DEMONSTRATIO MATHEMATICA
Vol. XIV No 2 1981

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STRUCTURAL NUMBERS AND GRAPHS

0. Introduction

The notion of structural number, resulting from the ideas of K.T.Wang (Wang [19]), was formally introduced by S.Bellert in 1962 (Bellert [1]) who applied the structural numbers to the analysis and synthesis of linear networks. This method, called the method of structural numbers, has later been developed by Bellert [2]-[4], Bellert and Woźniacki [5], Psarski [15] and Woźniacki [20], [21]. The principal features of the structural numbers method of network analysis are: simple and compact description of large network structures, algebraic computation of different structure transforms, computation of the expressions for the different parameters of a network without using the matrix and determinant techniques, easy implementation on computers. The present paper gives the mathematical foundations of the structural numbers method. The broad and precise definition of the structural number and an abstract characterisation of the ring of structural numbers (briefly: RSN) were given by Burakowski and Traczyk in 1972 ([7], [8]). In applications, the great importance have RSNs built on the family of all subsets of a finite set. Some structural numbers of such a RSN have a strong connexion with finite nondirected connected graphs. More precisely (compare [5], [10]), the product of one-line structural numbers representing elements of the basis of the cutset (circuit) space of a connected graph is the family of all trees (cotrees) of this graph. Then we say, that the graph is a geometric image (co-image) of the resulting structural number. Thus for a given graph it is easy to calculate the structural number for which the graph is a geometric image. However, there has not been known any efficient condition for existence of a geometric image for a given structural number. The present paper gives a solution of this problem. It must be pointed out, that the number of trees of a graph (network) is a non-polynomial function of the basic parameters of the graph which substantially restricts the direct application of the method of structural numbers. But the methods of decomposition of a network structure discovered in 1976 make the method entirely efficient.

In this paper the following problem of graph theory is considered: for the families of trees and cotrees of a finite connected graph find the families of circuits and cutsets of this graph. The solution of this problem is given in Theorems 2.5, 2.9 and 2.11 characterizing circuits and cutsets by the families of cotrees and trees of a graph. Then, basing on the above theorems, the necessary condition (Theorem 3.3) and the necessary and sufficient condition (Theorem 3.4) for the existence of a geometric image of a homogeneous structural number is specified. From these conclusions and the results of Burakowski [9] immediately arise a new characterization of planar graphs (Corollary 3.8) and the characterization of (0,1)-matrices having a ϱ-realization (Corollary 3.9) which, as author feels, is in itself an interesting result in graph theory (compare [11,12]). From these considerations an algorithm verifying existence of a geometric image of a homogeneous structural number is derived (Algorithm 4.2).

The paper contains a part of the results of the author's doctoral dissertation submitted in 1977 to the Institute of Mathematics, Technical University of Warsaw (the research supervised by Prof. Tadeusz Traczyk). Some of these results were presented in a preliminary version in [13], [16].

Throughout the text the standard mathematical notation is used. In particular $|X|$ denotes the cardinality of a set X, $\mathcal{P}(X)$ is powerset of X, $\forall$ and $\exists$ are universal and the existential quantifier, respectively. The symbol π denotes the product in RSN. The end of a proof or the fact that the proof is omitted is denoted by $\square$.

1. Rings of structural numbers

Let $(B, \leqslant)$ be a partially ordered set and let $a,b \in B$. For brevity we introduce the notations: $a \wedge b$ for the greatest lower bound of $\{a,b\}$ and $a \vee b$ for the least upper bound of $\{a,b\}$, whenever they exist in $(B, \leqslant)$. The following theorem due to Burakowski and Traczyk [7], [8] gives an abstract characterization of structural numbers. We shall use it as a definition of a structural number.

T h e o r e m 1.1. A commutative ring $(A;+,\cdot,0,1)$ is said to be a ring of structural numbers iff the following postulates are satisfied.

I. $a + a = 0$ for every $a \in A$.

II. There is partially ordered basis $(B, \leqslant)$, $B \subseteq A$, such that:

(i) $1 \in B$

(ii) for every $a \in A$, $a \neq 0$, there exists a unique representation $a = a_1 + \dots + a_n$, where $a_1,\dots,a_n$ are distinct elements of B ($a_1,\dots,a_n$ will be called components of a)

(iii) for every a,b in B, if $a \wedge b$ exists and is equal to 1 then $a \vee b$ exists and is equal to $a \cdot b$. In other cases $a \cdot b = 0$ for $a,b \in B$. $\square$

The next theorem, taken from [9], gives a representation of a RSN.

