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ON SOME DIFFERENTIAL INEQUALITIES FOR HOLOMORPHIC FUNCTIONS OF MANY VARIABLES

1. Introduction

Let $f(\zeta)$ be a holomorphic function in the unit disk $K \subset \mathbb{C}$. By h we denote a complex function defined at a some domain $\Delta \subset \mathbb{C}^3$. S.S.Miller and P.T.Mocanu indicated (see [1]) the conditions which should be satisfied by the function h in order that the inequality $\operatorname{re} f(\zeta) > 0$ resulted from the inequality $\operatorname{re} h(f(\zeta), \zeta f'(\zeta), \zeta^2 f''(\zeta)) > 0$.

In the present paper the authors extend this result to holomorphic functions of many variables.

Let $z = (z_1, \dots, z_n)$ and $\mathring{r} = (\mathring{r}_1, \dots, \mathring{r}_n)$. Let $P_{\mathring{r}} = \{z \in \mathbb{C}^n : |z_k| < \mathring{r}_k, \ k = 1, \dots, n\}$ and in particular

$$P = \{z \in \mathbb{C}^n : |z_k| < 1, \quad k = 1, \dots, n\}.$$

We shall consider the family H_P of the holomorphic functions $F : P \to \mathbb{C}$. Let $DF(z)(w)$ denote the value of the first differential of the function F at the point z on the vectors w and let $D^2F(z)(w,w)$ denote the value of the second differential at the point z on the vectors w and w, where $w = (w_1, \dots, w_n)$.

2. The main results of this paper is as follows:

Theorem 1. Let $F \not\equiv \text{const}$, $F(0) = 0$ and $F \in H_P$. If at the point

(1) $$\mathring{z} = \left(\mathring{r}_1 e^{i\mathring{\varphi}_1}, \dots, \mathring{r}_n e^{i\mathring{\varphi}_n}\right) \in P$$

the following equality holds

(2) $$|F(\mathring{z})| = \max_{z \in \bar{P}_{\mathring{r}}} |F(z)|,$$

then we have

(3) $$\frac{D\,F(\mathring{z})(\mathring{z})}{F(\mathring{z})} = m \geqslant 1$$

and

(4) $$\operatorname{re} \frac{D^2F(\mathring{z})(\mathring{z},\mathring{z})}{D\,F(\mathring{z})(\mathring{z})} + 1 \geqslant m.$$

Proof. Consider a point

(5) $$z = \left(\mathring{r}_1 e^{i\varphi_1}, \dots, \mathring{r}_n e^{i\varphi_n}\right) \in P,$$

where $\mathring{r}_k = |\mathring{z}_k|$, $k = 1,\dots,n$ and $\varphi_1,\dots,\varphi_n$ are arbitrary real numbers.

Let

(6) $$F(z) = R(\varphi_1,\dots,\varphi_n) e^{i\Phi(\varphi_1,\dots,\varphi_n)},$$

where R and Φ are real functions of the real variables $\varphi_1,\dots,\varphi_n$.

It is easy to see that

$$z_k F'_{z_k}(z) = e^{i\Phi}\left(\frac{1}{i}\frac{\partial R}{\partial \varphi_k} + R\frac{\partial \Phi}{\partial \varphi_k}\right), \quad k = 1,\dots,n.$$

Therefore we have

$$\frac{D\,F(z)(z)}{F(z)} = \sum_{k=1}^{n}\left(\frac{\partial\Phi}{\partial\varphi_k} - \frac{i}{R}\frac{\partial R}{\partial\varphi_k}\right).$$

From the condition (2) and the notations (5) and (6) it follows that the function R attains a local maximum at the point $(\mathring{\varphi}_1,\dots,\mathring{\varphi}_n)$. Hence we have

(7) $$\left.\frac{\partial R}{\partial\varphi_k}\right|_{(\mathring{\varphi}_1,\dots,\mathring{\varphi}_n)} = 0, \quad k = 1,\dots,n$$

and

$$\frac{D\,F(\mathring{z})(\mathring{z})}{F(\mathring{z})} = \operatorname{re}\frac{D\,F(\mathring{z})(\mathring{z})}{F(\mathring{z})} = \sum_{k=1}\left.\frac{\partial\Phi}{\partial\varphi_k}\right|_{(\mathring{\varphi}_1,\dots,\mathring{\varphi}_n)}.$$

Denote

$$\frac{D\,F(\mathring{z})(\mathring{z})}{F(\mathring{z})} = m.$$

We shall now prove, that $m \geqslant 1$. For this purpose let us consider the function

$$G(z) = \frac{F(z_1\mathring{z}_1,\dots,z_n\mathring{z}_n)}{F(\mathring{z})}, \quad z = (z_1,\dots,z_n) \in P.$$

From the maximum principle for the modulus of the holomorphic function of many variables it follows that $F(\mathring{z}) \neq 0$, therefore the function G is holomorphic on P. Moreover, $G(0) = 0$ and, according to (2), $|G(z)| < 1$ for $z \in P$.

