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ANALOGUES OF THE GARABEDIAN-SCHIFFER THEOREM FOR PAIRS OF VECTOR FUNCTIONS

1. Introduction, definitions and main results

Let S denote the class of all functions F , univalent in $\Delta = \{z : |z| < 1\}$, of the form

$$F(z) = z + \sum_{q=2}^{\infty} a_q z^q.$$

The classical Grunsky inequalities are based on the fact that the function F analytic in Δ is univalent in Δ if and only if the series

$$\log \frac{z - \zeta}{F(z) - F(\zeta)} = \sum_{q,p=0}^{\infty} a_{qp} z^q \zeta^p$$

is convergent in the bicylinder $\Delta \times \Delta$.

H. Grunsky [11] showed that this takes place if and only if, for any vectors $\lambda = [\lambda_1, \dots, \lambda_N] \in \mathbb{C}^N$, the inequality

$$(1.1) \quad \operatorname{Re} \left\{ \sum_{q,p=1}^N a_{qp} \lambda_q \lambda_p \right\} + \sum_{q=1}^N \frac{1}{q} |\lambda_q|^2 \geq 0$$

holds.

P.R. Garabedian and M. Schiffer [7] proved that, if $F \in S$, $u \neq v$, $1/u, 1/v \in E = \mathbb{C} \setminus F(\Delta)$, then, for the coefficients $a_{qp}(u, v)$ generated in the bicylinder $\Delta \times \Delta$ by the function

$$\log \frac{(z-\zeta) \left(\sqrt{[1-uF(z)][1-vF(\zeta)]} + \sqrt{[1-uF(\zeta)][1-vF(z)]} \right)}{\sqrt{[1-uF(z)][1-vF(\zeta)]} - \sqrt{[1-uF(\zeta)][1-vF(z)]}} =$$

$$= \sum_{q,p=0}^{\infty} a_{qp}(u, v) z^q \zeta^p$$

and for any vectors $\lambda = [\lambda_0, \lambda_1, \dots, \lambda_N] \in \mathbb{R} \times \mathbb{C}^N$, the inequality

$$(1.2) \quad \operatorname{Re} \left\{ \sum_{q,p=0}^N a_{qp}(u, v) \lambda_q \lambda_p \right\} + \sum_{q=1}^N \frac{1}{q} |\lambda_q|^2 \geq 0$$

holds.

Since $\infty \notin F(\Delta)$, and

$$\lim_{u, v \rightarrow 0} a_{qp}(u, v) = a_{qp}, \quad q+p > 0,$$

thereby, inequalities (1.2) are a considerable generalization of inequalities (1.1).

In the present paper we shall formulate and prove analogous generalizations for pairs of vector functions which are defined as follows.

For any nonnegative integers m, n such that $m+n > 0$, let

$$A_0 = [a_{01}, \dots, a_{0m}] \in \mathbb{C}^m, \quad m \geq 1 \quad B_0 = [b_{01}, \dots, b_{0n}] \in \mathbb{C}^n, \quad n \geq 1$$

$$= \emptyset, \quad m = 0 \quad = \emptyset, \quad n = 0,$$

with that

$$a_{0k} \neq a_{0j}, \quad k \neq j, \quad m \geq 1,$$

$$b_{0k} \neq b_{0j}, \quad k \neq j, \quad n \geq 1,$$

$$a_{0k} b_{0j} \neq 1, \quad m, n \geq 1,$$

and let $C_{m,n}(A_0, B_0)$ ([24]) stand for the class of all pairs (F, G) of vector functions F and G ,

$$F = [F_1, \dots, F_m]: \Delta \rightarrow \mathbb{C}^m, \quad m \geq 1 \quad G = [G_1, \dots, G_n]: \Delta \rightarrow \mathbb{C}^n, \quad n \geq 1$$

$$= 0, \quad m = 0, \quad = 0, \quad n = 0,$$

of the form

$$F(z) = A_0 + A_1 z + \dots + A_k z^k + \dots, \quad A_k = [a_{k1}, \dots, a_{km}], \quad m \geq 1,$$

$$G(z) = B_0 + B_1 z + \dots + B_k z^k + \dots, \quad B_k = [b_{k1}, \dots, b_{kn}], \quad n \geq 1,$$

with that $F_1, \dots, F_m$, $m \geq 1$, and $G_1, \dots, G_n$, $n \geq 1$, are univalent functions in Δ , such that

$$F_k(z) \neq F_j(\zeta), \quad k \neq j, \quad m > 1,$$

$$G_k(z) \neq G_j(\zeta), \quad k \neq j, \quad n > 1,$$

$$F_k(z)G_j(\zeta) \neq 1, \quad m, n \geq 1,$$

for all $(z, \zeta) \in \Delta \times \Delta$.

To shorten the notation, we denote $C_m(A_0) = C_{m,0}(A_0, \phi)$, $F = (F, 0)$. Of course, $(F, G) = (G, F)$.

Conditions for the univalence of Garabedian-Schiffer type for pairs (F, G) of classes $C_{m,n}(A_0, B_0)$ are formulated as follows.