T h e o r e m 1.2. Let $(B, \leqslant)$ be a poset with the least element $\wedge$ and with the following properties.

(i) There exists $a \vee b$ for every $a,b \in B$ for which $a \wedge b$ exists and is equal to $\wedge$.

(ii) For every a_1,a_2,a_3 in B, if $a_1 \wedge a_2 = a_2 \wedge a_3 = a_3 \wedge a_1 = \wedge$ then $a_1 \wedge (a_2 \vee a_3) = \wedge$ iff $(a_1 \vee a_2) \wedge a_3 = \wedge$.

Let $P_{fin}(B)$ be a family of all formal sums $\sum_{j \in J} a_j$, where $a_j \in B$ and J is a finite or empty set. We define the following operations on $P_{fin}(B)$.

(+) $\sum_{j\in J} a_j + \sum_{j\in K} a_j = \sum_{j\in J\Delta K} a_j$ where Δ denotes the symmetric difference of sets,

(•) $\sum_{j\in J} a_j \cdot \sum_{k\in K} a_k = \sum_{j\in J}\sum_{k\in K} a_j a_k$ where $a_j a_k = a_j \vee a_k$ if $a_j \wedge a_k = \Lambda$, and $a_j a_k = 0$ in other cases.

Then the system $(P_{fin}(B);+,\cdot)$ is a RSN.□

D e f i n i t i o n 1.3.

a. A structural number is <u>m-line</u> (<u>homogenous</u>) if each component of its unique representation by the elemenents of the basis B is the least upper bound of m (the same number of) distinct minimal and different from Λ elements of B.

b. <u>Algebraic</u> <u>derivative</u> ∂ and <u>coderivative</u> δ. Let $(B,\leqslant)$ be the basis of a RSN and let $a,b \in B$. Then

$$\frac{\partial a}{\partial b} = \begin{cases} a \setminus b & \text{for } b \leqslant a \text{ (a's complement of b)} \\ 0 & \text{in other cases} \end{cases}$$

$$\frac{\delta a}{\delta b} = \begin{cases} a & \text{if } a \wedge b = \Lambda \\ 0 & \text{in other cases.} \end{cases}$$

Let $a = \sum_{j\in J} a_j$, $a_j \in B$ for $j \in J$ and let $c \in B$. Then

$$\frac{\partial a}{\partial c} = \sum_{j\in J} \frac{\partial a_j}{\partial c} \quad \text{for } a \neq 0 \text{ and } \frac{\partial 0}{\partial c} = 0$$

$$\frac{\delta a}{\delta c} = \sum_{j\in J} \frac{\delta a_j}{\delta c} \quad \text{for } a \neq 0 \text{ and } \frac{\delta 0}{\delta c} = 0.$$

In the sequel we assume the following model of a RSN. $B = \mathcal{P}(X)$, where X is a finite set, and $\leqslant$ is the inclusion relation. Notice, that then each family $\mathfrak{M} \subseteq \mathcal{P}(X)$ is an element of this RSN and vice versa. The <u>complement</u> of the structural number (family) $\mathfrak{M} \subseteq \mathcal{P}(X)$ is the number (family) $\overline{\mathfrak{M}} = \{M: \bigcup\mathfrak{M} \setminus M \in \mathfrak{M}\}$

E x a m p l e 1.4. Let $X = \{1,2,3,4,5,6,7,8\}$, $B = \mathcal{P}(X)$. We shall use the same notation of structural numbers as was used in [5], for example in the present case $[i]$ denotes the family $\{\{i\}\}$.

$$\begin{bmatrix} 1 & 8 & 4 & 5 \\ 2 & 3 & 5 & \\ 5 & & 1 & \\ 7 & & 6 & \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \\ 7 \end{bmatrix} + \begin{bmatrix} 8 \\ 3 \end{bmatrix} + \begin{bmatrix} 5 \\ 6 \\ 4 \\ 1 \end{bmatrix} + [5]$$

$$\begin{bmatrix} 1 & 4 & 3 \\ 3 & 1 & 7 \\ 6 & 5 & 8 \end{bmatrix} \quad \text{a homogenous number}$$

$$\begin{bmatrix} 1 & 4 & 8 & 7 \end{bmatrix} \quad \text{an one-line number}$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 4 \\ & 3 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 2 & 4 \\ 4 & 5 & & 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 2 \\ 1 & 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 & 2 & 7 \\ & & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 4 & 7 \\ & 2 \end{bmatrix}.$$

Let $s = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ 3 & 2 & 2 & 3 & 3 & 3 & 3 & 5 \\ 5 & 4 & 5 & 4 & 4 & 4 & 5 & 7 \\ 7 & 6 & 6 & 5 & 6 & 6 & 6 & 8 \end{bmatrix}$ and let $d = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$.

