-1};t_n,x_n)f^{(n)}(t_n,x_n;t_{n+1},x)dx_n=$$

$$=\int_{U_\varepsilon(x)}\Big[g_n(t_n,x)+(x_n-x)\frac{\partial}{\partial x}g_n(t_n,x)+$$

$$+\frac{1}{2}(x_n-x)^2\frac{\partial^2}{\partial x^2}g_n(t_n,x)+o(x_n-x)^2\Big]g_{n+1}(t_{n+1},x)dx_n+$$

$$+\int_{V_\varepsilon(x)}f^{(n-1)}(t_{n-1},x_{n-1};t_n,x_n)f^{(n)}(t_n,x_n;t_{n+1},x)dx_n=$$

$$=\frac{1}{b_{n+1}}g_n(t_n,x)-\frac{1}{b_{n+1}^2}\left[x(b_{n+1}-1)+c_{n+1}\right]\frac{\partial}{\partial x}g_n(t_n,x)+$$

$$+\frac{1}{2b_{n+1}^3}\left[(x(b_{n+1}-1)+c_{n+1})^2+\sigma_{n+1}^2\right]\frac{\partial^2}{\partial x^2}g_n(t_n,x)+o(\Delta t_n)^2.$$

It follows from (6), (11) that

$$(28)\qquad \frac{\partial^2 g_n}{\partial t^2} = \frac{\partial}{\partial t} \quad -b_n' g_n - (b_n' x + c_n') \frac{\partial g_n}{\partial x} =$$

$$= (b_n')^2 g_n - \tilde{b}_n'' g_n + \left[3b_n'(b_n' x + c_n') - x\tilde{b}_n'' - \tilde{c}_n''\right] \frac{\partial g_n}{\partial x} + (b_n' x + c_n') \frac{\partial^2 g_n}{\partial x^2}.$$

Now expanding once more g into Taylor's formula writting $g_n = g_n(t_n, x)$ and taking into account (6), (11), we have

$$(29)\quad g_n(t_{n+1}, x) = g_n + \Delta t_n \frac{\partial g_n}{\partial t_n} + \frac{1}{2} (\Delta t_n)^2 \frac{\partial^2 g_n}{\partial t_n^2} + o(\Delta t_n)^2 =$$

$$= g_n - \Delta t_n \left[b_n' g_n + (b_n' x + c_n') \frac{\partial g_n}{\partial x}\right] + \frac{1}{2} (\Delta t_n)^2 \Big\{ (b_n')^2 g_n - \tilde{b}_n'' g_n +$$

$$+ \left[3b_n'(b_n' x + c_n') - x\tilde{b}_n'' - \tilde{c}_n'\right] \frac{\partial g_n}{\partial x} + (b_n' x + c_n') \frac{\partial^2 g_n}{\partial x^2} \Big\} + o(\Delta t_n)^2.$$

Taking into account (27)-(29), dividing by $(\Delta t_n)^2$ and passing to the limit when $\Delta t_n \to 0$, using (1'), (8'), (12'), (13) we get the following equation

$$d_n^2 \frac{\partial^2 g_n}{\partial x^2} + \left[(b_n')^2 + \tilde{b}_n'' - b_n''\right] \frac{\partial}{\partial x} (x g_n) + \left[c_n' b_n' + \tilde{c}_n'' - c_n''\right] \frac{\partial g}{\partial x} = 0.$$

This equation and formulas (15), (16) imply (22). Q.E.D.

P r o o f of Theorem 2. It follows from (15), (16), (20) that the general solution of (22) is given by

$$(30)\qquad f(t_{n-1}, x_{n-1}; t_n, x_n) = \left[\exp(-u^2)\right] \left[C_1 + C_2 \int \exp(u^2) du\right],$$

where $u = (x_n - \mu_n)/\sqrt{2}\sigma_n$.

Taking into account formula [3] we obtain

$$\int_0 \left[\exp(-b^2x^2)\right]\left[\int_0^x \exp(a^2x^2)\right]dx = \frac{1}{4b}\ln\frac{b+a}{b-a} \quad \text{for} \quad b>a$$

and passing to the limit when $b \to a$ we see that the function (30) can be a density function iff $C_2 = 0$, $C_1 = (\sqrt{2\pi}\,\sigma_n)^{-1}$.

Finally the unique solution of (22) which is a density function is given by

$$f^{(n-1)}(t_{n-1},x_{n-1};t_n,x_n) = 1/\sigma_n\left[\exp-(x_n-\mu_n)^2/2\sigma_n^2\right].$$

It is evident that if μ_n, σ_n^2 are expressed by the covariance function K in the manner indicated by formulas (1), (2) then these functions satisfy the consistency conditions.

