DEMONSTRATIO MATHEMATICA
Vol. XIV No 2 1981

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SEQUENCES OF CONTRACTIONS AND RANDOM FIXED POINT THEOREMS IN DYNAMIC PROGRAMMING

In this paper we consider a non-stationary discounted dynamic programming model with a random parameter. We associate with this model a backward sequence of decision problems with increasing planning horizon, and study the asymptotic behaviour of optimal rewards. In this analysis we use some results on multifunctions and sequences of contractions, and apply a random analogue of the Banach fixed point theorem. Similar problems were investigated by Çapar [3], and we generalize his results.

1. Preliminaries

Throughout this section (X,d) is a metric space, and $(\Omega, \mathcal{U}, P)$ a complete probability space. A function $f : \Omega \to X$ is measurable if for any Borel set $B \subset X$, $f^{-1}(B) \in \mathcal{U}$. By $C(X)$ we denote the Banach space of all real-valued bounded continuous functions on X with the sup norm. A mapping $g : X \to X$ is called a k-contraction, where $0 \leqslant k < 1$, if for every $x,y \in X$, $d(g(x),g(y)) \leqslant kd(x,y)$.

Let (g_i) be a sequence of functions $g_i : X \to X$ such that g_i is a k_i-contraction. If X is complete, then each g_i has a unique fixed point a_i. Denote $k := \sup_{i \in N} k_i$.

The following theorem is a generalization of a result of Çapar ([3], Theorem 1).

T h e o r e m 1.1. Suppose X is complete, the sequence (g_i) converges pointwise to g, and $k<1$. Then the sequence (a_i) converges to a, the unique fixed point of g. Moreover, for any $x_0 \in X$,

$$\lim_{i\to\infty} g_i(g_{i-1}(\ldots g_1(x_0)\ldots)) = a.$$

P r o o f . By ([2], pp.6-7), $\lim_{i\to\infty} a_i = a$. Fix $x_0 \in X$ and denote

$$x_i := g_i(g_{i-1}(\ldots g_1(x_0)\ldots)), \quad i=1,2,\ldots$$

$$b := \limsup_{i\to\infty} d(x_i,a).$$

We have

$$d(x_i,a) = d(g_i(x_{i-1}),a) \leqslant d(g_i(x_{i-1}),g_i(a)) + d(g_i(a),a) \leqslant$$

$$\leqslant kd(x_{i-1},a) + d(g_i(a),g(a)).$$

This implies $b \leqslant kb$ and the boundness of the sequence $(d(x_i,a))$. Hence $b = 0$, which is equivalent to $\lim_{i\to\infty} x_i = a$.

A function $F : \Omega \times X \to X$ is a random contraction if for any $x \in X$, $F(\cdot,x)$ is measurable, and P-almost surely $F(\omega,\cdot)$ is a $k(\omega)$-contraction, where $k : \Omega \to [0, 1)$ is a measurable function. A measurable mapping $f : \Omega \to X$ is called a random fixed point of F if $F(\omega,f(\omega)) = f(\omega)$ a.s.

We will use the following random analogue of the Banach fixed point theorem:

T h e o r e m 1.2 (see [5], Theorem 5). Let X be a Polish space (i.e. a separable complete metric space), and $F : \Omega\times X \to X$ a random contraction. Then there exists a unique random fixed point f of F; that is, if $\tilde{f}$ is another fixed point, then $f(\omega) = \tilde{f}(\omega)$ a.s.

The following lemma is very useful:

L e m m a 1.3 (see [4], Lemma 16; [8], Prop.4.2).
(i) If $f : \Omega \to C(X)$ is measurable, then the function $g : \Omega \times X \to R$ defined by $g(\omega,x) := f(\omega)(x)$ is measurable in ω.

(ii) If X is compact, and $g : \Omega \times X \to R$ is measurable in ω and continuous in x, then the function $f : \Omega \to C(X)$ given by $f(\omega) := g(\omega,\cdot)$ is measurable.