The case $m, n \geq 1$. Let $(F, G) \in C_{m,n}(A_0, B_0)$, $m, n \geq 1$, and for any values $u, v \in E = \mathbb{C} \setminus \left[\bigcup_{k=1}^m F_k(\Delta) \cup \bigcup_{k=1}^n 1/G_k(\Delta) \right]$, $u \neq v$, let

$$\hat{F}_k(z) = \left[\frac{F_k(z) - u}{F_k(z) - v} \right]^{1/2}, \quad \hat{G}_k(z) = \left[\frac{1 - v G_k(z)}{1 - u G_k(z)} \right]^{1/2}$$

For (F, G) , we define the coefficients a_{qp}^{kj} , b_{qp}^{kj} and c_{qp}^{kj} generated in the bicylinder $\Delta \times \Delta$ by the functions:

$$(1.3) \quad \sum_{q,p=0}^{\infty} a_{qp}^{kj} z^q \zeta^p = \log \frac{\hat{F}_k(z) + \hat{F}_j(\zeta)}{\hat{F}_k(z) - \hat{F}_j(\zeta)}, \quad j \neq k, \quad m \geq 2$$

$$= \log \frac{(z-\zeta)[\hat{F}_k(z) + \hat{F}_k(\zeta)]}{\hat{F}_k(z) - \hat{F}_k(\zeta)}, \quad j = k,$$

$$(1.4) \quad \sum_{q,p=0}^{\infty} b_{qp}^{kj} z^q \zeta^p = \log \frac{\hat{G}_k(z) + \hat{G}_j(\zeta)}{\hat{G}_k(z) - \hat{G}_j(\zeta)}, \quad j \neq k, \quad n \geq 2$$

$$= \log \frac{(z-\zeta)[\hat{G}_k(z) + \hat{G}_k(\zeta)]}{\hat{G}_k(z) - \hat{G}_k(\zeta)}, \quad j = k,$$

$$(1.5) \quad \sum_{q,p=0}^{\infty} c_{qp}^{kj} z^q \zeta^p = \log \frac{1 + \hat{F}_k(z)\hat{G}_j(\zeta)}{1 - \hat{F}_k(z)\hat{G}_j(\zeta)}.$$

The system of vectors $\lambda_p = [\lambda_{p1}, \dots, \lambda_{pm}] \in \mathbb{C}^m$, $\mu_p = [\mu_{p1}, \dots, \mu_{pn}] \in \mathbb{C}^n$, $p = 0, 1, \dots, N$, will be called admissible if

$$(1.6) \quad \sum_{k=1}^m \lambda_{0k} + \sum_{k=1}^n \mu_{0k} = 0.$$

Let

$$a_q^k = \sum_{p=0}^N \left(\sum_{j=1}^m a_{qp}^{kj} \lambda_{pj} + \sum_{j=1}^n c_{qp}^{kj} \mu_{pj} \right), \quad k = 1, \dots, m,$$

$$q = 0, 1, \dots,$$

$$b_q^k = \sum_{p=0}^N \left(\sum_{j=1}^n b_{qp}^{kj} \mu_{pj} + \sum_{j=1}^m c_{qp}^{jk} \lambda_{pj} \right), \quad k = 1, \dots, n,$$

and let

$$p(u, v) = \sum_{k=1}^m \left(a_o^k \bar{\lambda}_{ok} + \sum_{q=1}^N e^{i\alpha_q^k} a_q^k \lambda_{qk} \right) + \\ + \sum_{k=1}^n \left(b_o^k \bar{\mu}_{ok} + \sum_{q=1}^N e^{i\beta_q^k} b_q^k \mu_{qk} \right),$$

where $0 \leq \alpha_q^k \leq 2\pi$, $k = 1, \dots, m$, $0 \leq \beta_q^k \leq 2\pi$, $k = 1, \dots, n$, $q = 1, \dots, N$.

With the above notation there holds

Theorem 1. If $(F, G) \in C_{m,n}(A_o, B_o)$, $m, n \geq 1$, $u, v \in E$, $u \neq v$, then, for any admissible system of complex vectors λ_p, μ_p , $p = 0, 1, \dots, N$, and for any $0 \leq \alpha_q^k \leq 2\pi$, $k = 1, \dots, m$, $0 \leq \beta_q^k \leq 2\pi$, $k = 1, \dots, n$, $q = 1, \dots, N$,

$$(1.7) \quad \operatorname{Re} p(u, v) + \sum_{k=1}^N \frac{1}{q} \left(\|\lambda_q\|^2 + \|\mu_q\|^2 \right) \geq 0,$$

where equality holds if and only if

$$\operatorname{Re} \left\{ \sum_{k=1}^m \bar{\lambda}_{ok} a_o^k + \sum_{k=1}^n \bar{\mu}_{ok} b_o^k \right\} = 0,$$

$$a_q^k = -e^{-i\alpha_q^k} \bar{\lambda}_{qk}, \quad q = 1, \dots, N, \quad b_q^k = -e^{-i\beta_q^k} \bar{\mu}_{qk}, \quad q = 1, \dots, N \\ = 0, \quad , \quad q = N+1, \dots, \quad = 0, \quad , \quad q = N+1, \dots \\ (k = 1, \dots, m) \quad (k = 1, \dots, n)$$