Q.E.D.

Ad.IV. First let us notice that if for a fixed i, a_{i2} exists, then $a_{i1} = 0$ and since (3'), (5) are satisfied for $r = 2$, hence they also hold for $r = 1$.

T h e o r e m 1 . Let (3'), (4), (5) be satisfied for $r = 2$. Then the following equations hold

$$(6')\qquad \frac{\partial}{\partial x_n}\left[a_{12}(t_n,x_n)\, f^{(n-1)}(t_{n-1},x_{n-1};t_n,x_n)\right] =$$

$$= \frac{1}{2}\frac{\partial^2}{\partial x_n^2}\left[a_{22}(t_n,x_n)f^{(n-1)}(t_{n-1},x_{n-1},t_n,x_n)\right],\quad n \geqslant 1$$

under the assumption that all the derivatives in (6') are continuous.

P r o o f . First let us notice that since (3'), (4), (5) are satisfied for $r = 2$, hence they also hold for $r = 1$. Next equation (6) and the fact $a_{i1} = 0$ for $i = 1,2$ imply

$$\frac{\partial}{\partial t_n} f^{(n-1)}(t_{n-1},x_{n-1};t_n,x_n) = 0$$

i.e. $f^{(n-1)}$ does not depend on t_n. In such a situation repeating the well-known reasoning used in the proof of prospective equations we get

$$\int_{(a,b)} \Big[f^{(n-1)}(t_{n-1},x_{n-1};t_n,x_n)\, \varphi'(x_n)Q_{12}(t_{n+1},x_n,U_\varepsilon(x_n)) +$$

$$+ \frac{1}{2}\varphi''(x_n)Q_{22}(t_{n+1},x_n,U_\varepsilon(x_n)) + O(\Delta t_n)\Big]dx_n = 0,$$

where (a,b) is an optional interval, φ - an optional function belonging to C^2 and satisfying the well-known assumptions. We pass to the limit with $\Delta t_n \to 0$ using relations (4), integrate by parts and finally we take into account that φ is an optional function. All this implies (6'). Q.E.D.

If $a_{11} = 0$, then $b'_n = c'_n = b''_n = c''_n = 0$, $a_{12} = 1/2\ \mu''_n = x_n b''_n + c''_n$. Equation (6') is then equivalent to equation (22). We have already shown that in the class of density functions there is a unique solution of (22) and it is a gaussian density. Thus we can formulate the following theorem.

Theorem 2'. If (1), (2) hold, $K \in C^2$, and $a_{11}(t_n,x_n) = 0 \pmod{P^{(0)}(t_n,\cdot)}$ then the unique explicit solution $f^{(n-1)}$ of (6), (6') is a gaussian density, the family $\{f^{(n-1)}\}_{n\in N}$ satisfies the consistency conditions.

Now we are going to give some examples.

Example 1. Let $K(t_1,t_2) = \exp(-(t_2-t_1)^2) = \exp(-(\Delta t_1)^2)$. Then $a_{12} = 0$, $a_{11}(t_1,x_1) = 0$, $a_{21}(t_2,x_2) = 2\Delta t_1(\exp(-2(\Delta t_1)^2)\left[x_1-x_2\exp(-(\Delta t_1)^2)\right]$, $a_{22}(t_1,x_1) = 2$, $a_{22}(t_2,x_2) = 2-2\exp(-2(\Delta t_1)^2) - 4(\Delta t_1)^2\exp(-2(\Delta t_1)^2)$

$$f^{(0)}(t_1,x_1) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x_1^2}{2}\right),$$

$$f^{(1)}(t_1,x_1;t_2,x_2) = \frac{1}{\sqrt{2\pi}\,\sigma_2}\exp\left(-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}\right)$$

and so on, where $\mu_2 = x_1 \exp(-(\Delta t_1)^2)$, $\sigma_2^2 = 1 - \exp(-2(\Delta t_1)^2$.

Therefore we see that for some n $a_1(t_n, x_n)$ may be equal to zero and for another n it may be different from zero.

E x a m p l e 2. Let $K(t_1,t_2) = \cos(t_2-t_1)$. Then $K^{(3)} = 0$. The main assumption of the present paper is not satisfied, the distribution of X_3 is improper. Only the cases $n = 1,2$ non-degenerate. For these cases all the assumptions are satisfied. The one and two-dimensional distributions are proper gaussian distributions.

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