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Z. Benedykt, J. Matkowski

REMARKS ON SOME FIXED POINT THEOREM

1. In [1] the second author of this note proved the following theorem.

Theorem 1. Let (X,d) be a nonempty complete metric space and let $T: X \to X$. If:

1^0 $\gamma: [0,\infty) \to [0,\infty)$ is nondecreasing,

2^0 $\lim_{n\to\infty} \gamma^n(t) = 0$ for $t \in [0,\infty)$,

3^0 $\lim_{t\to\infty} (t - \gamma(t)) = \infty$ for $t > 0$,

4^0 for every $x \in X$ there is a positive integer $n = n(x)$ such that for every $y \in X$

$$d(T^n(x), T^n(y)) \leq \gamma(d(x,y)), \tag{1}$$

then T has exactly one fixed point $a \in X$ and for every $x \in X$ $\lim_{k\to\infty} T^k(x) = a$.

(Here γ^k as well as T^k denotes the k-th iteration of γ and T, respectively).

Let us note that conditions 1^0 and 2^0 imply

$$t - \gamma(t) > 0 \quad \text{for} \quad t > 0$$

(cf. Lemma in [1]). On the other hand condition $3^{0'}$ requires that $\lim_{t\to\infty} (t-\gamma(t)) = \infty$. One can easily observe that condition 3^0 is superfluous for bounded metric d.

In this note we shall construct an example which shows that in general case condition 3^0 is essential and therefore cannot be omitted.

E x a m p l e 1. Let $X = \{a_1, a_2, \dots\}$ where

$$a_n = \sum_{k=1}^{n} \frac{1}{k}$$

and put

$$d(a_k, a_l) = |a_k - a_l|, \quad (k,l \in N).$$

Evidently, (X,d) is a complete metric space.

It is easily seen that the shifting map $T: X \to X$ defined as follows

$$T(a_n) = a_{n+1}, \quad (n \in N)$$

has no fixed point in X. We shall show that besides 3^0 all the conditions of Theorem 1 are fulfilled.

L e m m a . If $l,n \in N$, $l > 1$ and

$$s = s(n,l) = \sum_{i=1}^{l} \frac{1}{n+i}, \quad s^*(n,l) = \sum_{i=1}^{\left[\frac{l+1}{2}\right]} \frac{1}{n+i}$$

then

$$s^* < s - \frac{s}{3(1+s)} .$$

P r o o f . For an odd $l > 1$ we have

$$s = \sum_{i=1}^{2k+1} \frac{1}{n+i},$$

and

$$s^* = \sum_{i=1}^{k+1} \frac{1}{n+i} ,$$

where $k = \frac{1}{2}(l-1)$. Therefore, we have

$$s - s^* = \sum_{i=k+2}^{2k+1} \frac{1}{n+i} > \frac{k}{n+2k+1} > \frac{1}{\frac{n}{k}+3}$$

and

$$s < \frac{l}{n} = \frac{2k+1}{n} < \frac{3}{\frac{n}{k}} .$$

Hence we get

$$\frac{n}{k} < \frac{3}{s}$$

and consequently,

$$s - s^* > \frac{1}{\frac{3}{s}+3} = \frac{s}{3(s+1)} ,$$

which completes the proof for an odd l.

Suppose now that l is even. Then

$$s = \sum_{k=1}^{2k} \frac{1}{n+i} \quad \text{and} \quad s^* = \sum_{i=1}^{k} \frac{1}{n+i}$$

and we have

$$s < \frac{2k}{n} = \frac{2}{\frac{n}{k}} .$$

Hence $\frac{n}{k} < \frac{2}{s}$ and, consequently,

$$s - s^* = \sum_{i=1}^{k} \frac{1}{n+k+1} > \frac{k}{n+2k} = \frac{1}{\frac{n}{k}+2} > \frac{s}{2(1+s)} > \frac{s}{3(1+s)}$$

which completes the proof of the lemma.