Then

$$\frac{\partial s}{\partial d} = \begin{bmatrix} 1 & 3 \end{bmatrix} \quad \frac{\delta s}{\delta d} = \begin{bmatrix} 1 & 3 \\ 3 & 5 \\ 5 & 7 \\ 7 & 8 \end{bmatrix} \quad \bar{s} = \begin{bmatrix} 2 & 3 & 4 & 2 & 2 & 1 & 1 & 1 \\ 4 & 5 & 4 & 6 & 5 & 5 & 4 & 2 \\ 6 & 7 & 7 & 7 & 7 & 7 & 7 & 4 \\ 8 & 8 & 8 & 8 & 8 & 8 & 8 & 6 \end{bmatrix}.$$

2. Structural numbers and nondirected graphs

At the beginning we recall definitions of some graph-theoretical notions. A graph G is an ordered triple consisting of the set of vertices $V(G)$, the set of edges $E(G)$ and the function $P(G)$ from $E(G)$ in the family of two-element subsets of the set $V(G)$ $(V(G) \cap E(G) = \emptyset)$. In this paper only finite graphs are considered. A subgraph of a graph G is a graph G_1 such that $V(G_1) \subseteq V(G)$ and $P(G_1) \subseteq P(G)$ (if f is a function defined on X with values in Y then we can consider f as the set $\{(x,f(x)): x \in X\} \subseteq X \times Y$). A subgraph without isolated vertices of a graph G (briefly: WIV-subgraph) is a graph G_1 such that $P(G_1) \subseteq P(G)$ and for each vertex x of G_1 there exists an edge of G_1 which is incident at x. A component of a graph is a connected subgraph containing the maximal number of edges. An isolated vertex is a component. The rank and the nullity of a graph G with c components are defined as the numbers $r(G) = |V(G)| - c$ and $\lambda(G) = |E(G)| - |V(G)| + c$, respectively. A tree is a connected graph G such that $\lambda(G) = 0$. A circuit is a connected graph G such that $\lambda(G) = 1$ and if G_1 is a subgraph of G and $E(G_1) \subsetneq E(G)$ then $\lambda(G_1) = 0$. For a graph G, a cutset of G is a subgraph consisting of a minimal collection of edges whose removal reduces the rank of G by one, an incidence cut of G is the subgraph formed by the edges incident at a vertex of G (an incidence cut which is also a cutset is termed an incidence cutset), a tree of G is a subgraph t of G which is a tree and $V(t) = V(G)$, and a circuit of G is a subgraph of G which is a circuit. The complement of a tree of a graph G is called a cotree (i.e. if $\bar{t}$ is the cotree of a tree t, then $E(\bar{t}) = E(G) \setminus E(t)$). An edge in a cotree is called a chord.

For a given graph G the symbols $B(G)$, $I(G)$, $S(G)$, $T(G)$, $C(G)$, $\bar{T}(G)$ denote the class of cutsets, the class of incidence cuts, the class of incidence cutsets, the class of trees, the class of circuits and the class of cotrees of the graph G, respectively.

The fundamental circuits of a connected graph G with respect to a tree $t \in T(G)$, denoted $C_t(G)$, are the $\lambda(G)$ circuits, each being formed by a chord and the unique tree path connecting the two endpoints of the chord in t. The fundamental outsets of a connected graph G with respect to a tree $t \in T(G)$, denoted $B_t(G)$, are the $r(G)$ outsets, each containing exactly one edge of t. For a graph G, the vector space over the two-element field Z_2 generated by the class $B_t(G)$ in which addition is the symmetric difference is called the vector space of outs of G and is denoted $L(G)$.

For the sake of simplicity, the symbols denoting subgraphs will be also denoting the families of edges of these subgraphs, for example t may denote a tree or the set of edges of the tree t. Consequently $T(G)$, $C(G)$, etc. will be also denoting the families of sets of edges of suitable subgraphs of G, in other words a structural numbers of the ring $(P_{fin}(\mathcal{P}(E(G)));+,\cdot)$.

