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REMARKS ON STRICT CONVEXITY AND BETWEENNESS POSTULATES

In honor of Professor Kiyoshi Iséki
on his 60th birthday

Strict convexity in normed spaces was introduced by Clarkson in [1]. One of the characterizations of this concept states that a normed space is strictly convex if and only if metric betweenness in this space is equivalent to algebraic betweenness [6]. In this paper, strict convexity is studied in connection with the betweenness postulates given by Huntington and Kline in [4] and [5]. As a consequence, several new characterizations of strict convexity are developed.

Let X be a set and R be a relation on X×X×X. The notation R[abc], or simply abc, indicates that the elements a, b, c of X satisfy the relation R in the stated order. In this work, the relation abc will be restricted to certain ways of defining b to be "between" a and c. The following list of postulates for a betweenness relation abc was presented by Huntington and Kline in [4] and [5]. The first four are three-element postulates while the remaining nine are concerned with four elements.

A. axb implies that bxa.

B. For distinct a, b, c, at least one of the relations abc, acb, bac, bca, cab, or cba holds.

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C. For distinct a, x, y, both axy and ayx cannot be valid.

D. abc implies that a, b, and c are distinct.

For distinct elements a, b, x, y, c of X,

1. xab and aby imply that xay.
2. xab and ayb imply that xay.
3. xab and ayb imply that xyb.
4. axb and ayb imply that axy or ayx.
5. axb and ayb imply that axy or yxb.
6. xab and yab imply that xyb or yxb.
7. xab and yab imply that xya or yxa.
8. xab and yab imply that xya or yxb.
9. abc implies that abx or xbc.

For the remainder, we will assume that $(X, \|\cdot\|)$ is a normed linear space. If a and c are distinct points of X, we will say that b is algebraically between a and c, denoted $A[abc]$, if $b = \alpha a + (1-\alpha)c$ for some $\alpha \in (0,1)$. Also, for distinct $a, b, c \in X$, b is metrically between a and c, denoted $M[abc]$, if $\|c - a\| = \|b - a\| + \|c - b\|$. It is easily shown that $A[abc]$ implies $M[abc]$ but the converse is false in general. Those spaces for which the converse is true form the main focus of our study. $(X, \|\cdot\|)$ is strictly convex, or rotund, if the conditions $\|a + b\| = \|a\| + \|b\|$ and $a, b \neq 0$ imply that $a = \alpha b$ for some $\alpha > 0$. The following characterizations of strict convexity will be useful in later work. The proofs may be found in [6] and [7].

T h e o r e m 1. The following statements are equivalent.

1. $(X, \|\cdot\|)$ is strictly convex.
2. For $a, b, c \in X$, $M[abc]$ implies $A[abc]$.
3. For $a, b \in X$, the conditions $\|a\| = \|b\| = \left\|\frac{a+b}{2}\right\| = 1$ imply that $a = b$.

It is clear from the definitions of $A[abc]$ and $M[abc]$ that Postulates A, C, and D are always true for both concepts. Hence, none of these is sufficient for strict convexity. Our first result shows that Postulates B and 9 impose such severe restrictions on X that they cannot characterize strict convexity.

T h e o r e m 2. The following statements are equivalent.

1. X is 1-dimensional.
2. Postulate B or Postulate 9 holds for algebraic betweenness.
3. Postulate B or Postulate 9 holds for metric betweenness.

P r o o f . The proof that statement 1 implies statements 2 and 3 is straightforward. Therefore, we will show only that statements 2 and 3 each imply statement 1.

ad 1. If Postulate B holds for algebraic betweenness, let a be a fixed non-zero element of X and let x be any other non-zero element of X. Then, since Postulate A is always true for algebraic betweenness, Postulate B implies that $A[0xa]$, $A[0ax]$, or $A[x0a]$. In each case, x is dependent on a and hence, X must be 1-dimensional.