A multifunction φ from Y do X, where Y is an arbitrary set, is a function defined on Y, whose values are non-empty subsets of X. A multifunction φ is called bounded (closed, compact)-valued if $\varphi(y)$ is bounded (closed, compact) for all $y \in Y$. The family of all nonempty closed bounded subsets of X can be considered as a metric space, with the Hausdorff metric D induced by d. Thus, if Y is a topological space, then the continuity of a closed bounded-valued multifunction φ from Y to X is well defined.

T h e o r e m 1.4. (see [1], p.122). Let φ be a multifunction from Y to X, and $u : Y \times X \to R$. If φ is compact-valued and continuous, and u is continuous, then the function $v : Y \to R$ defined by

$$v(y) := \sup_{x \in \varphi(y)} u(y,x) \tag{1.1}$$

is continuous.

T h e o r e m 1.5. Let Y be a compact metric space, (φ_i) a sequence of compact-valued and continuous multifunctions from Y to X, and (u_i) a sequence of continuous functions $u_i : Y \times X \to R$. Define a new sequence (v_i) of functions $v_i : Y \to R$ by

$$v_i(y) := \sup_{x \in \varphi_i(y)} u_i(y,x).$$

If the sequence (φ_i) is uniformly convergent to a compact-valued multifunction φ, i.e. $\lim_{i \to \infty} D(\varphi_i(y),\varphi(y)) = 0$ uniformly on Y, and (u_i) is uniformly convergent to u, then the sequence (v_i) is uniformly convergent to the function v defined by (1.1).

P r o o f . For any $y \in Y$ we have

$$(1.2) \qquad |v(y) - v_i(y)| \leqslant |v(y) - \sup_{x\in\varphi_i(y)} u(y,x)| + $$
$$+ |\sup_{x\in\varphi_i(y)} u(y,x) - v_i(y)|.$$

We shall prove that both terms on the right hand side of this inequality are uniformly convergent to 0.

Note that the multifunction φ is continuous, as the uniform limit of continuous multifunctions. We shall show that its graph

$$G := \{(y,x) \in Y\times X : x \in \varphi(y)\}$$

is compact. Since φ is continuous and compact-valued, G is closed and the set

$$\varphi(Y) := \bigcup_{y\in Y} \varphi(y)$$

is compact (see [1], pp.116, 118). Thus G is compact, as a closed subset of $Y\times\varphi(Y)$.

Now we shall prove that for any $\varepsilon > 0$ there exists $n(\varepsilon)$ such that for all $i \geqslant n(\varepsilon)$, $y \in Y$,

$$(1.3) \qquad |v(y) - \sup_{x\in\varphi_i(y)} u(y,x)| \leqslant \varepsilon .$$

Since u is uniformly continuous on G, there exists $\delta > 0$ such that $d(x,x') < \delta$ implies $|u(y,x) - u(y,x')| < \varepsilon$ for all $y \in Y$, x, $x' \in \varphi(y)$. By the uniform convergence of (φ_i), there is n_0 such that for all $i \geqslant n_0$, $y\in Y$, $D(\varphi_i(y),\varphi(y)) < \delta$. Hence, for any $i \geqslant n_0$, $y \in Y$ and $x \in \varphi(y)$ there exists $x' \in \varphi_i(y)$ such that $d(x,x') < \delta$. For such y, x and x' we have $u(y,x) - u(y,x') < \varepsilon$, and consequently,

$$(1.4) \qquad v(y) \leqslant \sup_{x\in\varphi_i(y)} u(y,x) + \varepsilon.$$

Similary, for any $i > n_0$, $y \in Y$ and $x \in \varphi_i(y)$ there is $x' \in \varphi(y)$ such that $d(x,x') < \delta$. This implies

$$\sup_{x\in\varphi_i(y)} u(y,x) \leqslant v(y) + \varepsilon.$$

This inequality together with (1.4) give (1.3), where $n(\varepsilon) = n_0$.