The case $m \geq 1$, $n = 0$. Let $F \in C_m(A_o)$, $m \geq 1$, and if $u, v \in E = \mathbb{C} \setminus \bigcup_{k=1}^m F_k(\Delta)$, $u \neq v$, let us denote

$$\hat{F}_k(z) = \left[\frac{F_k(z) - u}{F_k(z) - v} \right]^{1/2}.$$

For F , we define the coefficients a_{qp}^{kj} of Garabedian-Schiffer type as follows

$$\sum_{q,p=0}^{\infty} a_{qp}^{kj} z^q \zeta^p = \log \frac{\hat{F}_k(z) + \hat{F}_j(\zeta)}{\hat{F}_k(z) - \hat{F}_j(\zeta)}, \quad j \neq k, \quad m \geq 2$$

$$= \log \frac{(z-\zeta) [\hat{F}_k(z) + \hat{F}_k(\zeta)]}{\hat{F}_k(z) - \hat{F}_k(\zeta)}, \quad j = k.$$

The system of vectors $\lambda_p = [\lambda_{p1}, \dots, \lambda_{pm}]$, $p = 0, 1, \dots, N$, will be called admissible if $\lambda_p \in \mathbb{C}^m$, $p = 0, 1, \dots, N$, and besides,

$$(1.8) \quad \sum_{k=1}^m \lambda_{ok} = 0$$

if $m > 2$.

Let

$$a_q^k = \sum_{p=0}^N \sum_{j=1}^m a_{qp}^{kj} \lambda_{pj}, \quad k = 1, \dots, m, \quad q = 0, 1, \dots,$$

and let

$$p(u, v) = \sum_{k=1}^m \left(a_0^k \bar{\lambda}_{ok} + \sum_{q=1}^N e^{i\alpha_q^k} a_q^k \lambda_{qk} \right),$$

where $0 \leq \alpha_q^k \leq 2\pi$, $k = 1, \dots, m$, $q = 1, 2, \dots, N$.

We shall now give a characterization of the function F , $m \geq 1$.

Theorem 2. If $F \in C_m(A_0)$, $m \geq 1$, $u, v \in E$, $u \neq v$, then, for any admissible system of complex vectors λ_p , $p = 0, 1, \dots, N$, and for any $0 \leq \alpha_q^k < 2\pi$, $k = 1, \dots, m$, $q = 1, 2, \dots, N$,

$$(1.9) \quad \operatorname{Re} p(u, v) + \sum_{q=1}^N \frac{1}{q} \|\lambda_q\|^2 \geq 0,$$

where equality takes place if and only if

$$\operatorname{Re} \left\{ \sum_{k=1}^m \bar{\lambda}_{ok} a_o^k \right\} = 0$$

and

$$a_q^k = -e^{-i\alpha_q^k} \bar{\lambda}_{qk}, \quad q = 1, \dots, N$$

$$(k=1, \dots, m)$$

$$a_q^k = 0, \quad q = N+1, \dots$$

R e m a r k . If $m = 1$, $a_{11} = 1$, $\lambda_{o1} = \bar{\lambda}_{o1}$, and if the condition $u, v \in E$ is replaced by the condition $1/u, 1/v \in E$, inequality (1.9) is identical with inequality (1.2) obtained by P.R.Garabedian and M.Schiffer who used the variational method.

2. Proofs

In the proofs of Theorem 1 and 2 we apply methods of Grunsky-Nehari type.

P r o o f of Theorem 1. In the case when $G_s(z_o) = 0$ for some $s = 1, \dots, n$ and $z_o \in \Delta$, and if $r \in (|z_o|; 1)$, let L_u be an analytic arc joining the point $1/G_s(r)$ with the point u , let L_v be an analytic arc joining the point u with the point v , let D_r be a simply connected domain bounded by the curves $\Gamma_k: w = F_k[\operatorname{rexp}(\varphi i)]$, $k = 1, \dots, m$ and the analytic arcs γ_k , $k = 1, \dots, m$, joining the point $1/G_s(r)$ with the points $F_k(r)$, $k = 1, \dots, m$, respectively, as well as by the curves $B_k: w = 1/G_k r[\exp(\varphi i)]$, $k = 1, \dots, n$, and the analytic arcs δ_k , $k = 1, \dots, n$, $k \neq s$, joining the point $1/G_s(r)$ with the points $1/G_k(r)$, $k = 1, \dots, n$, $k \neq s$, respectively, and let the arcs L_u , L_v and the domain D_r be such that the set $D_r^0 = D_r \setminus (L_u \cup L_v)$ is a simply connected domain. Since it does not cause any loss of generality of our considerations, we assume that $s = n$.

