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ON THE GREEN FUNCTIONS FOR THE HEAT EQUATION OVER THE m-DIMENSIONAL CUBCID

1. In this paper we shall construct by the method of symmetric images the first and second Green functions for the equation

$$(1) \qquad Pu(X,t) = f(X,t),$$

where

$$X = (x_1,\ldots,x_m), \quad m \geqslant 1, \quad t > 0, \quad P = \sum_{i=1}^{m} D^2_{x_i} - D_t,$$

and for the domain $B = B_1 \times T$, where $B_1 = \{X: |x_i| < a_i\ (i=1,\ldots,m)\}$, $T = \{t:\ t > 0\}$, a_i $(i=1,\ldots,m)$ are positive constants. We denote these functions by G and g.

We shall foresee the solutions of Dirichlet and Neumann problems for the equation (1) in the domain B by help of the functions G and g respectively, but omit the synthesis of the solutions of these problems. In [1] the construction of the second Green function for the biparabolic equation over the rectangular parallelepiped in E_3 was given. In [3] the construction of the Green function for the heat equation over the three-dimensional cuboid was given. In [2] the Green function for the heat equation over the rectangle is cited.

Let $Y \in \bar{B}_1$, $X \in B_1$ and $t > s \geqslant 0$. Let us consider the sequences $x^j_{n,i}$, where

$$(2) \qquad x^j_{n,i} = (-1)^n(x_i + (-1)^j 2a_i n), \; x^1_{0,i} = x^2_{0,i} = x_i$$

$(i=1,\dots,m;\ j=1,2;\ n=0,1,2,\dots)$
or

$$(3) \quad \begin{cases} x^1_{0,1} = x^2_{0,i} = x_i;\ x^1_{2n,i} = 4a_i n + x_i;\ x^1_{2n+1,i} = -4a_i n - x_i - 2a_i; \\ x^2_{2n,i} = -4a_i n + x_i;\ x^2_{2n+1,i} = 4a_i n + 2a_i - x_i \end{cases}$$

$(i=1,\dots,m;\ n=0,1,2,\dots)$.
Let

$$d^j_{n,i} = y_i - x^j_{n,i} \quad (i=1,\dots,m;\ j=1,2;\ n=0,1,2,\dots),$$

$$U_{n,i,j}(x_i,t;y_i,x) = (t-s)^{-\frac{1}{2}} \exp\left[(-4(t-s))^{-1}(d^j_{n,i})^2\right],$$

$$U_{0,i,1} = U_{0,i,2} = U_{0,i} = (t-s)^{-\frac{1}{2}} \exp\left[(-4(t-s))^{-1}(d^j_{0,i})^2\right],$$

where $i=1,\dots,m;\ j=1,2;\ n=0,1,2,\dots$.

Consider now the series

$$G_i(x_i,t;y_i,s) = U_{0,i} - U_{1,i,1} - U_{1,i,2} + \sum_{n=2}^{\infty} (-1)^n (U_{n,i,2} + U_{n,i,1}),$$

$$g_i(x_i,t;y_i,s) = U_{0,i} + U_{1,i,2} + \sum_{n=2}^{\infty} (U_{n,i,1} + U_{n,i,2})$$

for $i=1,\dots,m$, and their products

$$G(X,t;Y,s) = \prod_{i=1}^{m} G_i(x_i,t;y_i,s)$$

and

$$g(X,t;Y,s) = \prod_{i=1}^{m} g_i(x_i,t;y_i,s)$$

respectively. Denote

$$Q_i^j(x_i,t;y_i,x) = D_{x_i}^j(U_{0,i} - U_{1,i,1} - U_{1,i,2}) + S_i^j(x_i,t;y_i,s),$$

$$R_i^j(x_i,t;y_i,s) = D_{x_i}^j(U_{0,i} + U_{1,i,1} + U_{1,i,2}) + T_i^j(x_i,t;y_i,s),$$

