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ALMOST r-CONTACT STRUCTURE IN A PRODUCT MANIFOLD

An almost contact structure in a complex manifold and in a product manifold has been studied by Mishra [3], Upadhyay and Dube [4] respectively, by making use of index free notations. Dube and Niwas [1] defined and studied the almost r-contact structure in a product manifold. The purpose of the present paper is to study some properties of an almost r-contact structure in a product manifold.

1. Preliminaries

Let M^{n+r} be an (n+r)-dimensional differentiable manifold of class C^∞. Let there exist in M^{n+r} a (1,1) tensor field F of class C^∞, r-contravariant linearly independent vector fields $T^1, T^2, \ldots, T^r$ of class C^∞ and r 1-forms $A_1, A_2, \ldots, A_r$ of class C^∞ satisfying [1] the following condition

$$\bar{\bar{X}} = X + \sum_{p=1}^{r} A_p(X)T^p, \tag{1.1}$$

for an arbitrary vector field X on M^{n+r}, where

$$\bar{X} \overset{\text{def}}{=} F(X) \tag{1.2}$$

and

$$\bar{T}^p = 0, \quad 1 \leqslant p \leqslant r; \tag{1.3}$$

and

$$A_p(\bar{X}) = 0, \tag{1.4}$$

for an arbitrary vector field X and $1 \leqslant p \leqslant r$; then

$$A_p(T^q) + \delta_p^q = 0, \tag{1.5}$$

where $1 \leqslant p, q \leqslant r$ and δ denotes the Kronecker delta.

Thus M^{n+r} satisfying the conditions (1.1), (1.2), (1.3), (1.4) and (1.5) is said to possess an almost r-contact structure in a product manifold [1].

In the above and in what follows, the equations containing X,Y,Z hold for arbitrary vector fields X,Y,Z.

T h e o r e m 1.1. Let M^{n+r} be an almost r-contact product manifold; then there are r-eigenvalues zero corresponding to eigenvectors T^p $(1 \leqslant p \leqslant r)$ and if k values correspond to 1, then (n-k) eigenvalues correspond to -1.

P r oo f . Let λ be an eigenvalue of F and L be the corresponding eigenvector. Then

$$\bar{L} = \lambda L. \tag{1.6}$$

Barring (1.6) and making use of (1.1) and (1.6), we get

$$(\lambda^2 - 1)L = \sum_{p=1}^{r} A_p(L)T^p. \tag{1.7}$$

<u>Case</u> 1. Let $L = T^p$, then from (1.5) and (1.7) we get $\lambda = 0$. This gives the eigenvalue corresponding to the eigenvector T^p.

<u>Case</u> 2. If L and $T^1, \dots, T^r$ are linearly independent, then from (1.7) we have

$$A_p(L) = 0 \quad \text{and} \quad \lambda = \pm 1.$$

If k values correspond to the eigenvalue 1, then n-k values will correspond to the eigenvalue -1.

2. The metric tensor

Let the almost r-contact product manifold be endowed with a non-singular metric tensor g. From (1.1) and (1.4) we have

$$\bar{\bar{\bar{X}}} = \bar{X}. \tag{2.1}$$

Let us restrict g such that

$$g(\bar{X},\bar{Y}) = g(\bar{\bar{X}},\bar{\bar{Y}}) \tag{2.2}$$

and

$$g(T^p,X) = A_p(X). \tag{2.3}$$

Equation (2.2) in view of (1.1), (1.4), (2.1) and (2.3) yields

$$g(X,\bar{Y}) = g(\bar{X},Y). \tag{2.4}$$

Thus we have

$$g(\bar{X},\bar{Y}) = g(\bar{\bar{X}},Y) = g(X,\bar{\bar{Y}}). \tag{2.5}$$

In consequence of (1.1) and (2.3), the equation (2.5) is equivalent to

$$g(\bar{X},\bar{Y}) = g(X,Y) + \sum_{p=1}^{r} A_p(X)A_p(Y). \tag{2.6}$$

We call an almost r-contact product manifold M^{n+r} endowed with non-singular metric tensor g satisfying (2.6) as "an almost r-contact product metric manifold".

Let us define a tensor $'F$ by

$$'F(X,Y) \overset{\text{def}}{=} g(\bar{X},Y). \tag{2.7}$$

From (2.5) and (2.7) we have

$$'F(X,\bar{Y}) = g(\bar{X},\bar{Y}) = g(\bar{\bar{X}},Y) = {}'F(\bar{X},Y). \tag{2.8}$$

Putting $X = T^p$ in (2.7) and making use of (1.3), we get

(2.9) $$'F(T^p,Y) = 0.$$

T h e o r e m 2.1. The tensor $'F$ defined by (2.7) is hybrid in both the slots and is also symmetric.

