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BEHAVIOUR OF SOLUTIONS OF SOME HYPERBOLIC SYSTEMS WITH NON-LINEAR BOUNDARY CONDITIONS

Introduction

In the present paper boundeness, exponential convergence to zero, stability and asymptotic stability of solutions of some systems of two partial differential equations of the first-order with non-linear boundary conditions are investigated. In the papers [2] ÷ [6] are considered these properties (or only some of them) with linear and zero boundary conditions. Some results concerning the stability of solutions of such system were presented also in [5]. In that paper the author considered the equations of the transmission line with constant coefficients and with special type non-linear boundary conditions. He proved the stability of solutions of this system by the frequency method.

The present paper continues these investigations and generalizes results of papers [3], [4] and partly of [5]. The proofs in this paper make use of the second method of Liapunov type.

Let $R = (-\infty, \infty)$, $I = \langle 0, \infty)$, $X = \langle 0,1\rangle$ and assume that we are given functions c, l, i, u: $X\times I \to R$; g,r: $X\times I\times R\times R \to R$; a: $I \to R$; h_1,h_2: $X \to R$; f_1,f_2,f_3,f_4: $R \to R$.

We consider the following system of equations

$$(E)\quad \begin{cases} c(x,t)u_t(x,t)+g(x,t,u(x,t),i(x,t))u(x,t)+a(t)i_x(x,t) = 0 \\ l(x,t)i_t(x,t)+r(x,t,u(x,t),i(x,t))i(x,t)+a(t)u_x(x,t) = 0 \end{cases}$$
$$x\in X,\quad t\in I$$

with the initial conditions

(IC) $u(x,0) = h_1(x), \quad i(x,0) = h_2(x)$ for $x \in X$

and one of the following three types of boundary conditions

(BC1) $$\begin{cases} i(0,t) = -f_1(u(0,t)) \\ i(1,t) = f_2(u(1,t)) \end{cases}$$

or

(BC2) $$\begin{cases} u(0,t) = -f_3(i(0,t)) \\ u(1,t) = f_4(i(1,t)) \end{cases}$$

or

(BC3) $$\begin{cases} i(0,t) = -f_1(u(0,t)) \\ u(1,t) = f_4(i(1,t)) \end{cases}$$

for $t \in I$.

We suppose that the following compatibility conditions are respectively satisfied:

$$\begin{cases} h_2(0) = -f_1(h_1(0)) \\ h_2(1) = f_2(h_1(1)), \end{cases}$$

or

$$\begin{cases} h_1(0) = -f_3(h_2(0)) \\ h_1(1) = f_4(h_2(1)), \end{cases}$$

or

$$\begin{cases} h_2(0) = -f_1(h_1(0)) \\ h_1(1) = f_4(h_2(1)). \end{cases}$$

Physically we may interpret the function u as current and the function v as voltage. Then the boundary conditions (BCn), $n = 1,2,3$ describe the interdependence of the initial voltages and the initial currents. This dependence is described by the functions f_n, $n = 1,2,3$.

A pair of functions $w = [u,i]$ is said to be a classical solution of the problem (E), IC), (BCn), n = 1,2,3 defined on $X \times I$, if $u,i \in C^1(X \times I)$ and the functions u,i and their partial derivatives u_t, u_x, i_t, i_x satisfy the system of equations (E), the initial conditions (IC), and the boundary conditions (BCn), n = 1,2,3.

In the paper, it is assumed that there exists at least one non-zero classical solution of the problem (E), (IC), (BCn), n = 1,2,3 defined on $X \times I$. The problem of existence for this type of systems has been investigated, among others, by V.Barbu and I.Vrobie [1].

In order to define boundedness, exponential convergence to zero stability and asymptotic stability of solutions of the problem (E), (IC),(BCn), n = 1,2,3 we introduce the space V consisting of real functions $v = v(x,t)$, $x \in X$, $t \in I$, and Cartesian product $W = V \times V$.

If $V = C^0(X \times I)$ then we define in space W two norms

$$\| w(\cdot,t)\|_1 := \left((\max_{x\in X} |v_1(x,t)|)^2 + (\max_{x\in X} |v_2(x,t)|)^2 \right)^{1/2},$$

and

$$\| w(\cdot,t)\|_2 := \left(\int_0^1 \left[v_1^2(x,t) + v_2^2(x,t) \right] dx \right)^{1/2},$$

where $w = [v_1, v_2]$ and $v_1, v_2 \in V$, t is a parameter, and $t \in I$.