where

$$S_i^j(x_i,t;y_i,s) = \sum_{n=2}^{\infty} (-1)^n D_{x_i}^j(U_{n,i,1} + U_{n,i,2}),$$

$$T_i^j(x_i,t;y_i,s) = \sum_{n=2}^{\infty} D_{x_i}^j(U_{n,i,1} + U_{n,i,2}),$$

and $i=1,\dots,m;\ j=0,1,2.$

Let us consider the following sets

$$Z_i = \left\{(x_i,t;y_i,s):\ |x_i|<a_i,\ |y_i|<a_i,\ 0\leqslant s<t\right\} \quad (i=1,\dots,m),$$

$$x_{0,i,j} = (x_1^0,\dots,x_{i-1}^0,(-1)^j a_i,x_{i+1}^0,\dots,x_m^0) \quad (j=1,2;i=2,3,\dots,m-1),$$

$$x_{0,1,j} = ((-1)^j a_1,x_2^0,\dots,x_m^0) \quad (j=1,2),$$

$$x_{0,m,j} = (x_1^0,\dots,x_{m-1}^0,(-1)^j a_m) \quad (j=1,2).$$

2. In this chapter we examine the properties of the functions $U_{n,i,j}$ $(n=0,1,\dots,;\ i=1,\dots,m;\ j=1,2)$ and the series S_i^j, T_i^j $(i=1,\dots,m;\ j=0,1,2)$.

L e m m a 1. If $t>s\geqslant 0$, $Y\in\bar{B}_1$, $X\in B_1$, then the functions $U_{n,i,j}$ $(i=1,\dots,m;\ j=1,2;\ n=0,1,2,\dots)$, satisfy the equations $(D^2_{x_i} - D_t)U_{n,i,j} = 0$ $(i=1,\dots,m;\ j=1,2;$ $n=0,1,2,\dots)$.

We omit the simple proof of Lemma 1.

We shall prove the following lemma.

Lemma 2. If $t > s \geqslant 0$, $x_i, y_i \in [-a_i, a_i]$ $(i=1,\dots,m)$, then

$$(I) \qquad |U_{n,i,j}| \leqslant C_1 (t-s)^{\frac{1}{2}} (n-1)^{-2}$$

and

$$(I_1) \qquad |D^k_{x_i} U_{n,i,j}| \leqslant C(n-1)^{-2}$$

for $k=1,2$; $n=2,3,\dots$; $j=1,2$, C, C_1 being positive constants.

Proof. By (2), we obtain

$$|d^j_{n,i}| = |y_i - x^j_{n,i}| = |y_i + (-1)^{n+1} x_i + (-1)^{n+j+2} 2a_i n| \geqslant$$

$$\geqslant \| y_i + (-1)^{n+1} x_i| - |(-1)^{n+j+1} 2a_i n\| \ 2a_i n - 2a_i = 2a_i(n-1)$$

$(n=2,3,4,\dots;\ i=1,\dots,m;\ j=1,2)$ and

$$(4) \qquad (d^j_{n,i})^{-2} < (4a_i^2 (n-1)^2)^{-1} \quad (i=2,3,\dots,m;\ j=1,2).$$

The function $U_{n,i,j}$ can be represented in the form

$$U_{n,i,j} = (t-s)^{\frac{1}{2}} (d^j_{n,i})^{-2} ((d^j_{n,i})^2 (t-s)^{-1}) \exp\left[(-4(t-s))^{-1} (d^j_{n,i})^2\right]$$