Now put in our example

$$n(a_k) = k+1 \quad \text{and} \quad \gamma(t) = t - \frac{t}{3(1+t)} .$$

If $x = a_k$, $y = a_{k+1}$, $k \in N$, we have

$$d\left(T^{n(a_k)}(a_k), T^{n(a_k)}(a_{k+1})\right) = \frac{1}{2}\,\frac{1}{k+1} = \frac{1}{2}\, d(a_k, a_{k+1}) < \gamma(d(a_k, a_{k+1})).$$

If $x = a_k$, $y = a_{k+l}$ where $l > 1$, then we have

$$d\left(T^{n(a_k)}(a_k), T^{n(a_k)}(a_{k+l})\right) = a_{2k+l+1} - a_{2k+1} =$$

$$= \frac{1}{2(k+1)} + \frac{1}{2(k+1)+1} + \frac{1}{2(k+1)+2} + \frac{1}{2(k+1)+3} + \dots + \frac{1}{2(k+1)+(l-1)} =$$

$$= \frac{1}{k+1} + \frac{2}{k+2} + \dots + \frac{1}{k + \left[\frac{l+1}{2}\right]} = s^*(k,l).$$

Applying our Lemma we get

$$d\left(T^{n(a_k)}(a_k), T^{n(a_k)}(a_{k+l})\right) < s - \frac{s}{3(1+s)} = \gamma(s) = \gamma\left(d(a_k, a_{k+l})\right).$$

This shows that T satisfies condition 4^0.

It is easily seen that conditions 1^0 and 2^0 are fulfilled.

2. In this section we shall show that by a slight modification of inequality (1) condition 3^0 can be omitted.

T h e o r e m 2. Let (X,d) be a nonempty complete metric space and let $T: X \to X$. If $\gamma : [0,\infty) \to [0,\infty)$ fulfils 1^{o}, 2^{o} and the following condition
(5) there exist a function $n: X \to N$ such that for every $x \in X$ and $y \in X$

$$d(T^{n(x)}(x),T^{n(y)}(y)) \leqslant \gamma(d(x,y)),$$

then T has exactly one fixed point $a \in X$ and $\lim_{n\to\infty} T^{n}(x) = a$ for every $x \in X$.

P r o o f . Putting $S(x) = T^{n(x)}(x)$, $x \in X$ we obtain

$$d(S(x),S(y)) \leqslant \gamma(d(x,y)), \quad x,y \in X.$$

Theorem 1.2 in [2] implies that S has exactly one fixed point $a \in X$. One can easily verify that a is a unique fixed point of T and $\lim_{n\to\infty} T^{n}(x) = a$ for $x \in X$.

E x a m p l e 2. Let $X = [0, \infty)$, $d(x,y) = |x-y|$,

$$T(x) = \begin{cases} \sin x, & x \in [0,1], \\ k, & x \in (k,k+1], \quad k=1,2,\dots, \end{cases}$$

$$\gamma(t) = \begin{cases} 2\sin \frac{t}{2}, & t \in [0,1], \\ 1, & t > 1 \end{cases}$$

and

$$n(x) = \begin{cases} 1, & x \in [0,1], \\ k, & x \in (k,k+1], \quad k=1,2,\dots . \end{cases}$$

A simple calculation shows that all the conditions of Theorem 2 are fulfilled. On the other hand there is no such a function γ that condition 3^{o} and inequality (1) hold.

For an indirect proof suppose that there is such a function γ.

Thus, by inequality (1), for $x = 0$ there exist a positive integer $n = n(0)$ such that for every $y \in [0, \infty)$

$$|T^n y| < \gamma(y).$$

Hence, by 1^0, 2^0 and the definition of T we have

$$[y] - n < T^n y < \gamma(y) < y, \quad y > n = n(0)$$

which implies that

$$\limsup_{y \to \infty} (y - \gamma(y)) \leq \limsup_{y \to \infty} (y - [y] + n) \leq n+1.$$

This contradiction shows that condition 3^0 does not hold.

REFERENCES

[1] J. Matkowski: Fixed point theorems for the mappings with a contractive iterate at a point, Proc. Amer. Math. Soc., 2 (1977) 344-348.
[2] J. Matkowski: Integrable solutions of functional equations, Dissertationes Math. 127 (1975) 1-68.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY, ŁÓDŹ, BRANCH IN BIELSKO-BIAŁA
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