If Postulate 9 holds for algebraic betweenness, then for a fixed non-zero $a \in X$ and any x distinct from $-a$, 0, or a, $A[(-a)0a]$ implies either $A[(-a)0x]$ or $A[x0a]$. Once again, x must be dependent on a and X is 1-dimensional.

ad 2. Assume next that Postulate B holds for metric betweenness. If x and y are independent elements of X, let $u = \frac{x}{\|x\|}$ and $v = \frac{y}{\|y\|}$. Since Postulate A is always true for metric betweenness, Postulate B implies that either $M[0uv]$, $M[0vu]$, or $M[u0v]$. The conditions $M[0uv]$ and $M[0vu]$ each imply that $u = v$, which violates the independence of x and y. Therefore, we may assume that $M[u0v]$. Also, since x and $(-y)$ must be independent, the same arguments show that $M[u0(-v)]$. As a result of these, it follows that $\|v - u\| = 2 = \|v + u\|$. Next, consider the distinct elements 0, $v - u$, $v + u$. By Postulates A and B, either $M[0(v-u)(v+u)]$, $M[(0(v+u)(v-u)]$, or $M[(v+u)0(v-u)]$. All of these are impossible since $\|v-u\| = 2 = \|v+u\|$ and u and v are non-zero. Therefore, there cannot be independent elements x and y in X and X must be 1-dimensional.

Finally, suppose that Postulate 9 holds and let $u = \frac{x}{\|x\|}$ and $v = \frac{y}{\|y\|}$ for independent elements x and y in X. Since u, $\frac{u+v}{2}$, $u+v$, and v are distinct, Postulate 9 implies that $M\left[u\left(\frac{u+v}{2}\right)v\right]$ or $M\left[u\left(\frac{u+v}{2}\right)(u+v)\right]$. The first condition implies that $2\|v\| = \|u-v\| + \|u+v\|$, while the second yields $\|u+v\| + \|u-v\| = 2\|u\| = 2\|v\|$. Therefore, each implies that $M[(u-v)0(u+v)]$. Since $(u-v)$, 0, u, and $(u+v)$ are distinct, Postulate 9 implies that $M[(u-v)0u]$ or $M[u0(u+v)]$. These conditions force $u = v$ or $u = -v$, both of which violate the independence of x and y. As in the previous argument, it follows that X is 1-dimensional. This completes the proof of Theorem 2.

Theorem 3 will give the connection between algebraic betweenness and the remaining postulates. Its proof is straightforward and will be omitted here. All subsequent theorems and examples will be concerned with metric betweenness.

T h e o r e m 3. For algebraic betweenness, Postulates 1 through 8 are true in any linear space.

We follow this with a result which shows that Postulates 2 and 3 fail to characterize strict convexity.

T h e o r e m 4. For metric betweenness,

1. Postulates 2 and 3 are equivalent.
2. Postulates 2 and 3 are true in any normed space.

P r o o f . 1. Let a, b, x, and y be distinct elements of X. It suffices to show that under the conditions $M[xab]$ and $M[ayb]$, $M[xay]$ if and only if $M[xyb]$. From $M[xab]$ and $M[ayb]$, we get the equations

$$\|b - x\| = \|a - x\| + \|b - a\|$$

and

$$\|b - a\| = \|y - a\| + \|b - y\|.$$

If we assume $M[xay]$, i.e., $\|y - x\| = \|a - x\| + \|y - a\|$, then

$$\|b - x\| = \|a - x\| + \|b - a\| = \|a - x\| + \|y - a\| + \|b - y\| =$$
$$= \|y - x\| + \|b - y\|.$$

Therefore, $M[xay]$ implies that $M[xyb]$.

On the other hand, if $M[xyb]$, i.e., $\|b - x\| = \|y - x\| + \|b - y\|$, then

$$\|y - x\| = \|b - x\| - \|b - y\| = \|a - x\| + \|y - a\|.$$

Thus, $M[xyb]$ implies that $M[xay]$ and part 1 is completed.

