DEMONSTRATIO MATHEMATICA
Vol. XIV No 2 1981

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ON A NONLINEAR EQUATION OF THE TYPE OF NONSTATIONARY FILTRATION

In this paper there is presented a method of separation of variables for some nonlinear parabolic equation of the type of nonstationary filtration. By this method we find the solution from which it follows that a support of this solution expands with the growth of the time. In the case when the nonlinear equation reduces to a linear equation, the support of the solution does not expand at all, this fact is well-known.

Consider the flow of a gas through a homogeneous porous medium. The density $u = u(r,t)$ of the gas in the case of radial or spherical symmetry satisfies the nonlinear parabolic equation

$$u_t = r^{-k}\left[r^k(u^m)_r\right]_r, \tag{1}$$

where t and r denote respectively time and space variable, the constant m is positive and $k+1$ is the dimension of the considered euclidean space.

Equation (1) is the mathematical description of several physical phenomena as for example: heat transfer in politropic medium, bomb blast, burst of radiation, seepage of liquids into porous bodies (see [3]). For that reason equation (1) is called the equation of nonstationary filtration.

We shall look for the exact solution of (1) by the method of separation of variables.

Let

(2) $$u(r,t) = R(z)T(t), \quad z = r/S(t),$$

where the functions $R(z)$, $T(t)$, $S(t)$ are different from constants. Substituting (2) into (1) we get

(3) $$T^m z^{-k}\left[z^k (R^m)'\right]' = \left[RST' - zTR'S'\right] S,$$

where a prime denotes differentiation with respect to argument. If we assume that

(4) $$TS' = -\beta ST', \quad \beta = \text{const.} > 0,$$

then the right side of (3) will take the form

$$S^2 T' \left[R + \beta z R'\right]$$

and equation (3) becomes

$$\frac{z^{-k}\left[z^k (R^m)'\right]'}{R + \beta z R'} = \frac{S^2 T'}{T^m} = -\alpha = \text{const.} > 0.$$

From (4) it follows that

$$S(t) = T^{-\beta}(t),$$

therefore

$$\frac{z^{-k}\left[z^k (R^m)'\right]'}{R + \beta z R'} = \frac{T'}{T^{m+2\beta}} = -\alpha.$$

If $\beta = 1/(k+1)$, then

$$T^{-\left(m + \frac{2}{k+1}\right)} T' = -\alpha,$$

hence

$$T(t) = \left[C_1 + \gamma \alpha t\right]^{-1/\gamma},$$

where C_1 is an arbitrary positive constant, $\gamma = m - \frac{k-1}{k+1}$ and

$$\frac{z^{-k}\left[z^k(R^m)'\right]'}{R+zR'/(k+1)} = -\alpha,$$

hence

$$\left[z^k(R^m)'\right]' = -\frac{\alpha}{k+1}\left[(k+1)z^kR + z^{k+1}R'\right] = -\frac{\alpha}{k+1}\left[(z^{k+1}R)\right]'.$$

If $\lim_{z\to 0} z^k(R^m)' = 0$ then $(R^m)' = -\frac{\alpha}{k+1} zR$.

A second integration yields

$$R(z) = \begin{cases} C_2 \exp - \frac{\alpha z^2}{2(k+1)} & \text{for } m = 1 \\ \left[C_2 - \frac{\alpha(m-1)}{2m(k+1)} z^2\right]^{1/(m-1)} & \text{for } m \neq 1, \end{cases}$$

where C_2 is an arbitrary positive constant.

Finally the solution of equation (1) has the form:

1° for $m = 1$

$$u(r,t) = C_2\left[S_1(t)\right]^{-(k+1)} \exp\left[-\frac{\alpha r^2}{2(k+1)S_1^2(t)}\right],$$

where $S_1(t) = \sqrt{C_1 + \frac{2\alpha t}{k+1}}$,

2° for $m \neq 1$

$$(5)\quad u(r,t) = \begin{cases} 0 & \text{for } r \geq \varrho(t) \\ \left[S(t)\right]^{-(k+1)}\left(C_2 - \frac{\alpha(m-1)r^2}{2m(k+1)S^2(t)}\right)^{\frac{1}{m-1}} & \text{for } 0 \leq r < \varrho(t), \end{cases}$$

where

$$S(t) = \left[C_1 + \gamma\alpha t\right]^{1/(k+1)}$$

and

$$\varrho(t) = S(t)\sqrt{\frac{2m(k+1)C_2}{\alpha(m-1)}} .$$

From (5) it follows that for $m>2$

$$\lim_{r\to\varrho^-} u_r(r,t) = -\infty, \quad \lim_{r\to\varrho^+} u_r(r,t) = 0$$

and $\lim\limits_{r\to\varrho(t)} u_r(r,t)$ non exist.