If $V = C^1(X \times I)$ then we define in the space W the following norm

$$\| w(.,t)\|_3 := \left(\int_0^1 \left[v_1^2(x,t)+v_2^2(x,t)+v_{1,t}^2(x,t)+v_2^2(x,t) \right] dx \right)^{1/2}.$$

D e f i n i t i o n 1. A solution $w = [u,i]$ of the problem (E), (IC), (BCn, n = 1,2,3) is said to be:

a) bounded in the norm $\|\cdot\|_1$ ($\|\cdot\|_2$) if there exists a positive constant M such that for every $t \in I$ we have

$$\| w(\cdot,t)\|_1 \leqslant M \quad (\| w(\cdot,t)\|_2 \leqslant M);$$

b) exponentially convergent to zero in the norm $\|\cdot\|_1$ ($\|\cdot\|_2$) for $t \to \infty$ if there exists positive constants M, K such that for every $t \in I$

$$\|w(\cdot,t)\|_1 \leq M \exp(-Kt), \quad (\|w(\cdot,t)\|_2 \leq M \exp(-Kt)),$$

Definition 2. The zero solution $w_0 = [0,0]$ of the problem (E), (IC), (BCn, n = 1,2,3) is said to be:

a) stable in the norm $\|\cdot\|_2$ ($\|\cdot\|_3$) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every classical solution $w = [u,i]$ of the considered system (E) which satisfies boundary conditions the inequality $\|w(\cdot,0)\|_2 < \delta$ ($\|w(\cdot,0)\|_3 < \delta$) implies $\|w(\cdot,t)\|_2 < \varepsilon$ ($\|w(\cdot,t)\|_3 < \varepsilon$) for $t \in I$;

b) stable in the norm $\|\cdot\|_1$ with respect to the norm $\|\cdot\|_3$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every classical solution $w = [u,i]$ of the considered system (E), which satisfies boundary conditions the inequality $\|w(\cdot,0)\|_3 < \delta$ implies $\|w(\cdot,t)\|_1 < \varepsilon$ for $t \in I$;

c) asymptotically stable in the norm $\|\cdot\|_2$ ($\|\cdot\|_3$) if it is stable and in notations of the definition 2a) we have

$$\lim_{t \to \infty} \|w(\cdot,t)\|_2 = 0 \quad \left(\lim_{t \to \infty} \|w(\cdot,t)\|_3 = 0\right);$$

d) asymptotically stable in the norm $\|\cdot\|_1$ with respect to the norm $\|\cdot\|_3$ if it is stable and

$$\lim_{t \to \infty} \|w(\cdot,t)\|_1 = 0.$$

In the part 1 of this paper a quasi-linear system (E) (with functions $g,r : X \times I \times R \times R \to R$ and $a: I \to R$) will be considered. In that part the above mentioned properties of solutions of the problem (E), (IC), (BCn, n = 1,2,3) will be investigated only in the norm $\|\cdot\|_2$.

Second part of this paper deals with a linear system (E), with $g,r : X \times I \to R$ and $a = A = \text{const} > 0$. In that part boundedness, exponential convergence to zero will be considered in the norm $\|\cdot\|_1$ or $\|\cdot\|_2$. The stability and asymptotic stability will be investigated in the norm $\|\cdot\|_2$ or $\|\cdot\|_3$ and in the norm $\|\cdot\|_1$ with respect the norm $\|\cdot\|_3$.

1. A quasi-linear system

Consider the system of equations (E) with initial conditions (IC) and boundary conditions (BCn, n = 1,2,3). Let the functions describing the problem (E), (IC), (BCn, n = 1,2,3) satisfy the following conditions:

A1. There exist classical solutions $w = [u,i]$ of the problem (E), (IC), (BCn, n = 1,2,3) defined on $X \times I$.

A2. $u,i \in C^1(X \times I)$; $a \in C^0(I)$; $c,l \in C^{0,1}(X \times I)$; $g,r \in C^0(X \times I \times R \times R)$; $h_1,h_2 \in C^1(X)$; f_1, f_2, f_3, $f_4 \in C^0(R)$.

A3. There exists positive constants C_1, L_1 such that for every $(x,t) \in X \times I$ the inequalities

$$c(x,t) \geqslant C_1, \quad l(x,t) \geqslant L_1$$

are satisfied.