2. It suffices to show that Postulate 2 is true in any normed space. If we assume $M[xab]$ and $M[ayb]$, then

$$\|b - x\| = \|a - x\| + \|b - a\|$$

and

$$\|b - a\| = \|y - a\| + \|b - y\|.$$

Therefore,

$$\|a - x\| + \|y - a\| = \|b - x\| - \|b - y\| \leq$$
$$\leq \|y - x\| \leq \|a - x\| + \|y - a\|.$$

Hence, $\|y - x\| = \|a - x\| + \|y - a\|$ and Postulate 2 holds.

The next sequence of theorems and examples concerns the relationship between Postulate 1 and strict convexity. We show first that strict convexity implies Postulate 1 and then, after reformulating Postulate 1, we give examples of a space which does not satisfy Postulate 1 and of a non-strictly convex space which does.

Theorem 5. If $(X, \|\cdot\|)$ is strictly convex, then Postulate 1 is true for metric betweenness.

Proof. Let a, b, x and y be distinct elements of X and assume that $M[xab]$ and $M[aby]$. Then, Theorems 1 and 3 imply that $A[xay]$. Since $A[xay]$ implies $M[xay]$, the proof is complete.

L e m m a . For u, v, and w in X and non-negative α, β, and δ,

1. $\|u+v\| = \|u\| + \|v\|$ implies that $\|\alpha u\|+\|\beta v\| = \alpha\|u\| + \beta\|v\|$.
2. $\|u+v+w\| = \|u\| + \|v\| + \|w\|$ implies that $\|\alpha u + \beta v + \delta w\| = \alpha\|u\| + \beta\|v\| + \delta\|w\|$.

P r o o f . A proof for 1. may be found in [3].

2. Assume $\|u + v + w\| = \|u\| + \|v\| + \|w\|$. By the symmetry of u, v, and w in statement 2, we may assume that $\beta \geq \alpha$. Also, since

$$\|u + v + w\| \leq \|u + v\| + \|w\| \leq \|u\| + \|v\| + \|w\| = \|u + v + w\|,$$

it follows that $\|u + v\| + \|u\| + \|v\|$ and $\|u + v + w\| = \|u + v\| + \|w\|$. Therefore, by part 1,

$$\|\alpha u + \beta v + \delta w\| = \|\beta(u + v) + \delta w - (\beta - \alpha)u\| \geq$$

$$\geq \|\beta(u + v) + \delta w\| - (\beta - \alpha)\|u\| = \beta\|u + v\| + \delta\|w\| - (\beta-\alpha)\|u\| =$$

$$= \beta(\|u\| + \|v\|) + \delta\|w\| - (\beta - \alpha)\|u\| = \alpha\|u\| + \beta\|v\| + \delta\|w\|.$$

The equality then follows directly from the triangle inequality.

T h e o r e m 6. Postulate 1 for metric betweenness is equivalent to the following statement:

(1′) The conditions $\|u\| = \|v\| = \left\|\frac{u+v}{2}\right\| = 1$ and $\|v\| = \|w\| = \left\|\frac{v+w}{2}\right\| = 1$ imply that $\|u + v + w\| = 3$.

P r o o f . Postulate 1 states that for distinct a,b,x, and y in X, the conditions $\|b - x\| = \|a - x\| + \|b - a\|$ and $\|y - a\| = \|b - a\| + \|y - b\|$ imply that $\|y - x\| = \|a - x\| + \|y - a\|$. We first note that this is equivalent to the following statement: (*) For non-zero u, v, and w, if $\|u + v\| = \|u\| + \|v\|$ and $\|v + w\| = \|v\| + \|w\|$, then $\|u + v + w\| = \|u\| + \|v\| + \|w\|$. To show this, substitute $u = a - x$, $v = b - a$, and $w = y - b$ for one direction, and for the converse, let $x = 0$, $a = u$, $b = u + v$, and $y = u + v + w$. Since (*) implies (1′), we need only show that (1′) implies (*). If u, v, and w are non-zero and satisfy the conditions $\|u + v\| = \|u\| + \|v\|$ and $\|v + w\| = \|v\| + \|w\|$, then the Lemma implies that for $u' = \frac{u}{\|u\|}$, $v' = \frac{v}{\|v\|}$, and $w' = \frac{w}{\|w\|}$,