$(n=2,3,\dots;\ i=1,\dots,m;\ j=1,2)$. By the inequality

$$(5) \qquad z^k \exp[-z] < \left(\frac{k}{2}\right)^{\frac{k}{2}} \quad \text{for} \quad z \geqslant 0,\ k > 0,$$

and by (4), we obtain (I). Moreover

$$D_{x_i} U_{n,i,j} = 2^{-1} (-1)^{n+1} d^j_{n,i} (t-s)^{-\frac{3}{2}} \exp((-4(t-s))^{-1} (d^j_{n,i})^2),$$

$$D^2_{x_i} U_{n,i,j} =$$

$$= \left[-2^{-1}+2^{-2}(t-s)^{-1}(d^j_{n,i})^2\right](t-s)^{-\frac{3}{2}} \exp((-4(t-s))^{-1}(d^j_{n,i})^2)$$

for $n=2,3,\dots$, $i=1,\dots,m$, $j=1,2$. Applying the inequalities (4) and (5) we obtain the following estimates

$$|D_{x_i} U_{n,i,j}| \leqslant C_2(n-1)^{-2}, \ |D^2_{x_i} U_{n,i,j}| \leqslant C_3(n-1)^{-2}$$

($n=2,3,\dots$, $i=1,\dots,m$, $j=1,2$), and for $C = \max(C_2,C_3)$, we obtain the assertion (I_1).

L e m m a 3. The series $S^j_i(x_i,t;y_i,s)$ and $T^j_i(x_i,t;y_i,s)$ ($i=1,\dots,m$, $j=0,1,2$) are uniformly convergent at every point $(x_i,t;y_i,s) \in Z_i$ ($i=1,\dots,m$).

P r o o f . By Lemma 2, the series

$$S = C_1(t-s)^{\frac{1}{2}} \sum_{n=2}^{\infty} (n-1)^{-2}$$

is the absolute majorant of the series S^o_i and T^o_i in the sets Z_i ($i=1,\dots,m$), respectively. Howeover the series of the form

$$S_1 = C \sum_{n=2}^{\infty} (n-1)^{-2}$$

is the absolute majorant of the series S^j_i and T^j_i in the sets Z_i ($i=1,\dots,m$; $j=1,2$), respectively.

By Lemma 3 we obtain the following lemma.

L e m m a 4. The series $Q^j_i(x_i,t;y_i,s)$ and $R^j_i(x_i,t;y_i,s)$ are uniformly convergent at every point $(x_i,t;y_i,s) \in Z_i$, for $i=1,\dots,m$, $j=0,1,2$. As a consequence of Lemma 1 and 4 we obtain the following lemma.

L e m m a 5. If $t > s \geqslant 0$, $X \in B_1$, $Y \in \bar{B}_1$, then the functions G_i, g_i are continuous with the derivatives $D^k_{x_i} G_i$,

$D_{x_i}g_i, D_tG_i, D_tg_i$ for $i=1,2,\dots,m$, $k=1,2$, and satisfy the homogeneous equation $Pu(X,t)=0$ for $(X,t)\in B$.

Now we shall prove the following theorem.

T h e o r e m 1. If $X\in B$, $Y\in \bar{B}_1$, $0\leqslant s<t$, then

(a) $$P_{X,t}G(X,t;Y,s)=0; \quad \text{(b)} \quad P_{X,t}g(X,t;Y,s)=0.$$

P r o o f . By definition of the function G, we have

$$P_{X,t}G=\sum_{k=1}^{m} D^2_{x_k}\left(\prod_{i=1}^{m} G_i\right)-D_t\left(\prod_{i=1}^{m} G_i\right)=$$

$$=\left(D^2_{x_1}G_1\right)\prod_{i=2}^{m} G_i+\sum_{k=2}^{m-1}\left(D^2_{x_k}G_k\right)\prod_{i=1}^{m} G_i+D^2_{x_m}G_m\prod_{i=1}^{m} G_i-$$

$$-D_tG_1\prod_{i=2}^{m} G_i-\sum_{k=2}^{m-1}(D_tG_k)\prod_{i=1}^{m} G_i-D_tG_m\prod_{i=1}^{m} G_i=$$

$$=\left[(D^2_{x_1}-D_t)G_1\right]\prod_{i=2}^{m} G_i+\sum_{k=2}^{m-1}(D^2_{x_k}-D_t)G_k\prod_{i=1}^{m} G_i+(D^2_{x_m}-D_t)G_m\prod_{i=1}^{m} G_i$$

and, by Lemma 5, we obtain the assertion (a). The proof of the assertion (b) is similar.