For other graph-theoretical notions the reader is referred to [10], [22].

From the definition of a graph immediately results

L e m m a 2.0. If G_1 is a subgraph of a graph G, G_2 is a WIV-subgraph of G and $E(G_2) \subseteq E(G_1)$, then G_2 is a subgraph of the graph G_1. □

We shall use this lemma in the proof of the following important propositions.

L e m m a 2.1. If G' is such a subgraph of a graph G that $E(G') = E(G) \setminus \{e\}$ and $V(G') = V(G)$, then

(i)
$$T(G') = \frac{\delta T(G)}{\delta \{e\}}$$

and

(ii)
$$\bar{T}(G') = \frac{\partial \bar{T}(G)}{\partial \{e\}} .$$

P r o o f . (i). If $t \in T(G)$ and $e \notin t$ then t is a WIV-subgraph of G and $t \subseteq E(G')$. Since G' is a subgraph of G then in virtue of Lemma 2.0 and in the presence of the obvious identities $V(t) = V(G) = V(G')$ we have $t \in T(G')$. On the other hand, if $t \in T(G')$ then $e \notin t \in T(G)$. Thus $t \in T(G')$ iff $t \in T(G)$ and $e \notin t$, i.e. $T(G') = \frac{\delta T(G)}{\delta \{e\}}$.

(ii). Let $\bar{t}$ be such a cotree of G that $e \in \bar{t}$ and let $s = \bar{t} \setminus \{e\}$. Since $E(G') \setminus s = E(G) \setminus \bar{t}$, then $E(G') \setminus s \in T(G)$ and $e \notin E(G') \setminus s$. In virtue of proposition (i), $E(G') \setminus s \in T(G')$, hence $s \in \bar{T}(G')$. Let now s be any element of $\bar{T}(G')$. Then $e \notin s$ and $E(G) \setminus (\{e\} \cup s) \in T(G')$. In the presence of proposition (i) we have $E(G) \setminus (\{e\} \cup s) \in T(G)$ hence $\{e\} \cup s \in \bar{T}(G)$. So we obtain that if $s \in \bar{T}(G')$ then there exists $\bar{t} \in \bar{T}(G)$ such that $s = \bar{t} \setminus \{e\}$ and $e \in \bar{t}$. This completes the proof of the proposition (ii). □

T h e o r e m 2.2. (Berge [6]). Let G be a graph and let G_1 be a graph obtained by addition a new edge between vertices $x, y \in V(G)$. If the vertices x and y are in the same component of the graph G, then

$$r(G_1) = r(G) \text{ and } \lambda(G_1) = \lambda(G) + 1,$$

otherwise

$$r(G_1) = r(G) + 1 \text{ and } \lambda(G_1) = \lambda(G). \square$$

L e m m a 2.3. If G is a connected graph then $\bigcup C(G) = \bigcup \bar{T}(G)$.

P r o o f . From the definition of fundamental circuits with respect to a tree t it follows immediately that if $\bar{t} \in \bar{T}(G)$ and $e \in \bar{t}$, then there exists a circuit of G containing the edge e. On the other hand, in virtue of the connectedness of G and Theorem 2.2, if a circuit c of the graph G contains an edge e, then there exists a subgraph G_1 of the graph G such that $E(G_1) = E(G) \setminus \{e\}$ and $T(G_1) \neq \emptyset$. Therefore the result of Lemma 2.1(i) is the existence of a tree $t \in T(G)$ such that $e \notin t$, in other words there exists a cotree $\bar{t} \in \bar{T}(G)$ such that $e \in \bar{t}$. This completes the proof. □

From Lemmas 2.0 and 2.1 in a similar way we obtain

L e m m a 2.4 (Rajkow-Krzywicki [17, Lemma 1.4]). If $e \in \bar{t} \in \bar{T}(G)$, $t = E(G) \setminus \bar{t}$ and G' is such a subgraph of G that $E(G') \setminus \{e\}$, then $C_t(G') = \{c : c \in C_t(G) \ \& \ e \in c\}$. □

The above lemmas permit one to prove the following theorem characterizing the fundamental circuits of a graph by the family of cotrees of the graph.

T h e o r e m 2.5. If $\lambda(G) \geqslant 1$, $\bar{t} \in \bar{T}(G)$, $t \in T(G)$ and $t = E(G) \setminus \bar{t}$, then $C_t(G) = \left\{ \bigcup \frac{\partial \bar{T}(G)}{\partial(\bar{t} \setminus \{e\})} : e \in t \right\}$.