From Schwarz's lemma for functions of many variables, we obtain

$$|G(z)| \leqslant \|z\| = \max_{k=1,\dots,n} |z_k|.$$

In particular, for $0<\varrho<1$ we have $|G(\varrho,\dots,\varrho)|\leqslant\varrho$ and as a consequence $\operatorname{re} G(\varrho,\dots,\varrho)\leqslant\varrho$ we obtains

$$\operatorname{re}\frac{F(\varrho\mathring{z})}{F(\mathring{z})}\leqslant\varrho,\qquad 0<\varrho<1.$$

It can be observed that

$$\frac{d}{d\varrho}\left(\frac{F(\varrho\mathring{z})}{F(\mathring{z})}\right)=\frac{\sum_{k=1}^{n}\varrho\mathring{z}_k F'_{\varrho\mathring{z}_k}(\varrho\mathring{z})}{\varrho F(\mathring{z})}$$

hence we have

$$m=\frac{D\,F(\mathring{z})(\mathring{z})}{F(\mathring{z})}=\frac{d}{d\varrho}\left(\frac{F(\varrho\mathring{z})}{F(\mathring{z})}\right)\Bigg|_{\varrho=1}=\lim_{\varrho\to1^-}\left(\frac{1}{F(\mathring{z})}\,\frac{F(\varrho\mathring{z})-F(\mathring{z})}{\varrho-1}\right)=$$

$$=\lim_{\varrho\to1^-}\frac{1}{1-\varrho}\left(1-\frac{F(\varrho\mathring{z})}{F(\mathring{z})}\right)=\operatorname{re}\lim_{\varrho\to1^-}\frac{1}{1-\varrho}\left(1-\frac{F(\varrho\mathring{z})}{F(\mathring{z})}\right)=$$

$$=\lim_{\varrho\to1^-}\frac{1}{1-\varrho}\left(1-\operatorname{re}\frac{F(\varrho\mathring{z})}{F(\mathring{z})}\right)\geqslant\lim_{\varrho\to1^-}\frac{1}{1-\varrho}\,(1-\varrho)=1.$$

Hence $m\geqslant1$.

In order to prove the second part of the theorem let us note at first that

$$\operatorname{re}\frac{D\,F(z)(z)}{F(z)}=\sum_{k=1}^{n}\frac{\partial\Phi}{\partial\varphi_k},$$

$$\operatorname{re}\left(\frac{D\,F(z)(z)}{F(z)}\right)^2=\left(\sum_{k=1}^{n}\frac{\partial\Phi}{\partial\varphi_k}\right)^2-\frac{1}{R^2}\left(\sum_{k=1}^{n}\frac{\partial R}{\partial\varphi_k}\right)^2,$$

$$\operatorname{re}\frac{D^2F(z)(z,z)}{F(z)}=\left(\sum_{k=1}^{n}\frac{\partial\Phi}{\partial\varphi_k}\right)^2-\sum_{k=1}^{n}\frac{\partial\Phi}{\partial\varphi_k}-\frac{1}{R}\sum_{j,k=1}^{n}\frac{\partial^2R}{\partial\varphi_j\partial\varphi_k}$$

for $z=\left(\mathring{r}_1e^{i\varphi_1},\dots,\mathring{r}_ne^{i\varphi_n}\right)$.