L e m m a 6. Let $|x_i|<a_i$, $|y_i|\leqslant a_i$ $(i=1,\dots,m)$, $0\leqslant s<t$. Then $G_i(x_i,t;y_i,s)\longrightarrow 0$ when $x_i\longrightarrow \pm a_i$ $(i=1,\dots,m)$.

P r o o f . Let us write the function $G_i(x_i,t;y_i,s)$ in the form

(6) $$G_i(x_i,t;y_i,s)=\sum_{n=0}^{\infty}(U_{2n,i,1}-U_{2n+1,i,2})+$$

$$+\sum_{n=1}(U_{2n,i,2}-U_{2n-1,i,1})$$

or

$$(6') \qquad G_i(x_i,t;y_i,s) = \sum_{n=0}^{\infty} (U_{2n,i,2} - U_{2n+1,i,1}) + \sum_{n=1}^{\infty} (U_{2n,i,1} - U_{2n,i,2}).$$

Now in order to prove Lemma 6 we observe, by (3), that

$$(7) \begin{cases} d^2_{2n+1,i} = d^1_{2n,i} \ (n=0,1,2,\dots); \ d^2_{2n,i} = d^1_{2n-1,i} \ (n=1,2,\dots), \\ \text{for } x_i = a_i \quad (i=1,\dots,m); \\ d^2_{2n,i} = d^1_{2n+1,i} \ (n=0,1,\dots); \ d^2_{2n-1,i} = d^1_{2n,i} \ (n=1,2,3,\dots), \\ \text{for } x_i = -a_i \quad (i=1,\dots,m). \end{cases}$$

By (6), (6'), (7), we obtain the thesis of Lemma 6.

L e m m a 7. Let $|x_i| < a_i$, $|y_i| \leqslant a_i$ $(i=1,\dots,m)$, $0 \leqslant s < t$. Then $D_{x_i} g_i(x_i,t;y_i,s) \longrightarrow 0$ when $x_i \longrightarrow \pm a_i$.

P r o o f . Since

$$D_{x_i} U_{n,i,j} = (-1)^n 2^{-1} (t-s)^{-\frac{3}{2}} d^j_{n,i} \exp((-4(t-s))^{-1} (d^j_{n,i})^2)$$

$n=0,1,\dots;$ $i=1,\dots,m;$ $j=1,2,$ thus by formula (7), we obtain

$$(8) \begin{cases} \left.\begin{array}{l} D_{x_i} U_{2n+1,i,2} = -D_{x_i} U_{2n,i,1} \\ D_{x_i} U_{2n,i,2} = -D_{x_i} U_{2n-1,i,1} \end{array}\right\} (n=0,1,2,\dots) \text{ for } x_i = a_i, \\ D_{x_i} U_{2n,i,2} = -D_{x_i} U_{2n+1,i,1} \quad (n=0,1,\dots) \quad \text{for } x_i = -a_i, \\ D_{x_i} U_{2n-1,i,2} = -D_{x_i} U_{2n,i,1} \quad (n=1,2,\dots) \quad \text{for } x_i = -a_i, \\ i=1,\dots,m. \end{cases}$$

Let us write the function $g_i(x_i,t;y_i,s)$ in the form

$$(9)\qquad g_i(x_i,t;y_i,s) = \sum_{n=0}^{\infty} (U_{2n+1,i,2} + U_{2n,i,1}) + $$

$$+ \sum_{n=1}^{\infty} (U_{2n,i,2} + U_{2n-1,i,1})$$

or

$$(9')\qquad g_i(x_i,t;y_i,s) = \sum_{n=0}^{\infty} (U_{2n,i,2} + U_{2n+1,i,1}) + $$

$$+ \sum_{n=1}^{\infty} (U_{2n-1,i,2} + U_{2n,i,1})$$

$(i=1,\dots,m)$. By (8) and (9), (9') we obtain the thesis of Lemma 7.