From the obtained result it follows that for $m>1$ the support of the function $u(r,t)$ expands with the increasing of time whereas for $m\leqslant 1$ this support is the set $(r,t) \in [0,\infty)\times[0,\infty)$ and it does not change at all.

It is generally known that the equation (1) for $m>1$ describes the behaviour of sudden phenomena (for example - an explosion) occuring in the diffusion, radiation or filtration which has according to the experiment the finite speed of expansion of this phenomena. Our result attest it.

Now, we shall show that every solution of (1) which satisfies some conditions has also such property as (5).

We shall assume that there exists a classical solution of (1). By a classical solution we understood such a solution which is continuous and has continuous partial derivatives.

L e m m a . Let u_1 and u_2 be bounded and positive solutions of the equation (1) in a set $Q_R^{\varepsilon} = (\varepsilon,R)\times(0,T]$, where $\varepsilon > 0$, R and T are arbitrary positive constants. If on $\Gamma = \{\varepsilon\}\times[0,T]\cup[\varepsilon,R]\times\{0\}\cup\{R\}\times[0,T]$ the inequality $u_1|_\Gamma \leqslant u_2|_\Gamma$ is satisfied then also

$$u_1 \leqslant u_2 \quad \text{everywhere in} \quad \bar{Q}_R = [0,R]\times[0,T].$$

P r o o f . Substituting $u^m = v$ into (1) we obtain

$$mv^{(m-1)/m}\left[v_{rr} + \frac{k}{r}v_r\right] = v_t, \quad m \neq 1. \tag{6}$$

Replacing in (6) the function v by v_1 and next by v_2 and subtracting, we see that the difference $w = v_1 - v_2$ satisfies the equation

$$mv_1^{(m-1)/m}\left[w_{rr} + \frac{k}{r} w_r\right] - w_t + (m-1)\tilde{v}^{-1/m}\left[(v_2)_{rr} + \frac{k}{r}(v_2)_r\right]w = 0,$$

where $\tilde{v}$ is some value between v_1 and v_2.

Now we assume that

$$(m-1)v^{-1/m}\left[(v_2)_{rr} + \frac{k}{r}(v_2)_r\right] \tag{7}$$

is bounded in $\bar{Q}_R^{\varepsilon}$. Let M be a constant greater than the absolute value of (7). We assert that $w = v_1 - v_2 \leqslant 0$. If this is not true, then the function $z = we^{-Mt}$ is positive in $\bar{Q}_R^{\varepsilon}$ and satisfies the equation

$$\begin{aligned} mv_1^{(m-1)/m}\left[z_{rr} + \frac{k}{r} z_r\right] - z_t &= \\ &= \left[- M + (m-1)\tilde{v}^{-1/m}\left[(v_2)_{rr} + \frac{k}{r}(v_2)_r\right]\right]z, \end{aligned} \tag{8}$$

which right side is negative for $z > 0$. Hence it follows that the function z cannot attain its minimum in $\bar{Q}_R^{\varepsilon}$. If it is attained on the boundary Γ of the set $\bar{Q}_R^{\varepsilon}$ then

$$z|_\Gamma = we^{-Mt}|_\Gamma = (v_1 - v_2)|_\Gamma \leqslant 0.$$

Thus $v_1 \leqslant v_2$ in $\bar{Q}_R^{\varepsilon}$, and this is equivalent to the inequality $u_1 \leqslant u_2$. If we suppose that u_2 is the solution defined by (5) then

$$(m-1)\tilde{v}^{-1/m}\left[(v_2)_{rr} + \frac{k}{r}(v_2)_r\right]$$

is bounded for $r = 0$, and for $\varepsilon \to 0$ we have $\bar{Q}_R^{\varepsilon} \to \bar{Q}_R$.

Now we consider the next problem:

$$(9)\qquad \begin{cases} r^{-k}\left[r^k(u^m)_r\right]_r = u_t & \text{for } (r,t)\in Q_R,\quad m>1,\\ u(0,t) = \varphi_1(t) > 0 & \text{for } t\in[0,T],\\ u(R,t) = \varphi_2(t)\geqslant 0 & \text{for } t\in[0,T],\\ u(r,0) = \varphi(r)\geqslant 0 & \text{for } r\in[0,R] \end{cases}$$

and besides that the following compatibility conditions

$$\varphi_1(0) = \varphi(0),\quad \varphi_2(0) = \varphi(R)$$

are satisfied.