From the above we obtain

$$\mathrm{re}\left(\frac{D^2F(z)(z,z)}{F(z)} + \frac{D\,F(z)(z)}{F(z)} - \left(\frac{D\,F(z)(z)}{F(z)}\right)^2\right) =$$

$$= \mathrm{re}\left(\left(\frac{D^2F(z)(z,z)}{D\,F(z)(z)} + 1\right)\frac{D\,F(z)(z)}{F(z)} - \left(\frac{D\,F(z)(z)}{F(z)}\right)^2\right) =$$

$$= \frac{1}{R^2}\left(\sum_{k=1}^{n}\frac{\partial R}{\partial \varphi_k}\right)^2 - \frac{1}{R}\sum_{j,k=1}^{n}\frac{\partial^2 R}{\partial\varphi_j\partial\varphi_k}\,.$$

We already know that the function R attains a local maximum at the point $(\mathring{\varphi}_1,\dots,\mathring{\varphi}_n)$. Hence we have

$$\left.\frac{\partial R}{\partial\varphi_k}\right|_{(\mathring{\varphi}_1,\dots,\mathring{\varphi}_n)} = 0, \quad k = 1,\dots,n,$$

and

$$\left.\left(\sum_{j,k=1}^{n}\frac{\partial^2 R}{\partial\varphi_j\partial\varphi_k}\right)\right|_{(\mathring{\varphi}_1,\dots,\mathring{\varphi}_n)} \leqslant 0.$$

Therefore we obtain

$$\mathrm{re}\left[\left(\frac{D^2F(\mathring{z})(\mathring{z},\mathring{z})}{DF(\mathring{z})(\mathring{z})} + 1\right)\frac{DF(\mathring{z})(\mathring{z})}{F(\mathring{z})} - \left(\frac{DF(\mathring{z})(\mathring{z})}{F(\mathring{z})}\right)^2\right] \geqslant 0.$$

Hence taking into consideration the inequality (3), which was already proved, we can obtain the inequality (4).

Before we formulate the next theorem, we shall prove the following lemma.

L e m m a . Let $g, g: \overline{K} \to \mathbb{C}$, $g(0) = a$, be an univalent and holomorphic function without at most one point ζ, $|\zeta| = 1$, which is a single pole. Let $f \not\equiv \mathrm{const}$, $f(0) = a$ and $f \in H_P$. Suppose that there is a point $\mathring{z} = \left(\mathring{r}_1 e^{i\mathring{\varphi}_1},\dots,\right.$ $\left.\dots,\mathring{r}_n e^{i\mathring{\varphi}_n}\right) \in P$ for which

(8) $$f(\mathring{z}) \in g(|\zeta|=1)$$

and

(9) $$f(\bar{P}_{\mathring{r}}) \subset g(\bar{K}).$$

Then there exists $m \geqslant 1$ such that

(10) $$Df(\mathring{z})(\mathring{z}) = m\,\mathring{\zeta}\,g'(\mathring{\zeta})$$

and

(11) $$\operatorname{re}\frac{D^2f(\mathring{z})(\mathring{z},\mathring{z})}{Df(\mathring{z})(\mathring{z})} + 1 \geqslant m \operatorname{re}\left(\frac{\mathring{\zeta}g''(\mathring{\zeta})}{g'(\mathring{\zeta})} + 1\right),$$

where $\mathring{\zeta} = g^{-1}(f(\mathring{z}))$.

P r o o f . Put

(12) $$F(z) = g^{-1}(f(z)), \quad z \in \bar{P}_{\mathring{r}}.$$

The function F is holomorphic as a superposition of the holomorphic functions. Moreover $F(0) = 0$. From (12), by (8) and (9), we have $|F(\mathring{z})| = 1$ and $|F(z)| \leqslant 1$ for $z \in \bar{P}_{\mathring{r}}$, and so

$$|F(\mathring{z})| = \max_{z \in \bar{P}_{\mathring{r}}} |F(z)|.$$

It is easy to see that the function F of (12) satisfies the assumptions of Theorem 1 and so for the function F inequalities (3) and (4) hold.

At the same time we can see that

(13) $$g'(F(\mathring{z}))DF(\mathring{z})(\mathring{z}) = Df(\mathring{z})(\mathring{z})$$

and also

$$\frac{g'(F(\mathring{z}))F(\mathring{z})DF(\mathring{z})(\mathring{z})}{F(\mathring{z})} = D\,f(\mathring{z})(\mathring{z}).$$

Taking

$$(14) \qquad \mathring{\zeta} = g^{-1}(f(\mathring{z})) = F(\mathring{z})$$

we have

$$\mathring{\zeta}\, g'(\mathring{\zeta}) \frac{DF(\mathring{z})(\mathring{z})}{F(\mathring{z})} = Df(\mathring{z})(\mathring{z}).$$

From the above in view of (3) we obtain the equality (10).