Now we shall prove the following lemma.

Lemma 8. Let $|x_i| < a_i$, $|y_i| \leqslant a_i$, $0 \leqslant s < t$ $(i=1,\dots,m)$. Then $D_{y_i} G_i(x_i,t;\pm a_i,s) \longrightarrow 0$ when $x_i \longrightarrow \pm a_i$.

Proof. Let us write the function $G_i(x_i,t;y_i,s)$ in the form

$$G_i(x_i,t;y_i,s) = \sum_{n=0}^{\infty} (U_{2n,i,2} - U_{2n+1,i,2}) + $$

$$+ \sum_{n=1}^{\infty} (U_{2n,i,1} - U_{2n-1,i,1})$$

or

$$G_i(x_i,t;y_i,s) = \sum_{n=0}^{\infty} (U_{2n,i,1} - U_{2n+1,i,1}) + $$

$$+ \sum_{n=1}^{\infty} (U_{2n,i,2} - U_{2n-1,i,2})$$

$(i=1,2,\dots,m)$. Since

$$D_{y_i}U_{n,i,j} = -\frac{1}{2}(t-s)^{-\frac{3}{2}}\, d^j_{n,i}\, \exp((-4(t-s))^{-1}(d^j_{n,i})^2)$$

$$(i=1,\dots,m;\ j=1,2),$$

thus, by formula (3) we obtain

$d^2_{2n,i} = -d^2_{2n+1,i}$ $(n=0,1,\dots)$; $d^1_{2n,i} = -d^2_{2n,i}$ $(n=1,2,\dots)$ for $y_i=a$,

and

$d^1_{2n,i} = d^1_{2n+1,i}$ $(n=0,1,\dots)$; $d^2_{2n,i} = -d^2_{2n-1}$ $(n=1,2,\dots)$ for $y_i=-a$

which implies

$$(10)\quad D_{y_i}G_i(x_i,t;a_i,s) = 2\sum_{n=0}^{\infty} D_{y_i}U_{2n,i,2}(x_i,t;\pm a_i,s) + $$

$$+ 2\sum_{n=1}^{\infty} D_{y_i}U_{2n,i,1}(x_i,t;\pm a_i,s)$$

$(i=1,\dots,m)$. By (3), we obtain

$$(11)\quad \begin{cases} d^2_{2n-2,i} = -d^1_{2n,i} & (n=1,2,\dots) \text{ for } y_i=a_i,\ x_3=-a_i, \\ d^2_{2n,i} = -d^1_{2n-2,i} & (n=1,2,\dots) \text{ for } y_i=-a_i,\ x_i=a_i, \\ d^1_{2n,i} = -d^2_{2n,i} & (n=1,2,\dots) \text{ for } y_i=x_i=a_i, \\ d^2_{2n,i} = -d^2_{2n,i} & (n=1,2,\dots) \text{ for } y_i=x_i=-a_i. \end{cases}$$

By (10) and (11), we have the thesis of Lemma 8.

From Lemmas 6, 7 and 8 we get the following theorem.

Theorem 2. Let $X\in B_1$, $Y\in\bar{B}_1$, $|x_i^0|<a_i$ $(i=1,\dots,m)$, $t>s\geqslant 0$ and let

$$Y_{i,j}=(y_1,\dots,y_{i-1},(-1)^j a_i,y_{i+1},\dots,y_m)\quad (i=2,\dots,m-1;\ j=1,2),$$

$$Y_{1,j}=((-1)^j a_1,y_2,\dots,y_m),\ Y_{m,j}=(y_1,y_2,\dots,(-1)^j a_m)\quad (j=1,2).$$