A4. For every $(x,t,y,z) \in X \times I \times R \times R$ we have

$$2g(x,t,y,z) \geqslant c_t(x,t), \qquad 2r(x,t,y,z) \geqslant l_t(x,t).$$

A5. For every $t \in I$ the inequality

$$a(t) \geqslant 0 \text{ is satisfied.}$$

A6. For every $v \in R$ and $j \in \{1,2,3,4\}$ we have

$$f_j(v) \cdot v \geqslant 0.$$

A3. There exists positive constants C_1, C_2, L_1, L_2 such that for every $(x,t) \in X \times I$ we have

$$C_1 \leqslant c(x,t) \leqslant C_2, \quad L_1 \leqslant l(x,t) \leqslant L_2.$$

A4. There exist positive constants K_1, K_2 such that for every $(x,t,y,z) \in X \times I \times R \times R$ we have

$$2g(x,t,y,z) \geqslant K_1 + c_t(x,t),\ 2r(x,t,y,z) \geqslant K_2 + l_t(x,t).$$

R e m a r k 1. It is easy to check that the functions $f_j(v) = v^{2p_j+1}$ $p_j \in N$ satisfy the assumption A6.

T h e o r e m 1.1. If the assumption A1-A6 are satisfied then every solution $w = [u,i]$ of the problem (E), (IC), (BCn, n = 1,2,3) is bounded in the norm $\|\cdot\|_2$.

P r o o f . Let $w = [u,i]$ be an arbitrary solution of the system (E) with conditions (IC) and (BCn, n = 1,2,3) and let the function of Liapunov type for this solution be of the form

$$(1.1) \qquad k_1(t) := \int_0^1 \left[c(x,t)u^2(x,t) + l(x,t)i^2(x,t)\right]dx \text{ for } t \in I.$$

In virtue of the assumptions A3 we have

$$k_1(t) \geqslant 0 \quad \text{for} \quad t \in I.$$

Next, by the assumption A2 the integrals in (1.1) are of class C^1 with respect to t and of class C^0 with respect to x. Thus we may interchange integration and differentiation with respect to t. After differentiation of k_1 with respect to t we obtain for $t \in I$

$$\dot{k}_1(t) = \int_0^1 (c_t u^2 + l_t i^2)dx + 2\int_0^1 (cuu_t + lii_t)dx.$$

Computing cu_t and li_t from the system (E) and introducing them into the previous formula we see that the last equality becomes

$$(1.2) \qquad \dot{k}_1(t) = \int_0^1 \left[(c_t - 2g)u^2 + (l_t - 2r)i^2\right]dx +$$

$$- 2a(t)\left[u(1,t)\ i(1,t) - u(0,t)\ i(0,t)\right].$$

Taking into account the boundary conditions (BCn, n=1,2,3) we have

$$(1.3)\qquad u(1,t)\,i(1,t) - u(0,t)\,i(0,t) =$$

$$= \begin{cases} f_2(u(1,t))u(1,t) + f_1(u(0,t))u(0,t) & \text{for (BC1)} \\ f_4(i(1,t))i(1,t) + f_3(i(0,t))i(0,t) & \text{for (BC2)} \\ f_4(i(1,t))i(1,t) + f_1(u(0,t))u(0,t)) & \text{for (BC3).} \end{cases}$$

By the formulae (1.2) and (1.3) and the assumptions A5, A6 we obtain the following inequality

$$(1.4)\qquad \dot{k}_1(t) \leqslant \int_0^1 \left[(c_t - 2g)u^2 + (l_t - 2r)i^2\right]dx \quad \text{for} \quad t \in I.$$

In view of the assumption A4 we have

$$\dot{k}_1(t) \leqslant 0 \quad \text{for} \quad t \in I.$$

Thus

$$(1.5)\qquad k_1(t) \leqslant k(0) \quad \text{for} \quad t \in I,$$

where

$$(1.6)\qquad k_1(0) = \int_0^1 \left[c(x,0)h_1^2(x) + l(x,0)h_2^2(x)\right]dx.$$

By the assumptions A3 and the definition (1.1) we get for $t \in I$

$$C_3\|w(\cdot,t)\|_2^2 \leqslant k_1(t),$$

where

$$(1.7)\qquad C_3 := \min\,(C_1,\, L_1).$$

From the last inequality and the inequality (1.5) it follows that

$$\|w(\cdot,t)\|_2 < M_1 \quad \text{for} \quad t \in I, \tag{1.8}$$

where

$$M_1 := \left(\frac{k_1(0)}{C_3}\right)^{\frac{1}{2}}. \tag{1.9}$$

This completes the proofs of Theorem 1.1.