P r o o f . Let $\lambda(G) = k$ and let $\bar{t} = \{e_1, e_2, \dots, e_k\}$. For every $e_i \in \bar{t}$, let G_i be a subgraph of G such that $E(G_i) = t \cup \{e_i\}$. If $k = 1$ then $G_1 = G$, otherwise G_i can be obtained by consecutive deletion of edges $e_1, \dots, e_{i-1}$, $e_{i+1}, \dots, e_k$ from G. Thus $|C_t(G_i)| = 1$ and $\{\bigcup \bar{T}(G_i)\} = C_t(G_i)$ by Lemmas 2.4 and 2.3. From Lemma 2.1 and Definition 1.3(b) we obtain that for every $e_i \in \bar{t}$ $e_i \in \bigcup \frac{\partial \bar{T}(G)}{\partial(\bar{t} \setminus \{e_i\})} \in C_t(G)$ which, since $|\bar{t}| = \lambda(G)$, implies the thesis of the theorem. □

The following lemmas imply the next important theorem characterizing the fundamental cutsets of a graph by the family of trees of the graph.

L e m m a 2.6. If $t, t_1 \in T(G)$, $e \in t$, $e_1 \in t_1$, $t \setminus \{e\} = t_1 \setminus \{e_1\}$ and $e \neq e_1$, then there exists a circuit $c \in C_t(G)$ such that $e, e_1 \in c$. □

L e m m a 2.7. If $t \in T(G)$, $c \in C_t(G)$, $e, e_1 \in c$, $e \in t$ and $e_1 \notin t$, then there exists a tree $t_1 \in T(G)$ such that $t_1 = (t \setminus \{e\}) \cup \{e_1\}$. □

L e m m a 2.8. Let e be any edge of a tree t of a graph G. Then the fundamental cutset containing e consists of these and only these chords which belong to fundamental circuits containing e. □

T h e o r e m 2.9. If $t \in T(G)$ and $r(G) \geqslant 1$, then $B_t(G) = \left\{ \bigcup \frac{\partial T(G)}{\partial(t \setminus \{e\})} : e \in t \right\}$.

P r o o f . Since $|t| = r(G) > 1$, then having in mind the Lemma 2.8 it is sufficient to show that for every $e \in t$

$$(*) \quad \{a : (\exists c \in C_t(G))[a, e \in c \ \& \ a \notin t \ \& \ e \in t] \text{ or } a = e\} = \bigcup \frac{\partial T(G)}{\partial(t \setminus \{e\})} .$$

Indeed, for every $e \in t$, $a \in \bigcup \frac{\partial T(G)}{\partial(t \setminus \{e\})}$ iff there exists a tree $t_1 \in T(G)$ such that $t \setminus \{e\} \subseteq t_1$ and $a \in t_1 \setminus (t \setminus \{e\})$. But $|t| = |t_1|$, therefore if $a \in t_1 \setminus (t \setminus \{e\})$ then $t \setminus \{e\} = t_1 \setminus \{a\}$ and $a \in t_1$. Hence, in virtue of Lemma 2.6, if $a \in \bigcup \frac{\partial T(G)}{\partial(t \setminus \{e\})}$, then either there exists a circuit $c \in C_t(G)$ such that $e, a \in c$, $e \in t$, $a \notin t$ or $e = a$. On the other hand, for every $e \in t$, if $t_1 \in T(G)$ is such that $t_1 = (t \setminus \{e\}) \cup \{a\}$, then $a \in t_1 \setminus (t \setminus \{e\})$ and $t \setminus \{e\} \subseteq t_1$, so in virtue of Lemma 2.7, if either $a \in c \in C_t(G)$ and $a \notin t$, $e \in c$, $e \in t$ or $e = a$, then there exists a tree $t_1 \in T(G)$ such that $a \in t_1 \setminus (t \setminus \{e\})$ and $t \setminus \{e\} \subseteq t_1$. Thus (*) is valid for every $e \in t$ and this completes the proof. □

L e m m a 2.10. For every cutset $b \in B(G)$ there exists a tree $t \in T(G)$ such that $b \in B_t(G)$.

