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EMBEDDINGS OF TOPOLOGICAL SEMIGROUPS IN TOPOLOGICAL GROUPS AND SEMIGROUP-VALUED MEASURES

Two simple theorems on embeddings of uniform or topological commutative semigroups in topological groups are proved. As an application, some recent results on semigroup-valued measures are quickly derived from their (earlier known and easier to prove) counterparts for group-valued measures.

Throughout this note, S is a commutative semigroup under the operation of addition $+$, with the zero element 0.

Theorem 1. For any Hausdorff uniformity $\mathcal{U}$ on S the properties (a), (b) and (c) below are mutually equivalent.

(a) There exists a Hausdorff topological commutative group G and a map $h: S \longrightarrow G$ which is an algebraic isomorphism and simultaneously a uniform homeomorphism onto its image, where $h(S)$ is considered with the uniformity induced by the natural uniformity of G.

(b) For every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that if $(x_1, x_2) \in V$ and $(x_1+y_1, x_2+y_2) \in V$, then $(y_1, y_2) \in U$.

(c) The following two conditions hold:

(c′) The map $+: (S,\mathcal{U}) \times (S,\mathcal{U}) \longrightarrow (S,\mathcal{U})$ is uniformly continuous.

(c″) For every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that if $(x+z, y+z) \in V$, then $(x,y) \in U$.

P r o o f . (a) $\Longrightarrow$ (b) is obvious.

(b) $\Longrightarrow$ (c): Take any $U \in \mathcal{U}$ and select $V \in \mathcal{U}$ according to (b). Then to this V choose $W \in \mathcal{U}$ so that the implication in (b) holds with V, W in place of U, V. We may assume that W is symmetric and contained in V.
Suppose

$$(x,x') \in W, \ (y,y') \in W \quad \text{but} \quad (x+y, x'+y') \notin U.$$

Since

$$(x',x) \in W \subset V, \quad (x+y, x'+y') \notin U,$$

it follows by (b) that

$$(x'+x+y, x+x'+y') \notin V.$$

This together with $(y',y) \in W$, by (b) again, imply

$$(y'+x'+x+y, y+x+x'+y') \notin W,$$

which is false. This proves (c$'$). The argument showing (c$''$) is even simpler: If U, V are as in (b), $(x+z, y+z) \in V$ but $(x,y) \notin U$, then $(z,z) \notin V$ by (b), which is impossible.

(c) $\Longrightarrow$ (a) will immediately follow from [6], Proposition 1.2 and Remark 1 on p.711, if we show that the cancellation law

$$x+z = y+z \Longrightarrow x = y$$

holds in S, and that

for every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that whenever $(x,y) \in V$, then $(x+z, y+z) \in U$, for all $x,y,z \in S$.

Since $\mathcal{U}$ is Hausdorff, the cancellation law is easily seen to follow from (c$''$); and the above condition is an obvious consequence of (c$'$).

T h e o r e m 2. Let τ be a Hausdorff topology on S under which S is a topological semigroup, i.e., the map

$$+:(S,\tau)\times(S,\tau) \longrightarrow (S,\tau)$$

is continuous.

(a) There exists a Hausdorff uniformity $\mathcal{U}$ on S which is the strongest among the uniformities $\mathcal{U}'$ on S such that

1° the map $+:(S,\mathcal{U})\times(S,\mathcal{U}) \longrightarrow (S,\mathcal{U})$ is uniformly continuous, and

2° the identity map of (S,τ) into S equipped with the topology induced by $\mathcal{U}'$ is continuous at 0.

(b) If (S,τ) satisfies the following condition

(*) for every $x\in S$ and every τ-neighbourhood U of 0 the set $x + U$ is a τ-neighbourhood of x,

and if $\tilde{\tau}$ denotes the topology associated with $\mathcal{U}$, then $\tilde{\tau}\subset\tau$. Moreover, if A is an s-complete subset of (S,τ), then $\tilde{\tau}\mid A = \tau\mid A$. (A is s-complete if every $\mathcal{U}$-Cauchy net in A is τ-convergent to a point in A; cf. [5]).

(c) Suppose that the cancellation law holds in S and that condition (*) is satisfied. Then there exists a Hausdorff topological commutative group G and an algebraic isomorphic embedding $h:S \longrightarrow G$ such that $h:(S,\tau) \longrightarrow G$ is continuous and a homeomorphism on every s-complete subset of S, and $h:(S,\mathcal{U}) \longrightarrow G$ is a uniform homeomorphism.

P r o o f . Let $\mathcal{U}$ be the filter of all τ-neighbourhoods of 0. For each U in $\mathcal{U}$ let

$$\tilde{U} = \{(x,y)\in S\times S:\ (x+U)\cap(y+U) \neq \emptyset\}.$$

Then

$$\tilde{\mathcal{U}} = \{\tilde{U}:\ U\in\mathcal{U}\}$$

is a base for a Hausdorff uniformity, $\mathcal{U}$, on S.