Next, we shall prove the inequality (11). Since

$$D^2f(\mathring{z})(\mathring{z},\mathring{z}) = g''(F(\mathring{z}))(DF(\mathring{z})(\mathring{z}))^2 + g'(F(\mathring{z}))D^2F(\mathring{z})(\mathring{z},\mathring{z})$$

according to (13), we obtain

$$\frac{D^2f(\mathring{z})(\mathring{z},\mathring{z})}{Df(\mathring{z})(\mathring{z})} = \frac{g''(F(\mathring{z}))}{g'(F(\mathring{z}))} DF(\mathring{z})(\mathring{z}) + \frac{D^2F(\mathring{z})(\mathring{z},\mathring{z})}{DF(\mathring{z})(\mathring{z})} =$$

$$= \frac{DF(\mathring{z})(\mathring{z})}{F(\mathring{z})} \frac{F(\mathring{z})g''(F(\mathring{z}))}{g'(F(\mathring{z}))} + \frac{D^2F(\mathring{z})(\mathring{z},\mathring{z})}{DF(\mathring{z})(\mathring{z})}$$

and next, in view of (14)

$$\frac{D^2f(\mathring{z})(\mathring{z},\mathring{z})}{Df(\mathring{z})(\mathring{z})} = \frac{DF(\mathring{z})(\mathring{z})}{F(\mathring{z})} \frac{\mathring{\zeta} g''(\mathring{\zeta})}{g'(\mathring{\zeta})} + \frac{D^2F(\mathring{z})(\mathring{z},\mathring{z})}{DF(\mathring{z})(\mathring{z})} .$$

Hence, using the theorem 1 we have

$$\operatorname{re} \frac{D^2f(\mathring{z})(\mathring{z},\mathring{z})}{Df(\mathring{z})(\mathring{z})} \geqslant m \operatorname{re} \frac{\mathring{\zeta} g''(\mathring{\zeta})}{g'(\mathring{\zeta})} + m - 1$$

and next the inequality (11).

Theorem 2. Let $f \not\equiv \text{const}$, $f(0) = a$, $f \in H_P$ and $\mathring{z} = \left(\mathring{r}_1 e^{i\mathring{\varphi}_1}, \ldots, \mathring{r}_n e^{i\mathring{\varphi}_n}\right) \in P$. If

$$(15) \qquad \operatorname{re} f(\mathring{z}) = \min_{z \in \bar{P}_{\mathring{r}}} \operatorname{re} f(z),$$

then

$$(16)\qquad Df(\mathring{z})(\mathring{z}) \leqslant -\frac{|a-f(\mathring{z})|^2}{2\mathrm{re}(a-f(\mathring{z}))},$$

$$(17)\qquad \mathrm{re}\,\frac{D^2f(\mathring{z})(\mathring{z},\mathring{z})}{Df(\mathring{z})(\mathring{z})}+1 \geqslant 0$$

and

$$(18)\qquad \mathrm{re}(D^2f(\mathring{z})(\mathring{z},\mathring{z}) + Df(\mathring{z})(\mathring{z})) \leqslant 0.$$

P r o o f . Let us observe that the function

$$g(\zeta) = \frac{a-(2\mathrm{re}\; f(\mathring{z})-\bar{a})\zeta}{1-\zeta}$$

satisfies the assumptions of the lemma. Since the function g maps the unit disk $\bar{K}$ on to half-plane $\{\zeta\in\mathbf{C} : \mathrm{re}\,\zeta \geqslant \mathrm{re}\; f(\mathring{z})\}$, so $f(\mathring{z})\in g(|\zeta|=1)$ for $\mathring{z}\in P$ and according to (15) we have $f(\bar{P}_{\mathring{r}})\subset g(\bar{K})$. Therefore the function f satisfies the assumptions of lemma, too. Hence, there exists $m \geqslant 1$ such that the inequalities (10) and (11) hold.