P r o o f . Let $b \in B(G)$ and $e \in b$. Then $e \in \bigcup T(G)$ and, by Lemma 2.1, $e \in \bigcup \frac{\delta T(G)}{\delta(b \setminus \{e\})}$. Hence, by Definition 1.3(b), there exists a tree $t \in T(G)$ such that $b \cap t = \{e\}$ and this ends the proof. □

From Lemma 2.10 and Theorem 2.9 immediately results

T h e o r e m 2.11.

$$B(G) = \left\{ \bigcup \frac{\partial T(G)}{\partial(t \setminus \{a\})} : t \in T(G) \ \& \ a \in t \right\}. \ \square$$

3. Geometric images of homogenous structural numbers

Let X be a finite set and let $\mathfrak{M}$ be a homogenous structural number of the ring $(P_{fin}(\mathcal{P}(X));+,\cdot)$. $\mathfrak{M}$ is said to have a geometric image if there exists a connected graph G such that $T(G) = \mathfrak{M}$. The problem of the existence of geometric image of a homogenous structural number is of essential importance in the analysis and synthesis of electrical networks. The results obtained in Section 2 let us to solve this problem. At the beginning the following two theorems are needed.

T h e o r e m 3.1. If G is a connected graph then there exists a family K of linearly independent cutsets of G such that $K \subseteq S(G)$ and $|K| = r(G)$.

P r o o f . Since $B(G) \subseteq L(G)$ and the dimension of $L(G)$ is $r(G)$, for every family K of linearly independent elements from $L(G)$ and such that $K \subseteq S(G)$ we have inequality $|K| \leqslant r(G)$. Let now K be such a maximal family of linearly independent elements from $L(G)$ that $K \subseteq S(G)$. Every cutset belonging to $S(G)$ is then a linear combination of the elements of K. Since every element of $I(G)$ is a linear combination of elements of the family $S(G)$, then every element of $I(G)$ is a linear combination of the elements of the family K. Thus $I(G)$ is contained in the subspace of the space $L(G)$, generated by K. Since in $I(G)$ there exist $r(G)$ linearly independent elements, we have $|K| \geqslant r(G)$. This completes the proof. □

In the sequel we will denote by $H(A)$ the family of single element subsets of a set A, for example $H(\{1,2,3\}) = \{\{1\},\{2\},\{3\}\}$.

T h e o r e m 3.2. (Chen [10]). Let CI be the set of $\lambda(G)$ linearly independent circuits or linear combinations of edge-disjoint circuits and let CU be the set of $r(G)$ linearly independent cutsets or linear combinations of edge-disjoint cutsets. Then

$$\bar{T}(G) = \prod_{c \in CI} H(c) \quad \text{and} \quad T(G) = \prod_{b \in CU} H(b). \; \square$$

The following two theorems give necessary as well as necessary and sufficient conditions for the existence of a geometric image for a homogenous structural number.

T h e o r e m 3.3. If for a given structural number $\mathfrak{M}$ there exists a connected graph G such that $T(G) = \mathfrak{M}$, then

(i) for every $A \in \mathfrak{M}$, $\prod_{x \in A} \frac{\partial \mathfrak{M}}{\partial (A \setminus \{x\})} = \mathfrak{M}$

(ii) for every $A \in \bar{\mathfrak{M}}$, $\prod_{x \in A} \frac{\partial \bar{\mathfrak{M}}}{\partial (A \setminus \{x\})} = \bar{\mathfrak{M}}$.

P r o o f . The theorem results immediately from Theorems 2.5, 2.9 and 3.2. □

T h e o r e m 3.4. For a given structural number $\mathfrak{M}$ there exists a connected graph G such that $T(G) = \mathfrak{M}$ iff there is a family K fulfilling the conditions:

(i) $K \subseteq \left\{ \bigcup \frac{\partial \mathfrak{M}}{\partial (a \setminus \{z\})} : a \in \mathfrak{M} \ \& \ z \in a \right\}$

(ii) elements of the family K are linearly independent

(iii) $\prod_{p \in K} H(p) = \mathfrak{M}$

(iv) every element of $\bigcup \mathfrak{M}$ belongs to at most two sets of the family K.

P r o o f . Let G be such a connected graph that $T(G) = \mathfrak{M}$. Then, basing on Theorems 2.11, 3.1 and 3.2, there exists a family K fulfilling conditions (i), (ii), (iii). The condition (iv) results from the facts that each element of E(G) appears in exactly two elements of I(G) and that each element of I(G) is either a cutset or a linear combination of edge-disjoint cutsets.