In the case when $G_k(z) \neq 0$ for $k = 1, \dots, n$ and $z \in \Delta$, let L_0 be a circle $|w| = R$ negatively oriented with respect to its interior, containing in the interior the points u, v , and such that the set D_r , $0 < r < 1$, bounded by the curves $\Gamma_k : w = F_k[r \exp(\varphi i)]$, $k = 1, \dots, m$, and the analytic arcs γ_k , $k = 1, \dots, m$, joining the point R with the points $F_k(r)$, $k = 1, \dots, m$, respectively, as well as by the curves $B_k : w = 1/G_k[r \exp(\varphi i)]$, $k = 1, \dots, n$, and the analytic arcs δ_k , $k = 1, \dots, n$, joining the point R with the points $1/G_k(r)$, $k = 1, \dots, n$, respectively, is a simply connected domain. Let L_u be an analytic arc joining the point R with the point u , let L_v be an analytic arc joining the point u with the point v , and let the domain D_r and the arcs L_u, L_v be such that the set $D_r^0 = D_r \setminus (L_u \cup L_v)$ is a simply connected domain.

In both the cases we assume that the orientation of the curves Γ_k , $k = 1, \dots, m$, and B_k , $k = 1, \dots, n$, is determined by the run of the parameter φ from 0 to the point 2π .

We adopt the notation

$$\hat{w}(w) = \left[\frac{w-u}{w-v} \right]^{1/2}, \quad w \in D_r^c$$

and let

$$(2.1) \quad \log \frac{\hat{w}(w) + \hat{F}_j(\zeta)}{\hat{w}(w) - \hat{F}_j(\zeta)} = P_{0j}(w) + \sum_{p=1}^{\infty} \frac{1}{p} [(w-u)(w-v)]^{1/2} P_{pj}(w) \zeta^p,$$

$$\log \frac{1 + \hat{w}(w) \hat{G}_j(\zeta)}{1 - \hat{w}(w) \hat{G}_j(\zeta)} = Q_{0j}(w) + \sum_{p=1}^{\infty} \frac{1}{p} [(w-u)(w-v)]^{1/2} Q_{pj}(w) \zeta^p.$$

It is easily verified that

$$P_{0j}(w) = \log \frac{\hat{w}(w) + \hat{a}_{0j}}{\hat{w}(w) - \hat{a}_{0j}} =$$

$$= 2 \log \left\{ [(w-u)(a_{0j}-v)]^{1/2} + [(w-v)(a_{0j}-u)]^{1/2} \right\} - \log [(w-a_{0j})(u-v)],$$

$$Q_{oj}(w) = \log \frac{1+\hat{w}(w)\hat{b}_{oj}}{1-\hat{w}(w)\hat{b}_{oj}} =$$

$$= 2 \log \left\{ \left[(w-v)(1-ub_{oj}) \right]^{1/2} + \left[(w-u)(1-vb_{oj}) \right]^{1/2} \right\} - \log \left[(1-wb_{oj})(u-v) \right],$$

where

$$\hat{a}_{oj} = \left[\frac{a_{oj} - u}{a_{oj} - v} \right]^{1/2}, \quad \hat{b}_{oj} = \left[\frac{1 - vb_{oj}}{1 - ub_{oj}} \right]^{1/2}.$$

We introduce the following normalization

$$P_{oj}(v) = 0, \quad Q_{oj}(u) = 0.$$

Let

$$P_j(w) = \lambda_{oj} P_{oj}(w) + \sum_{p=1}^N \frac{\lambda_{pj}}{p} \left[(w-u)(w-v) \right]^{1/2} P_{pj}(w),$$

$$Q_j(w) = \mu_{oj} Q_{oj}(w) + \sum_{p=1}^N \frac{\mu_{pj}}{p} \left[(w-u)(w-v) \right]^{1/2} Q_{pj}(w)$$

and

$$H(w) = \sum_{j=1}^m P_j(w) + \sum_{j=1}^n Q_j(w).$$

Since

$$\hat{F}'_k(\zeta) = \frac{u-v}{2} F'_k(\zeta) \left[F_k(\zeta) - u \right]^{-1/2} \left[F_k(\zeta) - v \right]^{-3/2},$$

$$\hat{G}'_k(\zeta) = \frac{u-v}{2} G'_k(\zeta) \left[1 - vG_k(\zeta) \right]^{-1/2} \left[1 - uG_k(\zeta) \right]^{-3/2}$$

therefore

$$\sum_{p=1}^{\infty} P_{pk}(w) \zeta^{p-1} = \frac{F'_k(\zeta)}{[w - F_k(\zeta)] \{ [F_k(\zeta) - u] [F_k(\zeta) - v] \}^{1/2}}$$

$$\sum_{p=1}^{\infty} Q_{pk}(w) \zeta^{p-1} = \frac{G'_k(\zeta)}{[1 - wG_k(\zeta)] \{ [1 - vG_k(\zeta)] [1 - uG_k(\zeta)] \}^{1/2}},$$

and in consequence,

$$H'(w) = \frac{\hat{H}(w)}{[(w-u)(w-v)]^{1/2}},$$

where

$$\hat{H}(w) = \sum_{j=1}^m \hat{P}_j(w) + \sum_{j=1}^n \hat{Q}_j(w),$$

while

$$\begin{aligned} \hat{P}_j(w) &= \lambda_{oj} \frac{[(a_{oj}-u)(a_{oj}-v)]^{1/2}}{a_{oj}^{-w}} + \\ &+ \sum_{p=1}^N \frac{\lambda_{pj}}{p} \left[\left(w - \frac{u+v}{2} \right) P_{pj}(w) + (w-u)(w-v) P'_{pj}(w) \right], \\ \hat{Q}_j(w) &= \mu_{oj} \frac{[(1-ub_{oj})(1-vb_{oj})]^{1/2}}{1-wb_{oj}} + \\ &+ \sum_{p=1}^N \frac{\mu_{pj}}{p} \left[\left(w - \frac{u+v}{2} \right) Q_{pj}(w) + (w-u)(w-v) Q'_{pj}(w) \right]. \end{aligned}$$