Theorem 1. If $u = u(r,t)$ is the solution of the problem (9) and $\varphi_1(t)\equiv 0$ for $t\geqslant t_0\geqslant 0$, $\varphi(r)\equiv 0$ for $r\geqslant r_0$ or $\varphi_2(t)\equiv 0$ for $t\geqslant t_1\geqslant 0$, $\varphi(r)\equiv 0$ for $r\geqslant r_1$ then this solution has a finite support i.e. there exists a set $S\subset \bar{Q}_R$ such that $u|_S\equiv 0$.

Proof. If $\varphi_1\geqslant 0$, $\varphi_2\geqslant 0$, $\varphi\geqslant 0$ then also $u(r,t)>0$ in $\bar{Q}_R$. This fact is well-known from the theory of parabolic equations. We now denote by v the solution (5) and express it by the formula

$$v(r,t) = \begin{cases} A(B+t)^{\delta}\left[C^2 - \dfrac{r^2}{(B+t)^2}\right]^{1/(m-1)} & \text{for } 0\leqslant r\leqslant C(B+t)^{\delta}\\ 0 & \text{for } r\geqslant C(B+t)^{\delta}, \end{cases}$$

where $\delta = -\dfrac{k+1}{m(k+1)-(k-1)}$ and A, B, C^2 are arbitrary positive constants.

If we choose A, B, C as follows

$$\begin{cases} v(0,t) = AB^{\delta}C^{2/(m-1)}\geqslant\varphi_1(t)\geqslant 0 & \text{for } t\in[0,T],\\ v(R,t) = A(B+t)^{\delta}\left[C^2 - \dfrac{R^2}{(B+t)^2}\right]^{1/(m-1)}\geqslant\varphi_2(t)\geqslant 0 & \text{for } t\in[0,T],\\ v(r,0) = AB^{\delta}\left[C^2 - \dfrac{r^2}{B^{2\delta}}\right]^{1/(m-1)}\geqslant\varphi(r)\geqslant 0 & \text{for } r\in[0,R], \end{cases}$$

then from the lemma we have

$$0 \leqslant u(r,t) \leqslant v(r,t) \quad \text{in} \quad \bar{Q}_R,$$

which means that $u(r,t)$ has a finite support.

As a corollary we obtain the following theorems.

Theorem 2. The solution of the problem

$$r^{-k}\left[r^k(u^m)_r\right]_r = u_t \qquad \text{in} \quad Q_R,$$

$$u(0,t) = \psi_1(t) > 0 \qquad \text{for} \quad t \in [0,T],$$

$$u(r,0) = \varphi(r) \geqslant 0 \qquad \text{for} \quad r \in [0,R],$$

$$\psi_1(0) = \varphi(0)$$

has a finite support if $\varphi(r) \equiv 0$ for $r \geqslant r_0 \geqslant 0$.

Theorem 3. If $\varphi(r) \equiv 0$ for $r \geqslant r_0 \geqslant 0$, then the solution of the Cauchy problem

$$r^{-k}\left[r^k(u^m)_r\right]_r = u_t \quad \text{in} \quad Q_\infty = [0,\infty) \times [0,T],$$

$$u(r,0) = \varphi(r) \geqslant 0 \quad \text{for} \quad r \geqslant 0$$

has a finite support i.e. there exists a point r_t such that for each $u \in [0,T]$ $u(r,t) \equiv 0$ if $r \geqslant r_t$.

In paper [1] J. Graveleau and P. Jamet have investigated the equation

$$u_t = f(x,t,u)u_{xx} + au_x^2,$$

where a is a constant and showed that for $a = 0$ the function support $u(\cdot,t)$, does not expand at all but for $a > 0$ it expands with a finite speed. From that it follows that a nonlinear component au_x^2 has an influence on the expansion of the function support $u(\cdot,t)$.

A. Kalashnikov has investigated in this paper [2] the equation

$$u_t = (u^m)_{xx}, \quad m \geqslant 2,$$

$$u(x,0) = u_0(x) \quad \text{for} \quad x \geqslant 0$$

and proved that if $u_0(x) = 0$ in a certain interval $a \leqslant x \leqslant b$ and $u_0(x) \not\equiv 0$, then for $t > 0$ a point of discontinuity of u_x can be found. In our case we have the same property.

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Received October 25, 1979.