Then

$$G(X,t;Y,s)\longrightarrow 0 \text{ and } D_{x_i}g(X,t;Y,s)\longrightarrow 0 \text{ when } X\longrightarrow X_{o,i,j}$$
$$(i=1,\dots,m;\ j=1,2)$$

and

$$D_{y_i}G(X,t;Y_{i,j},s)\longrightarrow C \quad \text{when} \quad X\longrightarrow X_{o,i,k}\quad (i=1,\dots,m;k=1,2).$$

3. At first we shall introduce the following notation. Let

$y^1=(y_2,\dots,y_m)$; $y^m=(y_1,\dots,y_{m-1})$; $y^i=(y_1,\dots,y_{i-1},y_{i+1},\dots,y_m)$ $(i=2,3,\dots,m-1)$,

$$S_{i,j}=\left\{(Y):y_i=(-1)^j a_i,\ |y_k|<a_k;\ (k=1,\dots,m;\ i=1,2,\dots,m;\ k\neq i;\ j=1,2\right\}.$$

Let

$$U(X,t)=U_o(X,t)+\sum_{i=1}^{m}\left(U_{i,1}(X,t)+U_{i,2}(X,t)\right)+U_m(X,t), \tag{12}$$

where

$$U_o(X,t)=A\int_{\bar{B}_1}f_o(Y)G(X,t;Y,0)dY,$$

$$U_{i,j}(X,t)=(-1)^{j+1}A\int_0^t\int_{S_{i,j}}f_{i,j}(Y_{i,j},s)D_{y_i}G(X,t;Y_{i,j},s)dy^i\,ds$$
$$(i=1,\dots,m;\ j=1,2),$$

$$U_m(X,t) = -A \int_0^t \int_{\bar{B}_1} f(Y,s)G(X,t;Y,s)dY\,ds,$$

and $A = (2\sqrt{\pi})^{-m}$, and let

$$(13) \qquad v(X,t) = v_o(X,t) + \sum_{i=1}^{m} (v_{i,1}(X,t) + v_{i,2}(X,t)) + v_m(X,t),$$

where

$$v_o(X,t) = \int_{\bar{B}_1} h_o(Y)g(X,t;Y,0)\,dY,$$

$$v_{i,j}(X,t) = \int_0^t \int_{S_{i,j}} h_{i,j}(Y_{i,j},s)g(X,t;Y_{i,j},s)dy^i\,ds \qquad (i=1,\dots,m;\ j=1,2),$$

$$v_m(X,t) = -\int_0^t \int_{\bar{B}_1} f(Y,s)g(X,t;X,s)\,dY\,ds,$$

f_o, h_o, f, $f_{i,j}$, $g_{i,j}$ $(i=1,\dots,m;\ j=1,2)$ being given functions.

Now we shall give without proof the theorem concerning the solutions of Dirichlet and Neumann problems.

Theorem. If the function f_o is continuous and bounded in the set B_1, the functions $f_{i,j}$ $(i=1,\dots,m;\ j=1,2)$ are continuous and bounded in the sets $S_{i,j}\times T$ $(i=1,\dots,m;$ $j=1,2)$ respectively, the functions $D^p_{y_k} f(y,s)$ $(k=1,\dots,m;$ $p=0,1)$ are continuous and bounded in the set B, then the function U given by formula (12) is the solution of the equation (1) for $(X,t)\in B$ and of the limit conditions

$$U(X,0) = f_o(X) \quad \text{for} \quad X\in B_1,$$

$$U(X,t) = f_{i,j}(X,t) \quad \text{for} \quad (X,t)\in S_{i,j}\times T \quad (i=1,\dots,m;\quad j=1,2),$$

and the function v given by formula (13) is the solution of the equation (1) for $(X,t) \in B$ and of the limit cohditions

$$v(X,0) = h_0(X) \quad \text{for} \quad X \in B_1,$$

$$D_{x_i} v(X,t) = h_{i,j}(X,t) \quad \text{for} \quad (X,t) \in S_{i,j} \times T, \quad i=1,\dots,m;\ j=1,2.$$

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