$$\|u'\| = \|v'\| = \left\|\frac{u' + v'}{2}\right\| = 1$$

and

$$\|v'\| = \|w'\| = \left\|\frac{v' + w'}{2}\right\| = 1.$$

Therefore, by (1′), $\|u' + v' + w'\| = 3 = \|u'\| + \|v'\| + \|w'\|$. Finally, since $u = \|u\|\, u'$, $v = \|v\|\, v'$, and $w = \|w\|\, w'$, the Lemma implies that $\|u + v + w\| = \|u\| + \|v\| + \|w\|$. Thus, (1′) and (*) are equivalent and the proof is complete.

E x a m p l e 1. Let X be 3-dimensional space and define $\|(x, y, z)\| = \|x\| + \sqrt{y^2 + z^2}$. Then, for $u = (1,0,0)$, $v = (0,1,0)$, and $w = (0,0,1)$, $\|u\| = \|v\| = \left\|\frac{u+v}{2}\right\| = 1$ and $\|v\| = \|w\| = \left\|\frac{v+w}{2}\right\| = 1$, but $\|u + v + w\| = 1 + \sqrt{2} \neq 3$. Hence, this space does not satisfy Postulate 1 for metric betweenness.

E x a m p l e 2. Again let X be 3-dimensional space but this time define $\|(x,y,z)\| = \sqrt{\left|z^2 - (x^2+y^2)\right| + 3z^2 + x^2 + y^2}$. To facilitate our work with $\|\cdot\|$, we will often refer to the following sets:

$$X_1 = \left\{(x, y, z) : z^2 \leq x^2 + y^2\right\}$$

and

$$X_2 = \left\{(x, y, z) : z^2 \geq x^2 + y^2\right\}.$$

Note that if $(x, y, z) \in X_1$, then $\|(x, y, z)\| = \sqrt{2}\sqrt{x^2+y^2+z^2}$ while if $(x, y, z) \in X_2$, then $\|(x, y, z)\| = 2|z|$. To show that $\|\cdot\|$ is a norm, the only property which is not immediate is the triangle inequality. The proof of this inequality involves all possible relationships between the vectors u, v, and $u + v$ and the sets X_1 and X_2. Since the steps are straightforward, they will be omitted here.

Our work with this example will be divided into 3 steps:

1. The non-strict convexity of $(X\|\cdot\|)$,

2. certain midpoint properties for the sets X_1 and X_2,

3. a proof that $(X,\|\cdot\|)$ satisfies Postulate 1 for metric betweenness.

ad.1. To show that $(X,\|\cdot\|)$ is not strictly convex, let $u = (0, 1/2, 1/2)$, and $v = (1/2, 0, 1/2)$. Then $\|u\| = \|v\| = \left\|\frac{u+v}{2}\right\| = 1$ but $u \neq v$. By Theorem 1, $(X,\|\cdot\|)$ fails to be strictly convex.

ad.2. Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ and assume that $\|u\| = \|v\| = \left\|\frac{u+v}{2}\right\| = 1$.

a) If $u, v \in X_2$, then $\frac{u+v}{2} \in X_2$.

P r o o f . Since $u, v \in X_2$ and $\|u\| = \|v\| = 1$, $|z_1| = |z_2| = 1/2$. If $z_1 = -z_2$, then $\frac{u+v}{2} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, 0\right)$ which implies that $\frac{u+v}{2} \in X_1$. Then, since $u, v \in X_2$,

$$\left\|\frac{u+v}{2}\right\| = \frac{1}{\sqrt{2}}\sqrt{(x_1+x_2)^2 \quad (y_1+y_2)^2} \leq \frac{1}{\sqrt{2}}\sqrt{x_1^2+y_1^2} + \sqrt{x_2^2+y_2^2} \leq$$