For any $y \in Y$,

$$|\sup_{x\in\varphi_i(y)} u(y,x) - v_i(y)| \leqslant \sup_{x\in\varphi_i(y)} |u(y,x) - u_i(y,x)|.$$

Since (u_i) is uniformly convergent to u, the right hand side of the last inequality uniformly converges to 0 on Y. This completes the proof.

A multifunction φ from Ω to X is measurable if for any open $B \subset X$, $\{\omega \in \Omega : \varphi(\omega) \cap B \neq \emptyset\} \in \mathcal{U}$. A mapping $f : \Omega \to X$ is a measurable selector of φ, if f is measurable and for any $\omega \in \Omega$, $f(\omega) \in \varphi(\omega)$.

T h e o r e m 1.6 (see [7], Prop.1). Let X be a Polish space, and φ, φ_i $(i=1,2,\dots)$ closed bounded-valued multifunctions from Ω to X. If φ_i are measurable and $\lim_{i\to\infty} D(\varphi_i(\omega),\varphi(\omega)) = 0$ a.s., then φ is measurable.

T h e o r e m 1.7 (c.f. [9], Theorem 9.1). Let X be a Polish space, φ a measurable compact-valued multifunction from Ω to X, u a real-valued function defined on $\Omega \times X$ such that for any $x \in X$, $u(\cdot,x)$ is measurable, and for almost all $\omega \in \Omega$, $u(\omega,\cdot)$ is continuous. Then the function

$$v(\omega) := \sup_{x\in\varphi(\omega)} u(\omega,x), \quad \omega \in \Omega,$$

is measurable.

R e m a r k . For the sake of simplicity we do not give the most general formulations of the theorems of this section.

2. Dynamic programming model

We study a dynamic programming model $(S,A,(\varphi_i),(T_i),(r_i),(\beta_i))$, where S, A are non-empty sets, φ_i is a multifunction from $\Omega\times S$ to A, T_i maps $\Omega\times S\times A$ into S, r_i is a real-valued function defined on $\Omega\times S\times A$, and $\beta_i : \Omega \to [0, 1)$ for $i \in N$.

We interpret S as the set of states of some controlled system, A as the set of actions, and Ω as the set of random parametrs (states of nature). By s_i and a_i we denote the state of the system and the action performed i steps backward from the end of the planning horizon. $\varphi_i(\omega,s_i)$ is interpreted as the set of all admissible actions at this step, when the state of nature is ω. The transition function T_i is the deterministic law of motion of the system between time i and $i-1$. Finally, r_i is interpreted as a reward function, and β_i as a discount factor at the i-th step before the end of the planning horizon.

We associate with our dynamic programming model a sequence of decision problems with increasing planning horizon. Now we describe the decision problem with the horizon n.

We start to control our system at time n, when it is in a state s_n. We assume that the random parameter ω is known before the decision making. We observe s_n and take an action $a_n\in \varphi_n(\omega,s_n)$, receive a reward $r_n(\omega,s_n,a_n)$, and the system moves to a new state $s_{n-1} = T_n(\omega,s_n,a_n)$, and so on until after $n-1$ of these steps we observe s_1, take an action $a_1\in\varphi_1(\omega,s_1)$, receive $r_1(\omega,s_1,a_1)$ and then the process stops. The reward of one unit at time $i-1$ is worth only $\beta_i(\omega)$ at time i. Our total discounted reward is given by

$$R_n(\omega,s_n,a_n,a_{n-1},\dots,a_1)=r_n(\omega,s_n,a_n)+\beta_n(\omega)(r_{n-1}(\omega,s_{n-1},a_{n-1}) +$$

$$+ \beta_{n-1}(\omega)(r_{n-2}(\omega,s_{n-2},a_{n-2})+ \dots + \beta_2(\omega)r_1(\omega,s_1,a_1))\dots)).$$

We are going to maximize this function for every $\omega\in\Omega$, $s_n\in S$ by the appropriate choice of $(a_n,a_{n-1},\dots,a_1)$. The optimal reward function V_n corresponding to the decision problem with horizon n is defined by

$$V_n(\omega,s) = \sup R_n(\omega,s,a_n,a_{n-1},\dots,a_1), \quad \omega\in\Omega,\ s\in S,$$

where supremum is taken over all sequences $(a_n,a_{n-1},\dots,a_1)$ such that $a_i\in\varphi_i(\omega,s_i)$ for $i = n,n-1,\dots,1$, and $s_n := s$, $s_{i-1} := T_i(\omega,s_i,a_i)$.