In fact, it is obvious that $\tilde{\mathcal{U}}$ is a filterbase on $S\times S$, each of whose members contains the diagonal of $S\times S$. If $U\in\mathcal{U}$, let $V\in\mathcal{U}$ be such that $V + V\subset U$; this is possible by the continuity of the map + at (0,0). Then $\tilde{V}\circ\tilde{V}\subset\tilde{U}$. Indeed, let (x_1,x_2) and (x_2,x_3) be in $\tilde{V}$, i.e., $x_1+v_1 = x_2+v_2$ and $x_2+w_2 = x_3+w_3$ for some v_i, $w_i\in V$, i=1,2. Then

$x_1+v_1+w_1 = x_2+v_2+w_1 = x_3+w_2+v_2$ and so $x_1+u_1 = x_2+u_2$, where $u_i = v_i+w_i \in U$, $i=1,2$. Thus $(x_1,x_3) \in \tilde{U}$. Since each $\tilde{U}$ is a symmetric subset of $S\times S$, we have thus checked all the conditions for $\tilde{\mathcal{U}}$ to be a base of a uniformity. The uniformity $\mathfrak{U}$ is Hausdorff: If $x,y \in S$ and $x \neq y$, then since τ is Hausdorff and the map $+$ is continuous, there exists $U \in \mathcal{U}$ such that $(x+U) \cap (y+U) = \emptyset$, i.e., $(x,y) \notin \tilde{U}$.

If $U,V \in \mathcal{U}$ are such that $V+V \subset U$, then $\tilde{V}+\tilde{V} \subset \tilde{U}$, and so $+: S\times S \longrightarrow S$ is uniformly continuous under $\mathfrak{U}$. It is also clear that condition 2^o is fulfilled for $\mathfrak{U}$.

Now let a uniformity $\mathfrak{U}'$ on S have the properties 1^o and 2^o. We have to prove that $\mathfrak{U}' \subset \mathfrak{U}$, i.e.,

$$\forall A \in \mathfrak{U}', \ \exists\, U \in \mathcal{U} : \tilde{U} \subset A.$$

Let $A \in \mathfrak{U}'$. Choose $A_1 \in \mathfrak{U}'$ so that $A_1 \circ A_1 \subset A$, and then (by 1^o) a symmetric $A_2 \in \mathfrak{U}'$ such that $A_2+A_2 \subset A_1$. By 2^o, there exists $U \in \mathcal{U}$ such that $U \subset A_2[0] = \{z \in S: (0,z) \in A_2\}$. Suppose $(x,y) \in \tilde{U}$, i.e., $x+u_1 = y+u_2$ for some $u_1,u_2 \in U$. Then (x,x) and $(0,u_1)$ are in A_2, whence $(x,x+u_1) \in A_1$; similarly, $(y+u_2,y) \in A_1$. Since $x+u_1 = y+u_2$, from $A_1 \circ A_1 \subset A$ we have $(x,y) \in A$.

This completes the proof of the part (a).

Part (b) follows easily from the relation $x+U \subset \tilde{U}[x]$ which holds true for all $x \in S$ and $U \in \mathcal{U}$, and from the definition of s-completeness.

Part (c) now follows by applying the implication (c)$\Longrightarrow$(a) of Theorem 1 to our uniform semigroup $(S,\mathfrak{U})$, and employing part (b) of the present Theorem. We only note that the cancellation law implies condition (c″) of Theorem 1: If $U \in \mathcal{U}$ and $(x+z,y+z) \in \tilde{U}$, then $(x,y) \in \tilde{U}$.

A p p l i c a t i o n to semigroup-valued measures. A set function μ is said to be exhaustive if $\mu(E_n) \longrightarrow 0$ for every sequence (E_n) of pairwise disjoint sets from the domain of μ. An assertion of the Brooks-Jewett type is that a convergent sequence of exhaustive set functions μ_n is uniformly exhaustive. A result of such a type was recently proved in [4] for exhaustive finitely additive measures from a sigma-ring to a uniform semigroup satisfying condition (b) of Theorem 1. Applying the implication (b) $\Rightarrow$ (a) of Theorem 1 we see that this result of [4] is immediately implied by the earlier extension of the Brooks-Jewett theorem to group-valued measures established in [3] (cf. also [1]).

A similar remark concerns the decomposition theorem (3.14) established in [7]. This result asserts the existence of a decomposition of a certain type for an exhaustive finitely additive measure μ from a sigma-ring to a topological semigroup S, under the following hypotheses: S satisfies the assumptions of Theorem 2(c), and the τ-closure of the range of μ is s-complete. Applying Theorem 2(c) we easily derive this result from the decomposition theorem for group-valued measures proved in [2].

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