$$\leq \frac{1}{\sqrt{2}}(|z_1| + |z_2|) = \frac{1}{\sqrt{2}}.$$

This is impossible since $\left\|\frac{u+v}{2}\right\| = 1$. Therefore, $z_1 = z_2 = \pm 1/2$ and

$$(x_1+x_2)^2 + (y_1+y_2)^2 \leq \left(\sqrt{x_1^2+y_1^2} + \sqrt{x_2^2+y_2^2}\right)^2 \leq (|z_1| + |z_2|)^2 =$$

$$= (z_1+z_2)^2,$$

which implies that $\frac{u+v}{2} \in X_2$.

b) Let $u, v \in X_1$.

If $\frac{u+v}{2} \in X_1$, then $u = v$.

If $\frac{u+v}{2} \in X_2$, then $u, v \in X_2$ also.

P r o o f . Since $u, v \in X_1$ and $\|u\| = \|v\| = 1$,

$$x_1^2 + y_1^2 + z_1^2 = 1/2$$

and

$$x_2^2 + y_2^2 + z_2^2 = 1/2.$$

If $\frac{u+v}{2} \in X_1$, then since $\left\|\frac{u+v}{2}\right\| = 1$,

$$\sqrt{(x_1+x_2)^2 + (y_1+y_2)^2 + (z_1+z_2)^2} =$$

$$= \sqrt{2} = \sqrt{x_1^2 + y_1^2 + z_1^2} + \sqrt{x_2^2 + y_2^2 + z_2^2}.$$

By the strict convexity of the usual Euclidean norm on 3-dimensional space, there is an $\alpha > 0$ such that $u = \alpha v$. Since $\|u\| = \|v\| = 1$, we must have $\alpha = 1$ and hence, $u = v$.

On the other hand, if $\frac{u+v}{2} \in X_2$, then the condition $\left\|\frac{u+v}{2}\right\| = 1$ implies that $|z_1+z_2| = 1$. Since $u \in X_1$ and $\|u\| = 1$,

$$1/2 = x_1^2 + y_1^2 + z_1^2 \geq 2z_1^2,$$

which implies that $|z_1| \leq 1/2$. A similar argument shows that $\|z_2\| \leq 1/2$. Therefore, $z_1 = z_2 = \pm 1/2$ and

$$x_1^2 + y_1^2 = 1/4 = z_1^2,$$

which shows that $u \in X_2$. The same argument applied to x_2, y_2, and z_2 shows that $v \in X_2$ also.

c) Let $u \in X_1$, and $v \in X_2$.

If $\frac{u+v}{2} \in X_1$, then $u = v$.

If $\frac{u+v}{2} \in X_2$, then $u \in X_2$ also.

P r o o f . In this case, since $\|u\| = \|v\| = 1$, $x_1^2 + y_1^2 + z_1^2 = 1/2$ while $|z_2| = 1/2$. If $\frac{u+v}{2} \in X_1$, then $(x_1+x_2)^2 + (y_1+y_2)^2 + (z_1+z_2)^2 = 2$. Therefore, since $v \in X_2$,

$$\sqrt{2} = \sqrt{(x_1+x_2)^2 + (y_1+y_2)^2 + (z_1+z_2)^2} \leq \sqrt{x_1^2+y_1^2+z_1^2} + \sqrt{x_2^2+y_2^2+z_2^2} \leq$$

$$\leq \sqrt{1/2} + \sqrt{2z_2^2} = \sqrt{2}.$$

This implies that

$$\sqrt{(x_1+x_2)^2 + (y_1+y_2)^2 + (z_1+z_2)^2} = \sqrt{x_1^2+y_1^2+z_1^2} + \sqrt{x_2^2+y_2^2+z_2^2}$$

and the same argument as used in part b) shows that $u = v$.