The functions (R_n) satisfy the equations

$$R_n(\omega,s_n,a_n,\dots,a_1) = r_n(\omega,s_n,a_n)+\beta_n(\omega)R_{n-1}(\omega,T_n(\omega,s_n,a_n),a_{n-1},\dots,a_1)$$

for $n\in\mathbb{N}$, where $R_0 := 0$. It follows from these relations that the optimal reward functions (V_n) satisfy the optimality equations

$$(2.1)\quad V_n(\omega,s) = \sup_{a\in\varphi_n(\omega,s)} \Big(r_n(\omega,s,a)+\beta_n(\omega)V_{n-1}(\omega,T_n(\omega,s,a))\Big),$$

$$V_0 := 0,\quad \omega\in\Omega,\quad s\in S,\ n\in\mathbb{N}.$$

3. Asymptotic behaviour of V_n

For the main result of this paper we shall assume:

A1. $(\Omega,\mathcal{U},P)$ is a complete probability space.

A2. S is a compact metric space, and A is a separable metric space.

A3. Multifunctions (φ_i) are compact-valued, measurable in ω and continuous in s.

A4. Transition functions (T_i) and rewards (r_i) are measurable in ω and continuous in (s,a).

A5. Discount factors (β_i) are measurable functions.

A6. For almost all $\omega\in\Omega$ we have:

a) the sequence $(\varphi_i(\omega,\cdot))$ is uniformly convergent on S to a compact-valued multifunction φ from $\Omega\times S$ to A;

b) the sequences $(T_i(\omega,\cdot,\cdot))$ and $(r_i(\omega,\cdot,\cdot))$ are uniformly convergent on $S\times A$ to functions $T : \Omega\times S\times A \to S$, and $r : \Omega\times S\times A \to R$, respectively;

c) $\lim_{i\to\infty} \beta_i(\omega) = \beta(\omega)$;

d) $k(\omega) := \sup_{i\in N} \beta_i(\omega) < 1$.

For $u \in C(S)$ we define

$$L_i(\omega,u)(s) := \sup_{a\in\varphi_i(\omega,s)} \Big(r_i(\omega,s,a)+\beta_i(\omega)u(T_i(\omega,s,a))\Big),$$

$$L(\omega,u)(s) := \sup_{a\in\varphi(\omega,s)} \Big(r(\omega,s,a)+\beta(\omega)u(T(\omega,s,a))\Big),$$

$$\omega\in\Omega,\ s\in S,\ i\in N.$$

L e m m a 3.1. If we assume A1-A5, then for any $i\in N$, L_i is a random contraction on $C(S)$. If we additionally assume A6, then L is a random contraction on $C(S)$.

P r o o f . For fixed $i\in N$ and $u\in C(S)$ we denote

$$v(\omega,s,a) := r_i(\omega,s,a) + \beta_i(\omega)u(T_i(\omega,s,a)),$$

$$w(\omega,s) := \sup_{a\in\varphi_n(\omega,s)} v(\omega,s,a), \quad \omega\in\Omega,\ s\in S,\ a\in A.$$

The function v is measurable in ω, and continuous in (s,a). From Theorems 1.4 and 1.7 we conclude that for any $\omega\in\Omega$, $w(\omega,\cdot)\in C(S)$, and for any $s\in S$, $w(\cdot,s)$ is measurable. In virtue of Lemma 1.3(i), the operator $L_i : \Omega\times C(S) \to C(S)$ is measurable in ω. For any $u_1,u_2\in C(S)$ and $s\in S$ we have

$$|L_i(\omega,u_1)(s)-L_i(\omega,u_2)(s)|\leqslant\beta_i(\omega)\cdot \sup_{a\in\varphi_i(\omega,s)} |u_1(T_i(\omega,s,a))-u_2(T_i(\omega,s,a))|\leqslant$$

$$\leqslant \beta_i(\omega)\,\|u_1-u_2\|.$$

Hence $L_i(\omega,\cdot)$ is a $\beta_i(\omega)$-contraction.