Hence, bearing in mind that $D_R^0 = D_R \setminus (L_u \cup L_v)$, we get

$$(2.2) \quad \iint_{D_R^0} |H'(w)|^2 d\Omega = \iint_{D_R} |H'(w)|^2 d\Omega.$$

Let us still observe that

$$(2.3) \quad \text{if } R \rightarrow \infty, \text{ then } \int_{L_0} \overline{H(w)} dH(w) \rightarrow 0,$$

which follows from the analyticality of H at the point $w = \infty$.

Denoting

$$\begin{aligned} \hat{P}_{pj}(w) &= [(w-u)(w-v)]^{1/2} P_{pj}(w), \\ \hat{Q}_{pj}(w) &= [(w-u)(w-v)]^{1/2} Q_{pj}(w), \end{aligned} \quad p=1,2,\dots,$$

from (1.3) - (1.5) and (2.1) we have

$$\begin{aligned} P_{oj} \circ F_k(z) &= \sum_{q=0}^{\infty} a_{qo}^{kk} z^q - \log z, \quad \hat{P}_{pj} \circ F_k(z) = p \sum_{q=0}^{\infty} a_{qp}^{kk} z^q + \frac{1}{z^q}, \quad j = k \\ &= \sum_{q=0}^{\infty} a_{qo}^{kj} z^q, \quad = p \sum_{q=0}^{\infty} a_{qp}^{kj} z^q, \quad j \neq k, \end{aligned}$$

$$P_{oj} \circ 1/G_k(z) = \sum_{q=0}^{\infty} c_{qo}^{jk} z^q, \quad \hat{P}_{pj} \circ 1/G_k(z) = p \sum_{q=0}^{\infty} c_{qp}^{jk} z^q, \quad (|z| = r)$$

$$\begin{aligned} Q_{oj} \circ 1/G_k(z) &= \sum_{q=0}^{\infty} b_{qo}^{kk} z^q - \log z, \quad \hat{Q}_{pj} \circ 1/G_k(z) = p \sum_{q=0}^{\infty} b_{qp}^{kk} z^q + \frac{1}{z^q}, \quad j = k \\ &= \sum_{q=0}^{\infty} b_{qo}^{kj} z^q, \quad = p \sum_{q=0}^{\infty} b_{qp}^{kj} z^q, \quad j \neq k, \end{aligned}$$

$$Q_{oj} \circ F_k(z) = \sum_{q=0}^{\infty} c_{qo}^{kj} z^q, \quad \hat{Q}_{pj} \circ F_k(z) = p \sum_{q=0}^{\infty} c_{qp}^{kj} z^q.$$

Consequently, adopting

$$\begin{aligned} a_{-q}^k &= \frac{1}{q} \lambda_{qk} & b_{-q}^k &= \frac{1}{q} \mu_{qk} & q &= 1, \dots, N \\ & (k=1, \dots, m), & & (k=1, \dots, n), & & \\ & = 0 & & = 0 & q &= N+1, \dots, \end{aligned}$$

we find that

$$(2.4) \quad \begin{cases} H \circ F_k(z) = -\lambda_{ok} \log z + \sum_{q=-\infty}^{\infty} a_q^k z^q, \\ H \circ 1/G_k(z) = -\mu_{ok} \log z + \sum_{q=-\infty}^{\infty} b_q^k z^q. \end{cases} \quad (|z| = r)$$

The functions P_j and Q_j are single-valued and analytic in D_F^0 . If we define P_j and Q_j on the arcs L_u and L_v as the limit from the right, then we obtain as the limit from the left, respectively:

$$\begin{aligned} -P_j & \quad \text{and} \quad -Q_j - 2\pi i \mu_{oj} \quad \text{on } L_v, \\ P_j - 2\pi i \lambda_{oj} & \quad \text{and} \quad Q_j - 2\pi i \mu_{oj} \quad \text{on } L_u. \end{aligned}$$

Hence $P_j(u) = \pi i \lambda_{oj}$, $Q_j(v) = -\pi i \mu_{oj}$ and, in consequence,

$$(2.5) \quad H(u) = \pi i \sum_{j=1}^m \lambda_{oj}, \quad H(v) = -\pi i \sum_{j=1}^n \mu_{oj}.$$