If $\frac{u+v}{2} \in X_2$, then $|z_1 + z_2| = 1$. By the same arguments as used in part b), we obtain $z_1 = \pm 1/2$ and hence, $x_1^2+y_1^2 = 1/4 = z_1^2$. This shows that $u \in X_2$ and the proof is complete.

ad.3. Now, we demonstrate that $(X,\|\cdot\|)$ satisfies Postulate 1 for metric betweenness.

Let $u = (x_1,y_1,z_1)$, $v = (x_2,y_2,z_2)$, and $w = (x_3,y_3,z_3)$ and assume that $\|u\| = \|v\| = \|w\| = \left\|\frac{u+v}{2}\right\| = \left\|\frac{v+w}{2}\right\| = 1$. By Part 2, all possible relationships between u, v, $\frac{u+v}{2}$, and $\frac{v+w}{2}$ and the sets X_1 and X_2 lead to 2 major cases: either $u, v, w \in X_2$ or at least 2 of these points are equal. If any 2 of these points are equal, then $\|u+v+w\| = 3$ by the lemma. Therefore, we need only consider the case where $u,v,w \in X_2$ and all 3 are distinct. By Part 2 a), $\frac{u+v}{2}$ and $\frac{v+w}{2}$ are in X_2 also. Since $\|u\| = \|v\| = \left\|\frac{u+v}{2}\right\| = \left\|\frac{v+w}{2}\right\| = 1$, it follows that $|z_1| = |z_2| = |z_3| = 1/2$ and $|z_1+z_2| = = |z_2+z_3| = 1$ and hence, $z_1 = z_2 = z_3 = \pm 1/2$. Further, since $u, v, w \in X_2$,

$$(x_1+x_2+x_3)^2 + (y_1+y_2+y_3)^2 \leq$$

$$\leq \left(\sqrt{x_1^2+y_1^2}+\sqrt{x_2^2+y_2^2}+\sqrt{x_3^2+y_3^2}\right)^2 \leq (|z_1|+|z_2|+|z_3|)^2 =$$

$$= (z_1 + z_2 + z_3)^2,$$

which implies that $u + v + w \in X_2$ also. Finally,

$$\|u + v + w\| = 2|z_1 + z_2 + z_3| = 2(3/2) = 3.$$

Therefore, Theorem 6 implies that $(X,\|\cdot\|)$ satisfies Postulate 1 for metric betweenness.

We conclude this paper with the main result, which states that each remaining postulate characterizes strict convexity.

T h e o r e m 7. Strict convexity of $(X,\|\cdot\|)$ is equivalent to each of Postulates 4 through 8 for metric betweenness.

P r o o f . Since Postulates A, C, 2 and 3 are true in any normed space, the equivalence of Postulates 4 and 5 and of Postulates 6, 7, and 8 can be deduced from the results in [5]. Further, it is shown in [2] that strict convexity implies Postulate 4. Therefore, it suffices to show that Postulate 5 implies strict convexity and that Postulate 6 is equivalent to strict convexity.

1. Assume Postulate 5 and let $\|x\| = \|y\| = \left\|\frac{x+y}{2}\right\| = 1$. Then, since this implies that $M[Ox(x+y)]$ and $M[Oy(x+y)]$, it follows by Postulate 5 that $M[Oxy]$ or $M[yx(x+y)]$. Since $\|x\| = \|y\|$, each possibility implies that $x = y$ and hence, $(X\|\cdot\|)$ is strictly convex.

2. The proof that the strict convexity of $(X,\|\cdot\|)$ implies Postulate 6 is the same as the proof of Theorem 5. For the

converse, assume that Postulate 6 holds and let $\|x\| = \|y\| = \left\|\frac{x+y}{2}\right\| = 1$. Since these conditions imply that $M[(-y)0x]$ and since $M[(-x)0x]$ is always true, Postulate 6 implies either $M[(-x)(-y)x]$ or $M[(-y)(-x)x]$. Because $\|x+y\| = 2\|x\|$, each of these implies that $x = y$. Therefore, $(X, \|\cdot\|)$ is strictly convex and the proof is complete.

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