It is easy to see that

$$\mathring{\zeta} = g^{-1}(f(\mathring{z})) = \frac{f(\mathring{z})-a}{f(\mathring{z}) - (2\mathrm{re}\; f(\mathring{z})-\bar{a})}$$

and so

$$\mathring{\zeta}\, g'(\mathring{\zeta}) = -\frac{|a-f(\mathring{z})|^2}{2\mathrm{re}(a-f(\mathring{z}))}$$

and

$$\mathrm{re}\,\frac{\mathring{\zeta}\, g''(\mathring{\zeta})}{g'(\mathring{\zeta})} + 1 = 0.$$

Therefore, taking into consideration (10) and (11), we obtain the inequality (17) and the equality

$$Df(\mathring{z})(\mathring{z}) = -m\frac{|a - f(\mathring{z})|^2}{2\operatorname{re}(a - f(\mathring{z}))} .$$

From the above equality, in view of the conditions: $m \geqslant 1$ and $\operatorname{re} a > \operatorname{re} f(\mathring{z})$ we can obtain inequality (16). Inequality (18) is a simple consequence of the inequalities (16) and (17).

Let us note also that in particular if $a = 1$ and $\operatorname{re} f(\mathring{z}) = 0$ the inequality (16) takes the form

$$(19) \qquad Df(\mathring{z})(\mathring{z}) \leqslant -\frac{1}{2}\left(1 + (\operatorname{im} f(\mathring{z}))^2\right) \leqslant -\frac{1}{2} .$$

Let $u = u_1 + iu_2$, $v = v_1 + iv_2$, $\tau = \tau_1 + i\tau_2$, $a \in \mathbf{C}$, $\operatorname{re} a > 0$. Let $\mathcal{H}(a,\Delta)$ denotes the family of all complex and continuous functions $h(u,v,\tau)$ defined on some domain $\Delta \subset \mathbf{C}^3$, $(a,0,0) \in \Delta$ and satisfying the following conditions

$$(20) \qquad \operatorname{re} h(a,0,0) > 0$$

and

$$(21) \qquad \operatorname{re} h(iu_2, v_1, \tau) \leqslant 0,$$

where

$$(iu_2, v_1, \tau) \in \Delta, \quad v_1 \leqslant -\frac{|a - iu_2|^2}{2\operatorname{re} a}, \quad v_1 + \tau_1 \leqslant 0.$$

Now, we shall show several examples of functions, which belong to the family $\mathcal{H}(a,\Delta)$ for certain a and Δ (see [1]). The functions

$$h(u,v,\tau) = \frac{u+1}{2} + \frac{v}{u+1} ,$$

$$h(u,v,\tau) = \frac{1}{2} + \frac{v}{u+1}$$

belong to the family $\mathcal{H}(1,\Delta)$, where $\Delta = (\mathbf{C} - \{-1\}) \times \mathbf{C} \times \mathbf{C}$. The functions

$$h(u,v,\tau) = u + v + \tau,$$

$$h(u,v,\tau) = ue^{v} + v + \tau$$

belong to the family $\mathcal{H}(a, \mathbb{C}^3)$, re $a > 0$, while the functions

$$h(u,v,\tau) = u + 2v + \tau + \frac{1}{2}(1-u)^2,$$

$$h(u,v,\tau) = u + v,$$

$$h(u,v,\tau) = 2v + \tau + \frac{1}{2}$$

belong to the family $\mathcal{H}(1, \mathbb{C}^3)$.

Theorem 3. Let $f \not\equiv$ const, $f(0) = a$, re $a > 0$ and $f \in H_P$. If there exists a function $h \in \mathcal{H}(a,\Delta)$ such that

$$\bigwedge_{z\in P} \left((f(z), Df(z)(z), D^2f(z)(z,z)) \in \Delta \wedge \right. \tag{22}$$

$$\left. \wedge \operatorname{re} h(f(z), Df(z)(z), D^2f(z)(z,z)) > 0 \right)$$

then

$$\operatorname{re} f(z) > 0, \quad z \in P. \tag{23}$$

Proof. Suppose that there exists a point $z^* \in P$ for which re $f(z^*) \leqslant 0$. Then according to the condition re $f(0) > 0$ we can find $P_{\mathring{r}} \subset P$ such that $z^* \notin P_{\mathring{r}}$ and $0 = \min_{z\in \overline{P}_{\mathring{r}}} \operatorname{re} f(z)$.

Let

$$\operatorname{re} f(\mathring{z}) = \min_{z\in \overline{P}_{\mathring{r}}} \operatorname{re} f(z) = 0.$$

Because the assumptions of Theorem 2 are satisfied, we have

$$Df(\mathring{z})(\mathring{z}) \leqslant -\frac{|a - f(\mathring{z})|^2}{2 \operatorname{re} a},$$

$$\operatorname{re}(D^2f(\mathring{z})(\mathring{z},\mathring{z}) + Df(\mathring{z})(\mathring{z})) \leqslant 0.$$

Let $v_1 = Df(\mathring{z})(\mathring{z})$, $iu_2 = f(\mathring{z})$, $\tau = D^2f(\mathring{z})(\mathring{z},\mathring{z})$.