T h e o r e m 1.2. If the assumptions A1, A2, $\overline{A3}$, $\overline{A4}$, A5, A6 hold, then every solution $w = [u,i]$ of the problem (E), (IC), (BCn), n = 1,2,3 is bounded and exponentially convergent to zero in the norm $\|\cdot\|_2$ for $t \to \infty$.

P r o o f. The boundedness of $w = [u,i]$ in the norm $\|\cdot\|_2$ follows from Theorem 1.1, as all its assertions are satisfied.

Now we prove the exponential convergence to zero in the norm $\|\cdot\|_2$ of any solution of the problem (E), (IC), (BCn), n = 1,2,3.

Taking the Liapunov type function k_1 in the form of (1.1) and making analogical transformations as in the proof of Theorem 1.1 we obtain the inequality (1.4). By the assumptions $\overline{A3}$, $\overline{A4}$ and the formula (1.1) (1.4) we obtain the inequality

$$-\dot{k}_1(t) \geq K_3\, k_1(t) \quad \text{for} \quad t \in I, \tag{1.10}$$

where

$$K_3 := \frac{1}{2} \min\,(K_1, K_2) \cdot \min\left(\frac{1}{C_2}, \frac{1}{L_2}\right).$$

From the last inequality we obtain

$$k_1(t) \leq k_1(0) \exp\,(-2K_3 t) \quad \text{for} \quad t \in I.$$

In virtue of the assumption $\overline{A3}$ and the formula (1.1) we get following estimation

(1.11) $$\|w(\cdot,t)\|_2 \leq M_1 \exp(-K_3 t) \quad \text{for} \quad t \in I,$$

where the constant M_1 is defined by the (1.9).

The inequality (1.11) implies exponential convergence to zero in the norm $\|\cdot\|_2$ of the solution $w = [u,i]$ for $t \to \infty$. This ends the proof of Theorem 1.2.

T h e o r e m 1.3. If the assumptions A1, A2, $\overline{A3}$, $\overline{A4}$, A5, A6 holds then the zero solution $w_0 = [0,0]$ of the problem (E), (BCn), n = 1,2,3, (IC: $h_1 \equiv h_2 \equiv 0$) is stable and asymptotically stable in the norm $\|\cdot\|_2$.

P r o o f . Consider any solution $w = [u,i]$ of the problem (E), (IC), (BCn), n = 1,2,3, such that

(1.12) $$\|w(\cdot,0)\|_2 < \delta .$$

Using the assumption $\overline{A3}$ from the identity (1.6) and the inequality (1.12) we have

(1.13) $$k_1(0) \leq C_4 \delta^2,$$

where

$$C_4 := \max(C_2, L_2).$$

From the inequality (1.11), (1.13) and the formula (1.9) we get

$$\|w(\cdot,t)\|_2 \leq \sqrt{\frac{C_4}{C_3}}\,\delta \exp(-K_3 t) \quad \text{for} \quad t \in I,$$

where the constant K_3 is defined by formula (1.10).

Putting $\delta = \frac{\varepsilon}{2}\sqrt{\frac{C_3}{C_4}}$ we obtain the estimation

(1.14) $$\|w(\cdot,t)\|_2 \leq \varepsilon \exp(-K_3 t) \quad \text{for} \quad t \in I.$$

From the inequality (1.14) it follows that the zero solution of the problem (E), (IC), (BCn), n = 1,2,3 is stable and asymptotically stable in the sense of definitions 2a and 2c, respectively.

2. A linear system

In this part we shall consider the system of equations

$$(E')\qquad \begin{cases} c(x,t)u_t(x,t) + g(x,t)u(x,t) + Ai_x(x,t) = 0 \\ l(x,t)i_t(x,t) + r(x,t)i(x,t) + Au_x(x,t) = 0, \end{cases}$$

(with functions $c,l,g,r\colon X\times I \to R$ and $A = \text{const} > 0$) with initial conditions (IC) and boundary conditions (BCn), n = 1,2,3.

We assume that:

A1′. There exist classical solutions $w = [u,i]$ of the problem (E′), (IC), (BCn), n = 1,2,3 defined on $X\times I$.

A2′. $u,i \in C^2(X\times I)$; $c,l \in C^{0,1}(X\times I)$, $g,r \in C^{0,2}(X\times I)$; $h_1,h_2 \in C^2(X)$; $f_1,f_2,f_3,f_4 \in C^1(R)$.

A3 . For every $(x,t)\in X\times I$ the inequalities

$$2g(x,t) \geqslant |c_t(x,t)|, \quad 2r(x,t) \geqslant |l_t(x,t)|$$

are satisfied.