Now we consider the operator L. By Theorem 1.6, the multifunction φ is measurable in ω. For almost all $\omega\in\Omega$, $\varphi(\omega,\cdot)$ is continuous, as the uniform limit of continuous multifunctions. By the same argument, T and r are measurable in ω, and continuous in (s,a) for almost all ω. It is obvious, that β is measurable, and $\beta(\omega) < 1$ a.s. The rest of the proof is the same as for the operator L_i, and we omit it.

By Theorem 1.2, each operator L_i has a unique random fixed point $f_i : \Omega \to C(S)$. Similary, L has a unique random fixed point $f : \Omega \to C(S)$. We associate with these fixed points functions $V, V_i^* : \Omega \times S \to R$ defined by

$$V(\omega,s) := f(\omega)(s),$$

$$V_i^*(\omega,s) := f_i(\omega)(s), \quad i \in N.$$

The following theorem interrelates the functions V_n, V_n^* and V.

T h e o r e m 3.2. (c.f. [3], Theorem 2). Assume A1-A6. For almost all $\omega \in \Omega$,

$$\lim_{n\to\infty} V_n(\omega,s) = \lim_{n\to\infty} V_n^*(\omega,s) = V(\omega,s),$$

uniformly on S.

P r o o f . There exists $U \in \mathcal{U}$ with $P(U) = 1$ such that for all $\omega \in U$ the following conditions are satisfied:

(i) $\varphi_n(\omega,\cdot) \to \varphi(\omega,\cdot)$ uniformly on S,

(ii) $T_n(\omega,\cdot,\cdot) \to T(\omega,\cdot,\cdot)$ and $r_n(\omega,\cdot,\cdot) \to r(\omega,\cdot,\cdot)$ uniformly on $S\times A$,

(iii) $\beta_n(\omega) \to \beta(\omega)$ and $k(\omega) < 1$,

(iv) for any $n \in N$, $s \in S$,

$$(3.1) \qquad V_n^*(\omega,s) = L_n(\omega,V_n^*(\omega,\cdot))(s), \quad V(\omega,s) = L(\omega,V(\omega,\cdot))(s).$$

By use of the operators (L_n), we can rewrite the optimality equations (2.1) in the form

$$(3.2) \qquad V_n(\omega,s) = L_n(\omega,V_{n-1}(\omega,\cdot))(s), \quad n \in N.$$

It can be compared with the equations (3.1) satisfied by the functions (V_n^*).

Fix $\omega \in U$ for the rest of the proof. It follows from (3.2) that

$$V_n(\omega,s) = L_n(\omega,L_{n-1}(\omega,\dots,L_1(\omega,V_0))\dots)(s), \quad n \in N.$$

In order to apply Theorem 1.1, we prove that

$$\lim_{n\to\infty} L_n(\omega,u) = L(\omega,u) \tag{3.3}$$

for $u \in C(S)$. It is not difficult to see that for any $u \in C(S)$,

$$r_n(\omega,\cdot,\cdot) + \beta_n(\omega)u(T_n(\omega,\cdot,\cdot)) \to r(\omega,\cdot,\cdot) + \beta(\omega)u(T(\omega,\cdot,\cdot)),$$

uniformly on $S\times A$. Thus Theorem 1.5 implies (3.3). By Theorem 1.1, the sequences $(V_n^*(\omega,\cdot))$ and $(V_n(\omega,\cdot))$ are convergent to $V(\omega,\cdot)$ in $C(S)$. This completes the proof.

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Received November 22nd, 1979.