Then for any function $h \in \mathcal{H}(a,\Delta)$ we obtain, according to (21), the following inequality

$$\operatorname{re} h(f(\mathring{z}),\ Df(\mathring{z})(\mathring{z}),\ D^2f(\mathring{z}),(\mathring{z},\mathring{z})) \leqslant 0,$$

which is contradictory to (22). Therefore our supposition was false, which means that $\operatorname{re} f(z) > 0$ for $z \in P$.

Let us observe moreover that the set of functions f satisfying the condition (22) is non-empty; for example the function

$$f(z) = a + a_1z_1 + \dots + a_nz_n, \quad \operatorname{re} a > 0$$

belongs to this set if $a_1,\dots,a_n$ are sufficiently small.

3. Some applications of Theorem 3

From Theorem 3 we can obtain easily the following theorem.

T h e o r e m 4. Let $h \in \mathcal{H}(1,\Delta)$, $F \in H_P$ and $\operatorname{re} F(z) > 0$ for z P. If the function f is a holomorphic solution of the differential equation

$$h(f(z),\ Df(z)(z),\ D^2f(z)(z,z)) = F(z), \quad z \in P, \quad f(0) = 1,$$

then we have

$$\operatorname{re} f(z) > 0, \quad z \in P.$$

Next, let us denote by C_α, V_α, M_α, N_α, $0 \leqslant \alpha < 1$ families of the functions $F \in H_P$, $F(0) = 1$ which satisfy the following conditions for $z \in P$, respectively

$$\operatorname{re} F(z) > \alpha,$$

$$\operatorname{re} L(F(z)) > \alpha,$$

$$\operatorname{re} \frac{L(F(z))}{F(z)} > \alpha ,$$

$$\operatorname{re} \frac{L(L(F(z)))}{L(F(z))} > \alpha ,$$

where

(24) $$L(F(z)) = F(z) + DF(z)(z).$$

These families of functions were considered by K.P.Bawrina in the paper [2].

From the definitions of the families C_α, V_α, M_α and N_α it follows that

(25) $$C_\alpha \subset C_\beta \subset C_0, \quad V_\alpha \subset V_\beta \subset V_0,$$
$$M_\alpha \subset M_\beta \subset M_0, \quad N_\alpha \subset N_\beta \subset N_0$$

for arbitrary α and β such that $0 \leqslant \beta < \alpha < 1$.

Theorem 5. For any $\alpha \in \langle 0,1)$ we have $V_\alpha \subset C_\alpha$.

Proof. Observe at first that $F \equiv 1$ belongs to the families C_α and V_α for $\alpha \in \langle 0,1)$. Let F be an arbitrary function of the family V_α and $F \not\equiv 1$. We put

(26) $$f(z) = \frac{\alpha}{\alpha - 1} - \frac{1}{\alpha - 1} F(z).$$

Then $f \in H_P$, $f(0) = 1$ and

(27) $$L(f(z)) = \frac{\alpha}{\alpha - 1} - \frac{1}{\alpha - 1} L(F(z)).$$

Since $F \in V_\alpha$ we have $\operatorname{re} L(F(z)) > \alpha$. From (27) it follows that

(28) $$\operatorname{re} L(f(z)) > 0 \quad \text{for} \quad z \in P.$$

Let $h(u,v,\tau) = u + v$. The function h belongs to the family $\mathcal{H}(1, \Delta)$ with $\Delta = \mathbb{C}^3$.

The inequality (28) can be written in the form

$$\mathrm{re}\, h(f(z),\, Df(z)(z),\, D^2f(z)(z,z)) > 0, \quad z \in P,$$

hence according to Theorem 3 we obtain

$$\mathrm{re}\, f(z) > 0, \quad z \in P.$$

From the above and from (26) it follows that

$$\mathrm{re}\, F(z) > \alpha, \quad z \in P.$$

This means that $F \in C_\alpha$, which ends this proof.