A4′. There exist positive constants G_1, R_1 such that for every $(x,t)\in X\times I$ we have

$$0 < g(x,t) \leqslant G_1, \quad 0 < r(x,t) \leqslant R_1.$$

A5′. For every $(x,t)\in X\times I$ the inequalities

$$g_t(x,t) \geqslant 0, \quad g_{tt}(x,t) \leqslant 0; \quad r_t(x,t) \geqslant 0, \quad r_{tt}(x,t) \leqslant 0$$

are satisfied.

A6′. For every $v\in R$ and $j\in \{1,2,3,4\}$ we have

$$f_j'(v) \geqslant 0.$$

A3'. There exist positive constants K_4, K_5 such that for $(x,t) \in X \times I$ we have

$$2g(x,t) \geqslant K_4 + |c_t(x,t)|, \quad 2r(x,t) \geqslant K_5 + |l_t(x,t)|.$$

$\overline{A5}'$. There exist positive constants G_2, R_2 such that for $(x,t) \in X \times I$ we have

$$g_{tt}(x,t) + G_2 g_t(x,t) \leqslant 0, \qquad r_{tt}(x,t) + R_2 r_t(x,t) \leqslant 0.$$

We also assume that the assumptions $\overline{A3}$ and A6 from § 1 hold.

T h e o r e m 2.1. If the assumptions A1' ÷ A6' and $\overline{A3}$, $\dot{A6}$ are satisfied, then every solution $w = [u,i]$ of the problem (E'), (IC), (BCn, n = 1,2,3) and its derivatives are bounded in the norm $\|\cdot\|_2$. The solution $w = [u,i]$ is bounded in the norm $\|\cdot\|_1$, too.

P r o o f . The boundedness of a solution of the problem (E'), (IC), (BCn), n = 1,2,3 in the norm $\|\cdot\|_2$ follows from Theorem 1.1, as all its assertions are satisfied. The estimation (1.8) is true, too. In order to prove second part of this Theorem we introduce for an arbitrary solution $w = [u,i]$ of the problem (E'), (IC), (BCn), n = 1,2,3 the function of Liapunov type:

$$(2.1) \qquad k_2(t) := \int_0^1 \Big[c(x,t)u_t^2(x,t) + l(x,t)i_t^2(x,t) + g_t(x,t)u^2(x,t) + r_t(x,t)i^2(x,t)\Big]dx \quad \text{for} \quad t \in I.$$

In virtue of the assumptions $\overline{A3}$, A5' we have

$$k_2(t) \geqslant 0 \quad \text{for} \quad t \in I.$$

In view of the assumption A2' the function k_2 belongs to $C^1(I)$ and

$$\dot{k}_2(t) = \int_0^1 \Big[c_t u_t^2 + l_t i_t^2 + 2(cu_t u_{tt} + li_t i_{tt} +$$

$$+ g_t uu_t + r_t ii_t)\Big]dx + \int_0^1 (g_{tt}u^2 + r_{tt}i^2)dx.$$

After differentiating the system (E′) with respect to t we obtain

$$cu_{tt} = -(c_t u_t + gu_t + g_t u + Ai_{xt})$$

$$li_{tt} = -(l_t i_t + ri_t + r_t i + Au_{xt}).$$

Thus

$$cu_t u_{tt} + li_t i_{tt} = - \Big[(c_t + g)u_t^2 + (l_t + r)i_t^2\Big] +$$

$$- (g_t uu_t + r_t ii_t) - A(u_t i_t)_x.$$

Taking into account the above identity for $t \in I$ we have

(2.2) $$\dot{k}_2(t) = \int_0^1 \Big[(2g + c_t)u_t^2 + (2r + l_t)i_t^2\Big]dx +$$

$$+ \int_0^1 (g_{tt}u^2 + r_{tt}i^2)dx - 2A\Big[u_t(1,t)i_t(1,t) +$$

$$- u_t(0,t)i_t(0,t)\Big].$$

Taking into consideration the boundary conditions (BCn), $n = 1,2,3$ we get

(2.3) $$u_t(1,t)i_t(1,t) - u_t(0,t)i_t(0,t) =$$

$$= \begin{cases} f_2'(u(1,t))u_t^2(1,t) + f_1'(u(0,t))u_t^2(0,t) & \text{for (BC1)} \\ f_4'(i(1,t))i_t^2(1,t) + f_3'(i(0,t))i_t^2(0,t) & \text{for (BC2)} \\ f_4'(i(1,t))i_t^2(1,t) + f_1'(u(0,t))u_t^2(0,t) & \text{for (BC3).} \end{cases}$$