3. Almost r-contact product manifold with specified affine connexion

Let us take the affine connexion D in M^{n+r} such that it satisfies the following conditions

(3.1) $$D_X T^p = \bar{X},$$

(3.2) $$D_X \bar{Y} = D_Y \bar{X} + \overline{[X,Y]} + A_p(Y)X,$$

(3.3) $$(D_X A_p)(Y) + (D_Y A_p)(X) = 0; \quad 1 \leqslant p \leqslant r.$$

Let S be the torsion tensor of D, K the curvature tensor with respect to D and Ric be the corresponding Ricci tensor. Then

(3.4) $$S(X,Y) \overset{\text{def}}{=} D_X Y - D_Y X - [X,Y];$$

(3.5) $$K(X,Y,Z) \overset{\text{def}}{=} D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z;$$

(3.6) $$\mathrm{Ric}(Y,Z) \overset{\text{def}}{=} (C_1^1 K)(Y,Z),$$

where $C_1^1 K$ denotes the contraction of K.

Now we shall prove the following theorems.

T h e o r e m 3.1. Let $1 \leqslant p \leqslant r$, then we have

(3.7) $$S(\bar{X},T^p) = \overline{[X,T^p]} - [\bar{X},T^p],$$

(3.8) $$\overline{D_{T^p}\bar{X}} = \bar{X} - [X,T^p] - \sum_{p=1}^{r} A_p([X,T^p])T^p,$$

$$(3.9)\qquad (D_XF)(\bar{Y}) - \overline{(D_XF)(Y)} = X\sum_{p=1}^{r} A_p(Y)T^p + A_p(Y)\bar{X} - \sum_{p=1}^{r} A_p(D_XY)T^p.$$

P r o o f . Putting T^p for Y in (3.4) and making use of (3.1), we get

$$(3.10)\qquad S(X,T^p) = \bar{X} - D_{T^p}(X) - [X,T^p].$$

Barring X in (3.2) and making use of (1.1) and (3.1), we obtain

$$D_{\bar{X}}\bar{Y} = D_YX + D_Y\left(\sum_{p=1}^{r} A_p(X)T^p\right) + \overline{[\bar{X},Y]} + \bar{X}\,A_p(Y) =$$

$$= D_YX + A_p(X)\bar{Y} + Y\sum_{p=1}^{r} A_p(X)T^p + \overline{[\bar{X},Y]} + \bar{X}\,A_p(Y).$$

Putting T^p for Y in the above equation and making use of (1.3), (1.5) and (3.3), we get

$$(3.11)\qquad D_{T^p}X = D_{\bar{X}}\bar{T}^p - \sum_{p=1}^{r} A_p(X)\bar{T}^p - T^p\sum_{p=1}^{r} A_p(X)T^p - \overline{[\bar{X},T^p]} - \bar{X}\sum_{p=1}^{r} A_p(T^p) =$$

$$= \bar{X} - \overline{[\bar{X},T^p]} - T^p\sum_{p=1}^{r} A_p(D_{T^p}(X)).$$

Therefore in consequence of (3.10) and (3.11), we have

$$S(\bar{X},T^p) = \overline{[\bar{\bar{X}},T^p]} + T^p \sum_{p=1}^{r} A_p(D_{T^p}(\bar{X})) - [\bar{X},T^p],$$

which in view of (1.1), (1.4) and (3.3) yields (3.7). Barring X in (3.11) and making use of (1.1), (1.4) and (3.3), we obtain

$$(3.12) \qquad D_{T^p}\bar{X} = \bar{\bar{X}} - \overline{[\bar{\bar{X}},T^p]} - T^p \sum_{p=1}^{r} A_p(D_{T^p}(\bar{X})) =$$

$$= X + \sum_{p=1}^{r} A_p(X)T^p - \overline{[X,T^p]}.$$

Barring (3.12) throughout and making use of (1.1) and (1.3) we get (3.8).

We know [5] that

$$(3.13) \qquad (D_X F)(Y) = D_X\bar{Y} + \overline{D_X Y}.$$

Barring Y in (3.13) and making use of (1.1) and (3.1), we obtain

$$(3.14) \qquad (D_X F)\bar{Y} = D_X Y + D_X\left(\sum_{p=1}^{r} A_p(Y)T^p\right) + \overline{D_X\bar{Y}} =$$

$$= D_X Y + X \sum_{p=1}^{r} A_p(Y)T^p + A_p(Y)\bar{X} + \overline{D_X\bar{Y}}.$$

Barring (3.13) throughout and making use of (1.1), we get

$$(3.15) \qquad \overline{(D_X F)Y} = \overline{D_X\bar{Y}} + D_X Y + \sum_{p=1}^{r} A_p(D_X Y)T^p.$$

Subtracting (3.15) from (3.14) we have (3.9).

T h e o r e m 3.2. If G is the Einstein tensor, then we have

$$G(\bar{X}, T^p) = 0. \tag{3.16}$$

P r o o f . We have [2]

$$G(X,Y) = \mathrm{Ric}(X,Y) - \frac{1}{2} k\, g(X,Y). \tag{3.17}$$

Putting T^p for Y in (3.17) and making use of (2.3), we get

$$G(X,T^p) = \left\{(n-1) - \frac{1}{2} k\right\} A_p(X), \tag{3.18}$$

since [3]

$$\mathrm{Ric}(X,T^p) = (n-1)A_p(X).$$

Barring X in (3.18) and making use of (1.4) we obtain (3.16).