Theorem 6. For any $\beta \in \langle 0, \frac{1}{2}\rangle$ we have

$$V_\alpha \subset C_\beta, \quad N_\alpha \subset V_\beta, \quad M_\alpha \subset C_\beta \quad \text{for} \quad \alpha \in \langle \tfrac{1}{2}, 1)$$

and

$$N_\alpha \subset M_\beta, \quad N_\alpha \subset C_\beta \quad \text{for} \quad \alpha \in \langle 0, 1).$$

Proof. In view of (25) we see that it is sufficient to show that

$$V_{\frac{1}{2}} \subset C_{\frac{1}{2}}, \quad N_{\frac{1}{2}} \subset V_{\frac{1}{2}}, \quad M_{\frac{1}{2}} \subset C_{\frac{1}{2}}$$

and

$$N_0 \subset M_{\frac{1}{2}}, \quad N_0 \subset C_{\frac{1}{2}},$$

respectively. Since $F \equiv 1$ belongs to each of the above considered families, we may assume $F \not\equiv 1$.

a) We prove at first that $V_{\frac{1}{2}} \subset C_{\frac{1}{2}}$. Let $F \in V_{\frac{1}{2}}$ and

(29) $$f(z) = 2F(z) - 1.$$

Then $f \in H_P$, $f(0) = 1$ and according to the definition V_α

$$\mathrm{re}(f(z) + Df(z)(z)) > 0, \quad z \in P,$$

we have

$$\mathrm{re}\, h(f(z), Df(z)(z), D^2f(z)(z,z)) > 0, \; z \in P,$$

where $h(u,v\tau) = u + v$. Therefore analogously as in Theorem 5 we get

$$\mathrm{re}\, f(z) > 0, \quad z \in P$$

and next, taking into consideration (29) we have

$$\mathrm{re}\, F(z) > \frac{1}{2}\, , \; z \in P.$$

Hence $F \in C_{\frac{1}{2}}$. This means that $V_{\frac{1}{2}} \subset C_{\frac{1}{2}}$.

b) Now we prove that $N_{\frac{1}{2}} \subset V_{\frac{1}{2}}$. Let $F \in N_{\frac{1}{2}}$ and

$$f(z) = 2L(F(z)) - 1. \tag{30}$$

Then $f \in H_P$, $f(0) = 1$ and

$$\frac{L(L(F(z)))}{L(F(z))} = 1 + \frac{D\, f(z)(z)}{1 + f(z)}\, .$$

From the above equality and from the definition of the family $N_{\frac{1}{2}}$ it follows that

$$\mathrm{re}\left(\frac{1}{2} + \frac{D\, f(z)(z)}{1 + f(z)}\right) > 0, \quad z \in P.$$

Taking $h(u,v,\tau) = \frac{1}{2} + \frac{v}{u+1}$ similarly as before basing on Theorem 3 we obtain the inequality

$$\mathrm{re}\, f(z) > 0, \quad z \in P$$

from which, according to (30), we get

$$\operatorname{re} L(F(z)) > \frac{1}{2}, \quad z \in P$$

and so $F \in V_{\frac{1}{2}}$.

c) Analogously we prove $M_{\frac{1}{2}} \subset C_{\frac{1}{2}}$. We only need to take

$$f(z) = 2F(z) - 1, \quad F \in M_{\frac{1}{2}}$$

and

$$h(u,v,\tau) = \frac{1}{2} + \frac{v}{u+1} .$$

d) In order to prove that $N_0 \subset M_{\frac{1}{2}}$, we take

$$f(z) = 2\,\frac{L(F(z))}{F(z)} - 1,$$

where $F \in N_0$. Then we have

$$\operatorname{re}\left(\frac{f(z)+1}{2} + \frac{D\, f(z)(z)}{1+f(z)}\right) > 0.$$

Putting $h(u,v,\tau) = \frac{u+1}{2} + \frac{v}{1+u}$ according to Theorem 3 we get $\operatorname{re} f(z) > 0$ and consequently

$$\operatorname{re} \frac{L(F(z))}{F(z)} > \frac{1}{2}, \quad z \in P$$

hence $F \in M_{\frac{1}{2}}$ and $N_0 \subset M_{\frac{1}{2}}$.

e) From c) and d) immediately it follows $N_0 \subset C_{\frac{1}{2}}$. This way Theorem 6 was proved.

Finally, let us add that the inclusion $N_{\alpha} \subset C_{\beta}$ for $\alpha \in <0,1)$ and $\beta \in <0, \frac{1}{2}>$ has been already proved by another method in [3].

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