In virtue of the identities (2.2), (2.3) and the assumption A6′ we have

$$(2.4) \qquad \dot{k}_2(t) \leqslant -\int_0^1 \left[(2g + c_t)u_t^2 + (2r + l_t)i_t^2\right]dx + \\ + \int_0^1 (g_{tt}u^2 + r_{tt}i^2)dx \quad \text{for} \quad t \in I.$$

By the assumptions A3′ and A5′ we get the inequality

$$\dot{k}_2(t) \leqslant 0 \quad \text{for} \quad t \in I.$$

Thus

$$(2.5) \qquad k_2(t) \leqslant k_2(0) \quad \text{for} \quad t \in I,$$

where

$$(2.6) \qquad k_2(0) = \int_0^1 \left[c(x,0)u_t^2(x,0) + l(x,0)i_t^2(x,0) + \\ + g_t(x,0)h_1^2(x) + r_t(x,0)h_2^2(x)\right]dx.$$

In view of the assumptions $\overline{A3}$, A5′ and the definition (2.1) it follows that

$$(2.7) \qquad \int_0^1 \left[u_t^2(x,t) + i_t^2(x,t)\right]dx \leqslant \frac{k_2(t)}{C_3} \quad \text{for} \quad t \in I,$$

where the constant C_3 is defined by the formula (1.7).

From the inequalities (2.7) and (2.5) we get the following estimation:

$$(2.8) \qquad \| w_t(\cdot,t) \| \leqslant M_2,$$

where

$$(2.9) \qquad M_2 := \sqrt{\frac{k_2(0)}{C_3}} \, .$$

Computing $u_x(x,t)$ and $i_x(x,t)$ from the system (E′) we obtain

$$\int_0^1 u_x^2(x,t)dx = \frac{1}{A^2}\int_0^1 \left[l(x,t)i_t(x,t) + r(x,t)i(x,t)\right]^2 dx \leqslant$$
$$\leqslant \frac{2}{A^2}\int_0^1 \left[l^2(x,t)i_t^2(x,t) + r^2(x,t)i^2(x,t)\right]dx$$

and

$$\int_0^1 i_x^2(x,t)dx = \frac{1}{A^2}\int_0^1 \left[c(x,t)u_t(x,t) + g(x,t)u(x,t)\right]^2 dx \leqslant$$
$$\leqslant \frac{2}{A^2}\int_0^1 \left[c^2(x,t)u_t^2(x,t) + g^2(x,t)u^2(x,t)\right]dx.$$

In virtue of the above, the estimations (1.8), (2.8) and the assumptions $\overline{A3}$, $A4'$ we have

$$(2.10) \qquad \|w_x(\cdot,t)\| \leqslant M_3,$$

where

$$(2.11) \qquad M_3 := \frac{\sqrt{2\left[(G_1^2 + R_1^2)M_1^2 + (C_2^2 + L_2^2)M_2^2\right]}}{A} \, .$$

Now we prove the boundedness of solutions of the problem (E′), (IC), (BCn), n = 1,2,3 in the norm $\|\cdot\|_1$. From the mean value theorem we have

$$\int_0^1 u(x,t)dx = u(\xi,t), \quad \text{where} \quad \xi \in X.$$

Making use of Schwarz s inequality we obtain

$$u^2(\xi,t) \leqslant \int_0^1 u^2(x,t)dx \quad \text{for} \quad t \in I. \tag{2.12}$$

The last inequality and the estimation (1.8) imply

$$|u(\xi,t)| \leqslant M_1 \quad \text{for} \quad t \in I,$$

where the constant M_1 is defined by the formula (1.9). On the other hand, making use of the formula

$$u(x,t) - u(\xi,t) = \int_\xi^x u_x(x,t)dx$$

and Schwartz's inequality we infer that

$$[u(x,t) - u(\xi,t)]^2 \leqslant |x - \xi| \left| \int_\xi^x u_x^2(x,t)dx \right| \leqslant \tag{2.13}$$
$$\leqslant \int_0^1 u_x^2(x,t)dx \quad \text{for} \quad t \in I,\ x,\ \xi \in X.$$

From the above and the inequality (2.10) we have

$$|u(x,t) - u(\xi,t)| \leqslant M_3 \quad \text{for} \quad t \in I,\ x, \xi \in X.$$

Thus

$$|u(x,t)| \leqslant M_1 + M_3 \quad \text{for} \quad (x,t) \in X \times I.$$

Similarly we can obtain the estimation

$$|i(x,t)| \leqslant M_1 + M_3 \quad \text{for} \quad (x,t) \in X \times I.$$

Therefore

$$\| w(.,t) \|_1 \leqslant M_4 \quad \text{for} \quad t \in I, \tag{2.14}$$

where

$$M_4 := (M_1 + M_3)\sqrt{2},$$

which proves the stated results.