Let now assume that for a given structural number $\mathfrak{M}$ there exists a family K fulfilling conditions (i) - (iv). Let $\langle x_1, \ldots, x_N \rangle$ be any one-to-one sequence of all elements of the set $\bigcup \mathfrak{M} = Y$. Let $\Phi : \mathcal{P}(Y) \to \{0,1\}^N$ be such that

$$(\forall p \in \mathcal{P}(Y)) \left[\Phi(p) = \langle u_1, \ldots, u_N \rangle \ \& \ u_i = \begin{cases} 1 & \text{if } x_i \in p \\ 0 & \text{if } x_i \notin p \end{cases} \right].$$

Finally, let W be a matrix, rows of which are vectors $\Phi(p)$, $p \in K$. From the conditions (i), (ii) and (iv) we conclude that the matrix obtained by joining to W the new row, elements of which are equal the sums (mod 2) of the elements of respective columns of W, is the incidence matrix of some connected graph G and therefore the elements of the family K are elements of the family I(G). From above statements, from (iii) and from Theorem 3.2 we have $\mathfrak{M} = T(G)$. □

We quote now three results from [9], which will allow us to derive two important corollaries from Theorem 3.4.

P r o p o s i t i o n 3.5 (Burakowski [9]). If two sets of structural numbers $\{a_1,\dots,a_p\}$, $\{c_1,\dots,c_p\}$ are basis of the same vector space over the field Z_2, then $\prod_{j=1}^{p} a_j = \prod_{j=1}^{p} c_j$. □

P r o p o s i t i o n 3.6 (Burakowski [9]). If a structural number $s \neq 0$ has a factoring $s = \prod_{j=1}^{p} a_j$, where each a_j is an one-line structural number, then every nonreversible one-line divisor of s belongs to the vector space over Z_2, generated by $\{a_1,\dots,a_p\}$. □

P r o p o s i t i o n 3.7 (Burakowski [9]). If a homogenous structural number s has a factoring on one-line factors, then the complement $\bar{s}$ has such a factoring too. □

From Whitney's characterization of planar graphs and by Proposition 3.7 we have

C o r o l l a r y 3.8. A connected graph G is planar iff the conditions (i), (ii) and (iv) of Theorem 3.4 are fulfilled for the structural number $\bar{T}(G)$. □

From this Corollary one can derive MacLane's characterization of planar graphs.

At this moment we need some additional definitions.

A <u>(0,1)-matrix</u> is a matrix $[a_{ij}]$ such that $a_{ij} \in \{0,1\}$ for all i,j. If $\{b_1,\dots,b_{r(G)}\}$ is a basis of the space $L(G)$, then a <u>cut-matrix</u> of G is a (0,1)-matrix $R = [a_{ij}]$ having $r(G)$ rows and $|E(G)|$ columns and such that

$$a_{ij} = \begin{cases} 1 & \text{if edge } e_j \text{ belongs to cutset } b_i \\ 0 & \text{otherwise.} \end{cases}$$

A connected graph G is a <u>ρ-realization</u> of a (0,1)-matrix A, if there exists a one-to-one correspondence between the edges of G and the columns of A such that A becomes a cut-matrix of G; we then say that <u>matrix A has a ρ-realization.</u>

Let A be a (0,1)-matrix with columns indexed by a set T. For every row w of A we define $q(w) = \{x \in T:$ the element in column x and in row w of A is equal to one$\}$.
Let $Q(A) = \{q(w): w$ is a row of $A\}$. At last we are ready to state

C o r o l l a r y 3.9. A (0,1)-matrix A has ϱ-realization iff the conditions (i), (ii) and (iv) of Theorem 3.4 are fulfilled for the structural number $\mathcal{M} = \prod_{u \in Q(A)} H(u)$.

P r o o f . Necessity: Let G be a ϱ-realization of a given matrix A. Then $\mathcal{M} = T(G)$ by Theorem 3.2, so the conditions (i), (ii) and (iv) of Theorem 3.4 are fulfilled for $\mathcal{M}$.

Sufficiency: If the conditions (i), (ii) and (iv) of Theorem 3.4 are fulfilled for the number $\mathcal{M}$ then, by Propositions 3.5 and 3.6, $\mathcal{M}$ fulfills condition (iii) of Theorem 3.4. Hence there exists a connected graph G such that $\mathcal{M} = T(G)$ and A is a cut-matrix of G (see the "if part" in the proof of Theorem 3.4 and Proposition 3.6). □

4. Algorithms

A l g o r i t h m 4.1. Basing on Theorem 2.9, the algorithm finds fundamental cutsets with respect to an arbitrary tree of a connected graph G using only the family T(G). This problem is of frequent occurence in the analysis and synthesis of electrical networks.