So, as the limit from the right we obtain, respectively:

$$\text{on } L_u \cup L_v : H,$$

$$\text{on } \gamma_1 \cup \Gamma_1 : H - 2\pi i \left(\sum_{k=1}^m \lambda_{ok} + \sum_{k=1}^n \mu_{ok} \right),$$

$$\begin{aligned}
 (2.6) \quad & \text{on } \gamma_{k+1} \cup \Gamma_{k+1}, \quad : H - 2\pi i \left(\sum_{q=1}^m \lambda_{oq} + \sum_{q=1}^n \mu_{oq} \right) - 2\pi i \sum_{q=1}^k \lambda_{oq}, \quad m \geq 2, \\
 & \quad k=1, \dots, m-1 \\
 & \text{on } \delta_1 \cup B_1 \quad : H - 2\pi i \left(2 \sum_{k=1}^m \lambda_{ok} + \sum_{k=1}^n \mu_{ok} \right), \\
 & \text{on } \delta_{k+1} \cup B_{k+1}, \quad : H - 2\pi i \left(2 \sum_{q=1}^m \lambda_{oq} + \sum_{q=1}^n \mu_{oq} \right) - 2\pi i \sum_{q=1}^k \mu_{ok}, \quad n \geq 2, \\
 & \quad k=1, \dots, n-1 \\
 & \text{on } L_o \quad : H - 4\pi i \left(\sum_{k=1}^m \lambda_{ok} + \sum_{k=1}^n \mu_{ok} \right), \quad n \geq 1,
 \end{aligned}$$

and as the limit from the left, respectively:

$$\begin{aligned}
 (2.7) \quad & \text{on } L_v \quad : -H - 2\pi i \sum_{k=1}^n \mu_{ok}, \\
 & \text{on } L_u \quad : H - 2\pi i \left(\sum_{k=1}^m \lambda_{ok} + \sum_{k=1}^n \mu_{ok} \right), \\
 & \text{on } \gamma_k, \quad : H - 2\pi i \left(\sum_{q=1}^m \lambda_{oq} + \sum_{q=1}^n \mu_{oq} \right) - 2\pi i \sum_{q=1}^k \lambda_{ok}, \quad m \geq 2, \\
 & \quad k=1, \dots, m-1 \\
 & \text{on } \gamma_m \quad : H - 2\pi i \left(2 \sum_{k=1}^m \lambda_{ok} + \sum_{k=1}^n \mu_{ok} \right), \\
 & \text{on } \delta_k \quad : H - 2\pi i \left(2 \sum_{q=1}^m \lambda_{oq} + \sum_{q=1}^n \mu_{oq} \right) - 2\pi i \sum_{q=1}^k \mu_{oq}, \quad n \geq 2, \\
 & \quad k=1, \dots, n-1 \\
 & \text{on } \delta_n \quad : H - 4\pi i \left(\sum_{k=1}^m \lambda_{ok} + \sum_{k=1}^n \mu_{ok} \right), \quad n \geq 1.
 \end{aligned}$$

In virtue of the Green formula ([16]) and equality (2.2), considering the orientation of the boundary ∂D_T^0 , we get

$$0 > -\frac{1}{\pi} \iint_{D_r} |H'(w)|^2 d\Omega = \operatorname{Re} \frac{1}{2\pi i} \int_{\partial D_r^0} \overline{H(w)} dH(w).$$

On account of (1.6), (2.3), (2.5)-(2.7), the equality

$$\int_{L_u} H'(w) dw = H(u) - H(\hat{w}), \text{ where } \hat{w} = 1/G_n(r) \text{ or } \hat{w} = R,$$

and the equality

$$\int_{L_v} H'(w) dw = H(v) - H(u),$$

the above inequality takes the form

$$\begin{aligned} & \operatorname{Re} \left\{ \frac{1}{2\pi i} \sum_{k=1}^m \left[\int_{\Gamma_k} \left(\overline{H(w)} + 2\pi i \sum_{q=1}^k \bar{\lambda}_{oq-1} \right) dH(w) - 2\pi i \bar{\lambda}_{ok} H \circ F(r) \right] + \right. \\ & + \frac{1}{2\pi i} \sum_{k=1}^n \left[\int_{B_k} \left(\overline{H(w)} + 2\pi i \sum_{q=1}^m \bar{\lambda}_{oq} + 2\pi i \sum_{q=1}^k \bar{\mu}_{oq-1} \right) dH(w) - \right. \\ & \left. \left. - 2\pi i \bar{\mu}_{ok} H \circ 1/G_k(r) \right] \right\} < 0, \end{aligned}$$

where $\lambda_{oo} = \mu_{oo} = 0$. So, taking account of (2.4), we have

$$\begin{aligned} & \operatorname{Re} \left\{ \frac{1}{2\pi i} \int_0^{2\pi} \sum_{k=1}^m \left[\sum_{q=-\infty}^{\infty} \overline{a_q^k} z^q - \bar{\lambda}_{ok} (\log r - \varphi_1) \right] \left[\sum_{q=-\infty}^{\infty} q a_q^k z^{q-1} - \frac{\lambda_{ok}}{z} \right] dz + \right. \\ & + \frac{1}{2\pi i} \int_0^{2\pi} \sum_{k=1}^n \left[\sum_{q=-\infty}^{\infty} \overline{b_q^k} z^q - \bar{\mu}_{ok} (\log r - \varphi_1) \right] \left[\sum_{q=-\infty}^{\infty} q b_q^k z^{q-1} - \frac{\mu_{ok}}{z} \right] dz + \\ & \left. + \sum_{k=1}^m \left[|\lambda_{ok}|^2 \log r - \bar{\lambda}_{ok} \sum_{q=-\infty}^{\infty} a_q^k r^q \right] + \sum_{k=1}^n \left[|\mu_{ok}|^2 \log r - \bar{\mu}_{ok} \sum_{q=-\infty}^{\infty} b_q^k r^q \right] \right\} < 0. \end{aligned}$$