Theorem 2.2. If the assumptions $A1'$, $A2'$, $\overline{A3}'$, $A4'$, $\overline{A5}'$, $A6$ and $\overline{A3}$, $A6$ are satisfied then any solution $w = [u,i]$ of the problem (E'), (IC), (BCn), $n = 1,2,3$ and its derivatives are exponentially convergent to zero in the norm $\|\cdot\|_2$ for $t \to \infty$ and any solution $w = [u,i]$ is convergent to zero in the norm $\|\cdot\|_1$ for $t \to \infty$.

Proof. The first part of this theorem follows from Theorem 1.2, as all its assertions are satisfied and the estimation (1.11) is true, too. Second part of the proof is similar to the proof of Theorem 1.2. Taking the Liapunov type function in the form of (2.1) and making analogical transformations we obtain the estimation (2.4). By the assumptions $\overline{A3}'$ and $\overline{A6}'$ we have from (2.4)

$$-\dot{k}_2(t) \geq \int_0^1 (K_4 u_t^2 + K_5 i_t^2)dx + \int_0^1 (G_2 g_t u^2 + R_2 r_t i^2)dx \quad \text{for} \quad t \in I,$$

thus

$$-\dot{k}_2(t) \geq \int_0^1 (K_4 u_t^2 + K_5 i_t^2)dx + K_6 \int_0^1 (g_t u^2 + r_t i^2)dx,$$

where

$$K_6 := \min (G_2, R_2).$$

Using the assumption $\overline{A3}$ we obtain

$$-\dot{k}_2(t) \geq K_7 \int_0^1 (c u_t^2 + l i_t^2)dx + K_6 \int_0^1 (g_t u^2 + r_t i^2)dx \quad \text{for} \quad t \in I,$$

where

$$K_7 := \min (K_4, K_5) \cdot \min \left(\frac{1}{C_2}, \frac{1}{L_2}\right).$$

From the above estimation it follows that

$$\dot{k}_2(t) \leqslant -2K_8\, k_2(t) \quad \text{for} \quad t \in I, \tag{2.15}$$

where

$$K_8 := \frac{1}{2} \min (K_6, K_7).$$

Thus

$$k_2(t) \leqslant k_2(0) \exp (-2K_3 t) \quad \text{for} \quad t \in I. \tag{2.16}$$

Applying the inequalities (2.7), (2.16) and the formula (2.9) we have

$$\| w_t(\cdot,t) \|_2 \leqslant M_2 \exp (-K_8 t) \quad \text{for} \quad t \in I. \tag{2.17}$$

Further the proof is similar to the one in Theorem 2.1. By the assumptions A4$'$, $\overline{A3}$ and the estimations (1.11), (2.17) we have

$$\int_0^1 u_x^2(x,t)dx \leqslant \frac{2}{A^2} \left[R_1^2 M_1^2 \exp (-2K_3 t) + L_2^2 M_2^2 \exp (-2K_8 t)\right],$$

$$\int_0^1 i_x^2(x,t)dx \leqslant \frac{2}{A^2} \left[G_1^2 M_1^2 \exp (-2K_3 t) + C_2^2 M_2^2 \exp (-2K_8 t)\right].$$

The above inequalities and formula (2.11) imply

$$\| w_x(\cdot,t) \|_2 \leqslant M_3 \exp (-K_9 t) \quad \text{for} \quad t \in I, \tag{2.18}$$

where

$$K_9 := \min (K_3, K_8). \tag{2.19}$$

In virtue of the inequalities (2.12), (2.13) and the estimations (1.11), (2.18) we also get

$$|u(x,t)| \leqslant M_1 \exp(-K_3 t) + M_3 \exp(-K_9 t)$$

$$|i(x,t)| \leqslant M_1 \exp(-K_3 t) + M_3 \exp(-K_9 t) \text{ for } (x,t) \in X \times I.$$

Denoting $K := \min (K_3, K_9)$, $M_4 := \max (M_1, M_3)$ and using the formula (2.14) we have

$$\| w(\cdot, t) \|_1 \leqslant M_4 \exp(-Kt). \tag{2.20}$$

The inequalities (2.17), (2.18), (2.20) imply exponential convergence to zero in the norm $\| \cdot \|_1$ for $t \to \infty$ of solution $w = [u, i]$ of the problem (E$'$), (IC), (BCn), n = 1,2,3 and its derivatives in the norm $\| \cdot \|_2$. This ends the proof of Theorem 2.2.