Let G be a connected graph, $r = r(G)$, $T(G) = \{t_1, \dots, t_m\}$.

1. Let $t_i = \{e_1, \dots, e_r\}$ be any (arbitrarily chosen) element of $T(G)$.
2. $B_{t_i}(G) := \emptyset$.
3. $j := 1$.
4. $b_j := \emptyset$ (b_j will be denoting the j-th fundamental cutset).
5. Find the set $d_j = t_i \setminus \{e_j\}$.
6. $k := 1$.
7. If $d_j \not\subset t_k$ then go to step 9.
8. $b_j := b_j \cup (t_k \setminus d_j)$.

9. $k := k+1$. If $k \leq m$ then go to step 7.

10. $B_{t_i}(G) := B_{t_i}(G) \cup \{b_j\}$.

11. $j := j+1$. If $j \leq r$ then go to step 4.

12. Stop.

In a similar way, replacing $T(G)$ by $\bar{T}(G)$ and $\bar{t}_i$ by t_i, we obtain the fundamental circuits with respect to a tree t_i. It should be emphasized that there are more efficient algorithms finding fundamental cutsets (circuits) directly from the structure of a graph (see e.g. [14, 18]).

Algorithm 4.2. Basing on Theorem 3.4, the algorithm tries whether a given homogenous structural number has a geometrical image. If the answer is positive, the algorithm gives the appropriate graph. Steps 1 - 3 may be used as a separate algorithm verifying a cannonical factoring of a homogenous structural number.

Let $\mathcal{M} = \{M_1, \ldots, M_r\}$ be an m-line structural number and let $X = \bigcup \mathcal{M}$.

1. $i := 1$.

2. Find the family $R_i = \left\{ \bigcup \frac{\partial \mathcal{M}}{\partial (M_i \setminus \{a\})} : a \in M_i \right\}$ in a similar way as in Algorithm 4.1 the family $B_{t_i}(G)$. The elements of the family R_i are linearly independent in virtue of the construction. If $i > 1$ then go to step 4.

3. Verify if $\prod_{p \in R_1} H(p) = \mathcal{M}$. If not, then go to step 9 (the number $\mathcal{M}$ has no cannonical factoring and then $\mathcal{M}$ has no geometrical image, see Theorem 3.3). Otherwise $\mathcal{M}$ has a cannonical factoring and, in virtue of Propositions 3.5, 3.6, the above identity will be valid for all R_i, $i=2,3,\ldots,r$; the problem of existence of a geometrical image is still open.

4. Verify if every element of X belongs to at most two sets of the family R_i. If so, then go to step 7.

5. $i := i+1$. If $i \leq r$ then go to step 2.

6. Find the family $F = \bigcup_{i=1}^{r} R_i$ and verify whether there is a family $P \subseteq F$ such that $P \neq R_i$ $(i = 1, \ldots, r)$ and the

elements of P are linearly independent and every element of X belongs to at most two sets of P. If not, then go to step 8.

7. Let us denote by Q the obtained family (R_i or P) and let $U = Q \cup \{\triangle Q\}$, where $\triangle Q$ denotes the symmetric difference of the sets of Q. Let $<x_1,\dots,x_k>$ be a one-to-one sequence of all elements of X and let $\Phi : \mathcal{P}(X) \longrightarrow \{0,1\}^k$ (see the proof of Theorem 3.4). A matrix which rows are vectors $\Phi(u)$, $u \in U$, is then the incidence matrix of a connected grap G such that $T(G) = \mathfrak{M}$. Therefore G is a geometrical image of the number $\mathfrak{M}$. Go to step 9.

8. The number $\mathfrak{M}$ has no geometrical image.

9. Stop.

Acknowledgement. The author would like to thank Prof. Tadeusz Trączyk for his guidance and valuable remarks.

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Added in the proof

Recently, an efficient algorithm for deciding whether a (0,1)-matrix has a ϱ-realization has been proposed by Satoru Fujishige, see: S.Fujishige, An efficient PQ-graph algorithm for solving the graph-realization problem, Journal of Computer and System Sciences 21(1980) 63-86.

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Received February 25, 1980.