and, in consequence, applying the formulae

$$\frac{1}{2\pi i} \int_0^{2\pi} \varphi z^{q-1} dz = \frac{r^q}{iq}, \quad q \neq 0 \quad (z = r \exp(i\varphi))$$

$$= \pi, \quad q = 0$$

and next, passing to the limit as $r \rightarrow 1$, we obtain

$$(2.8) \quad \sum_{q=1}^{\infty} q \left(\sum_{k=1}^m |a_q^k|^2 + \sum_{k=1}^n |b_q^k|^2 \right) \leq$$

$$\leq 2 \operatorname{Re} \left\{ \sum_{k=1}^m \bar{\lambda}_{ok} a_o^k + \sum_{k=1}^n \bar{\mu}_{ok} b_o^k \right\} +$$

$$+ \sum_{q=1}^{\infty} q \left(\sum_{k=1}^m |a_{-q}^k|^2 + \sum_{k=1}^n |b_{-q}^k|^2 \right).$$

Let us now assume that, for a real number M_0 and for complex numbers $K_q, T_q, q = 1, 2, \dots$, the inequality

$$(2.9) \quad \sum_{q=1}^{\infty} |K_q|^2 \leq 2 M_0 + \sum_{q=1}^{\infty} |T_q|^2$$

holds. Applying the Cauchy-Schwarz inequality and (2.9), we have

$$- M_0 - \operatorname{Re} \left\{ \sum_{q=1}^{\infty} K_q T_q \right\} \leq - M_0 + \left| \sum_{q=1}^{\infty} K_q T_q \right| \leq$$

$$\leq - M_0 + \left[\left(\sum_{q=1}^{\infty} |T_q|^2 + 2 M_0 \right) \sum_{q=1}^{\infty} |T_q|^2 \right]^{1/2} \leq \sum_{q=1}^{\infty} |T_q|^2,$$

with that equality hold if and only if $M_0 = 0$ and $K_q = -\bar{T}_q, q = 1, 2, \dots$.

Hence we obtain the proposition of Theorem 1 by adopting

$$M_0 = \operatorname{Re} \left\{ \sum_{k=1}^m \bar{\lambda}_{0k} a_0^k + \sum_{k=1}^n \bar{\mu}_{0k} b_0^k \right\},$$

$$K_{(q-1)m+(q-1)n+k} = \sqrt{q} a_q^k, \quad T_{(q-1)m+(q-1)n+k} = \sqrt{q} e^{i\alpha_q^k} a_{-q}^k, \quad k=1, \dots, m,$$

$$K_{qm+(q-1)n+k} = \sqrt{q} b_q^k, \quad T_{qm+(q-1)n+k} = \sqrt{q} e^{i\beta_q^k} b_{-q}^k, \quad k=1, \dots, n,$$

for $q = 1, 2, \dots$ and making use of inequality (2.8).

P r o o f of Theorem 2. Let L_0 be a circle $|w| = R$ negatively oriented with respect to its interior, such that $u, v \in \{w : |w| < R\}$ and such that the set D_r , $0 < r < 1$, bounded by the curves $\Gamma_k : w = F_k[r \exp(i\varphi)]$, $k=1, \dots, m$, and the analytic arcs γ_k , $k=1, \dots, m$, joining the point R with the points $F_k(r)$, $k=1, \dots, m$, is a simply connected domain. Let further L_u and L_v be analytic arcs joining the points R with u and u with v , respectively, lying in D_r and such that the set $D_r^0 = D_r \setminus (L_u \cup L_v)$ is a simply connected domain.

In this case, for the function H of the form

$$H(w) = \sum_{j=1}^m P_j(w),$$

as the limit from the right we obtain:

$$\text{on } L_u \cup L_v : H,$$

$$\text{on } \gamma_1 \cup \Gamma_1 : H - 2\pi i \sum_{k=1}^m \lambda_{0k},$$

$$\text{on } \gamma_{k+1} \cup \Gamma_{k+1}, \quad k=1, \dots, m-1 : H - 2\pi i \left(\sum_{q=1}^m \lambda_{0q} + \sum_{q=1}^k \lambda_{0q} \right), \quad m \geq 2,$$

$$\text{on } L_0 : H - 4\pi i \sum_{k=1}^m \lambda_{0k},$$

and as the limit from the left, respectively:

$$\text{on } L_v : -H,$$

$$\text{on } L_u : H - 2\pi i \sum_{k=1}^m \lambda_{ok},$$

$$\text{on } \gamma_k, \quad k=1, \dots, m-1 : H - 2\pi i \left(\sum_{q=1}^m \lambda_{oq} + \sum_{q=1}^k \lambda_{oq} \right), \quad m > 2,$$

$$\text{on } \gamma_m : H - 4\pi i \sum_{k=1}^m \lambda_{ok},$$

with that

$$H(u) = \pi i \sum_{j=1}^m \lambda_{oj}, \quad H(v) = 0.$$

And so, if we assume that $\lambda_{oo} = 0$ as $m = 1$, then

$$\begin{aligned} -\frac{1}{\pi} \iint_{D_r} |H'(w)|^2 d\Omega &= \operatorname{Re} \frac{1}{2\pi i} \left\{ \int_{L_u} \left[H' \bar{H} - H' \left(\bar{H} + 2\pi i \sum_{k=1}^m \bar{\lambda}_{ok} \right) \right] dw + \right. \\ &+ \int_{L_v} \left[H' \bar{H} - (-H')(-\bar{H}) \right] dw + \sum_{k=1}^m \int_{\gamma_k} \left[H' \left(\bar{H} + 2\pi i \sum_{q=1}^m \bar{\lambda}_{oq} + \sum_{q=1}^{k-1} \bar{\lambda}_{oq} \right) - \right. \\ &- H' \left(\bar{H} + 2\pi i \left(\sum_{q=1}^m \bar{\lambda}_{oq} + \sum_{q=1}^k \bar{\lambda}_{oq} \right) \right) \right] dw + \sum_{k=1}^m \int_{\gamma_k} H' \left(\bar{H} + 2\pi i \left(\sum_{q=1}^m \bar{\lambda}_{oq} + \sum_{q=1}^{k-1} \bar{\lambda}_{oq} \right) \right) dw + \\ &\left. + \int_{L_o} H' \left(\bar{H} + 4\pi i \sum_{k=1}^m \bar{\lambda}_{ok} \right) dw \right\} \end{aligned}$$

and, in consequence, taking account of equality (1.8) in the case $m \geq 2$,

$$\operatorname{Re} \left\{ \frac{1}{2\pi i} \int_0^{2\pi} \sum_{k=1}^m \left[\sum_{q=-\infty}^{\infty} \overline{a_q^k} z^q - \bar{\lambda}_{ok} (\log r - \varphi_1) \right] \left[\sum_{q=-\infty}^{\infty} a_q^k z^{q-1} - \frac{\lambda_{ok}}{z} \right] dz + \right. \\ \left. + \sum_{k=1}^m \left[|\lambda_{ok}|^2 \log r - \bar{\lambda}_{ok} \sum_{q=-\infty}^{\infty} a_q^k r^q \right] \right\} < 0.$$

As a result, when $r \rightarrow 1$,

$$\sum_{q=1}^{\infty} q \sum_{k=1}^m |a_q^k|^2 \leq 2 \operatorname{Re} \left\{ \sum_{k=1}^m \bar{\lambda}_{ok} a_0^k \right\} + \sum_{q=1}^{\infty} q \sum_{k=1}^m |a_{-q}^k|^2,$$

from which, after applying the Cauchy-Schwarz inequality, the proposition of Theorem 2 follows.

Under additional assumptions, inequalities (1.7) and (1.9) hold also when $u \neq v$, $u, v \in \mathbb{C} \setminus E$. This case as well as its applications will be given in our next paper.

3. Appendix

The importance of the conditions of Garabedian-Schiffer type in extremal problems is well known (see, e.g. R.N. Pederson [17], R.N. Pederson and M. Schiffer [18], P.L. Duren [5], Ch. Pommerenke [19, 20], A. Seiler [22], M. Schiffer and H. Schmidt [21]). Theorem 1 of the present paper, when $m = n$, implies analogous conditions for generalized pairs of Aharonov (D. Aharonov [1], J.A. Hummel [12], J.A. Hummel and M. Schiffer [14]) and if, moreover, $G = F$, or $G = \bar{F}$ and $|a_{ok}^k| < 1$ for $k = 1, \dots, m$, or $G = -\bar{F}$, where $\bar{F} = [\bar{F}_1, \dots, \bar{F}_m]$, $\bar{F}_k(z) = \overline{F_k(\bar{z})}$, $k = 1, \dots, m$, for generalized classes of Bieberbach-Eilenberg, bounded and Grunsky-Shah functions (L. Bieberbach [2], S. Eilenberg [6], H. Grunsky [10], Tao-Shing-Shah [23], D.W. De Temple [3], D.W. De Temple and D.B. Oulton [4], L.L. Gromova and N.A. Lebedev [9], N.A. Lebedev [15]), respectively, whereas Theorem 2 implies analogous conditions for sys-

tems of functions with disjoint images (N.A. Lebedev [15], D.W. De Temple [3]) and, in particular, for some extension of the class S when $m = 1$ and for Gel'fer functions when $m = 2$ and $F_2 = -F_1$ (S.A. Gel'fer [8], J.A. Hummel [13], D.W. De Temple and D.B. Oulton [4]).

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Received November 20, 1979.