T h e o r e m 2.3. If the assumptions A1$'$, A2$'$, $\overline{A3}'$, A4$'$, $\overline{A5}'$, A6$'$ and $\overline{A3}$, A6 hold, then every solution $w = [u, i]$ of the problem (E$'$), (IC), (BCn), n = 1,2,3 is stable and asymptotically stable in the norms $\| \cdot \|_2$, $\| \cdot \|_3$ and in the norm $\| \cdot \|_1$ with respect the norm $\| \cdot \|_3$.

P r o o f . As the system of the equations (E$'$) is linear, it is sufficient to prove the stability and asymptotic stability of zero solution of the problem (E$'$), (IC), (BCn), n = 1,2,3.

The stability and the asymptotic stability of zero solution our problem in the norm $\| \cdot \|_2$ follows from Theorem 1.3, as all its assertions are satisfied.

Now we prove this properties in the norm $\| \cdot \|_3$.

In virtue of the assumptions A2$'$ we infer that there exist positive constants G_3, R_3 such that for every $x \in X$ we have

(2.21) $$|g_t(x,0)| \leqslant G_3, \quad |r_t(x,0)| \leqslant R_3.$$

Consider any solution $w = [u,i]$ of the problem (E'), (IC), (BCn), n = 1,2,3 such that

(2.22) $$\| w(\cdot,0)\|_3 < \delta .$$

By the assumption $\overline{A3}$, the formula (2.6) and the inequalities (2.21), (2.22) we get

(2.23) $$k_2(0) \leqslant C_5\delta^2,$$

where $C_5 := \max\,(C_2,L_2,G_3,R_3)$.

Hence (2.22) implies $\| w(\cdot,0)\|_2 < \delta$. In view of the last inequality, the assumption $\overline{A3}$ and the formula (1.6) we obtain

(2.24) $$k_1(0) \leqslant C_5\delta^2.$$

The estimations (1.11), (2.17) and the formula (2.19) imply

(2.25) $$\| w(\cdot,t)\|_3 \leqslant \sqrt{M_1^2 + M_2^2}\, \exp\,(-K_9 t) \quad \text{for} \quad t \in I.$$

Using the formulae (1.9), (2.9) and the inequalities (2.23), (2.24) and (2.25) we obtain

$$\| w(\cdot,t)\|_3 \leqslant \delta\sqrt{\frac{C_5}{C_3}}\, \exp\,(-K_9 t) \quad \text{for} \quad t \in I.$$

Putting $\delta = \frac{\varepsilon}{2}\sqrt{\frac{C_3}{C_5}}$ we have

(2.26) $$\| w(\cdot,t)\|_3 < \varepsilon \exp\,(-K_9 t) \quad \text{for} \quad t \in I.$$

The inequality (2.26) implies the stability and the asymptotic stability the zero solution $w_0 = [0,0]$ of the considered problem in the norm $\| \cdot \|_3$.

The stability and the asymptotic stability of the zero solution $w_0 = [0,0]$ of the problem (E'), (IC), (BCn), n=1,2,3 in the norm $\|\cdot\|_1$ with respect the norm $\|\cdot\|_3$ follows from the inequality

$$\|w(\cdot,t)\|_1 < \varepsilon \exp(-Kt) \quad \text{for} \quad t \in I \tag{2.27}$$

(if $\|w(\cdot,t)\|_3 < \delta$).

We will show this inequality. Using the formulae (1.9), (2.9), (2.11), (2.14) and the inequalities (2.23), (2.24) we obtain

$$M_4 \leqslant \delta C, \tag{2.28}$$

where

$$C := \sqrt{\frac{2C_5}{C_3}} + \frac{2}{A}\sqrt{\frac{C_5}{C_3}\left(G_1^2 + R_1^2 + C_2^2 + L_2^2\right)}.$$

In view of (2.20) and (2.28) we have

$$\|w(\cdot,t)\|_1 < C\,\delta \exp(-Kt) \quad \text{for} \quad t \in I.$$

Finally, putting $\delta = \frac{\varepsilon}{2C}$ we get the intequality (2.27) and the desired properties. The proof is complete.

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