4. The Nijenhuis tensor

Let $N(X,Y)$ be the Nijenhuis tensor of F, then we have

$$N(X,Y) = [\bar{X},\bar{Y}] - \overline{[\bar{X},Y]} - \overline{[X,\bar{Y}]} + \overline{\overline{[X,Y]}}, \tag{4.1}$$

which in view of (1.1) becomes

$$N(X,Y) = [\bar{X},\bar{Y}] - \overline{[\bar{X},Y]} - \overline{[X,\bar{Y}]} + [X,Y] + \sum_{p=1}^{r} A_p([X,Y])T^p. \tag{4.2}$$

T h e o r e m 4.1. In an almost r-contact product manifold, we have

$$(4.3)\quad \overline{N(\bar{X},Y)} = -\left\{N(\bar{X},Y) + \sum_{p=1}^{r} A_p(X)N(T^p,Y) + \sum_{p=1}^{r} A_p([\bar{X},\bar{Y}])T^p\right\},$$

and

$$(4.4)\quad N(\bar{X},Y) - N(X,\bar{Y}) = \sum_{p=1}^{r} A_p(X)([T^p,\bar{Y}] - \overline{[T^p,Y]}) +$$

$$+ \sum_{p=1}^{r} A_p(Y)(\overline{[X,T^p]} - [\bar{X},T^p]) + \sum_{p=1}^{r} A_p([\bar{X},Y])T^p -$$

$$- \sum_{p=1}^{r} A_p([X,\bar{Y}])T^p.$$

P r o o f . Barring X in (4.2) and then barring throughout, we obtain

$$\overline{N(\bar{X},Y)} = [\bar{\bar{X}},\bar{Y}] - \overline{[\bar{\bar{X}},Y]} - \overline{[\bar{X},\bar{Y}]} + \overline{[\bar{X},Y]} + \sum_{p=1}^{r} \overline{A_p([\bar{X},Y])T^p},$$

which in view of (1.1) and (1.3) yields

$$(4.5)\quad \overline{N(\bar{X},Y)} = \overline{\left[X + \sum_{p=1}^{r} A_p(X)T^p,\bar{Y}\right]} - [\bar{\bar{X}},Y] - \sum_{p=1}^{r} A_p([\bar{\bar{X}},Y])T^p - [\bar{X},\bar{Y}] -$$

$$- \sum_{p=1}^{r} A_p([\bar{X},\bar{Y}])T^p + \overline{[\bar{X},Y]} = \overline{[X,\bar{Y}]} + \sum_{p=1}^{r} A_p(X)\overline{[T^p,\bar{Y}]} -$$

$$- [X,Y] - \sum_{p=1}^{r} A_p(X)\,[T^p,Y] -$$

$$- \sum_{p=1}^{r} A_p([X,Y] + \sum_{q=1}^{r} A_q(X)[T^q,Y])T^p -$$

$$- [\bar{X},\bar{Y}] - \sum_{p=1}^{r} A_p([\bar{X},\bar{Y}])T^p + \overline{[\bar{X},Y]} =$$

$$= -\Big\{[\bar{X},\bar{Y}] - \overline{[\bar{X},Y]} - \overline{[X,\bar{Y}]} + [X,Y] + \sum_{p=1}^{r} A_p([X,Y])T^p +$$

$$+ \sum_{p=1}^{r} A_p(X)\Big([T^p,Y] - \overline{[T^p,\bar{Y}]} + \sum_{q=1}^{r} A_q([T^p,Y])T^q\Big) +$$

$$+ \sum_{p=1}^{r} A_p([\bar{X},\bar{Y}])T^p\Big\}.$$

Putting T^p for X in (4.2) and making use of (1.3), we get

$$(4.6) \qquad N(T^p,Y) = [T^p,Y] - \overline{[T^p,\bar{Y}]} + \sum_{q=1}^{r} A_q([T^p,Y])T^q.$$

Equation (4.5) in consequence of (4.2) and (4.6) yields (4.3).

Barring X and Y respectively in (4.1) and making use of (1.1) we get (4.4).

Theorem 4.2. In an almost r-contact product manifold, we have

$$(4.7) \qquad A_pN(X,Y) = -\sum_{q=1}^{r} A_q(X)A_p(N(T^q,Y)) + A_p([\bar{X},\bar{Y}]),$$

and

$$(4.8) \qquad \overline{N(\bar{X},T^p)} = - N(X,T^p).$$

Proof. Operating (4.3) by A_p and making use of (1.5) and (1.4) we obtain (4.7). Replacing Y by T^p in (4.3) and making use of (1.3) and (4.1) we get (4.8).

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