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MEASURES OF NONCOMPACTNESS IN THE SPACE OF CONTINUOUS TEMPERED FUNCTIONS

1. Introduction

Up to now a lot of various definitions of the notion of a measure of noncompactness have been proposed (see e.g. bibliography in [8]). Almost in all definitions a measure of noncompactness is a function which is defined on the family of all nonempty and bounded subsets of a metric space with real nonnegative values and yet satisfies some other conditions (cf. [8]). Among those conditions the most characteristic is that which requires the measure of noncompactness to be equal to zero on the whole family of all relatively compact sets.

The most important measures of noncompactness are the Kuratowski measure α and the Hausdorff measure χ [8]. The last one is defined by the formula

$$\chi(X) = \inf \left[\varepsilon > 0 : X \text{ can be covered by a finite number of balls of radius } \varepsilon \right]$$

The above measures (i.e. the measures α and χ) was often used ([5], [6], [8]). In many works we can also find some exact formulas for these measures of noncompactness in a concrete metric spaces ([5], [6], [7], [8]).

In many situations the applications of such measures as the measures α, χ are not convenient because we do not always know a handy necessary and sufficient condition for relative compactness of subsets of a given space. B.N. Sadovskii was trying to overcome these difficulties, but he gave the axiomatic system of a measure of noncompactness which is too general and not very useful for applications.

In the paper [1] another definition of a measure of noncompactness was introduced. This one overcomes the mentioned above difficulties and seems to be useful for applications [2].

In this paper we accept this definition and we give a concrete realizations of it in the space of continuous tempered functions. Also, some comparisons between the measures defined here and Hausdorff's measure of noncompactness are given.

2. Notations and definitions

Let $(E, \| \|)$ be a given Banach space.

Throughout this paper we shall employ the same notations as in paper [1]. For instance, we shall denote:

$\mathfrak{M}_E$ - the family of all nonempty and bounded subsets of E,

$\mathfrak{N}_E$ - the family of all nonempty relatively compact subsets of E.

If $\mathfrak{X}$ is a nonempty family of sets then

$$\mathfrak{X}^c = [\bar{Z} : Z \in \mathfrak{X}] ,$$

where $\bar{Z}$ denotes the closure of the set Z.

We shall accept the definition of the measure of noncompactness of the paper [1].

D e f i n i t i o n . The function $\mu : \mathfrak{M}_E \to \langle 0,+\infty)$ will be called a measure of noncompactness if it subjects to the following conditions:

1^0 the family $\mathcal{P} = [X \in \mathfrak{M}_E : \mu(X) = 0]$ is nonempty and $\mathcal{P} \subset \mathfrak{N}_E$,

2^0 $X \subset Y \Longrightarrow \mu(X) \leq \mu(Y)$,

3^0 $\mu(\bar{X}) = \mu(X)$,

4^0 $\mu(\mathrm{Conv}\, X) = \mu(X)$,

5^0 $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$, for $\lambda \in \langle 0,1 \rangle$,

6^0 if X_n, $n = 1,2,\ldots,$ are closed sets such that $X_{n+1} \subset X_n$ and $\lim_{n\to\infty} \mu(X_n) = 0$ then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is non-empty.

The family $\mathcal{P}$ described in the axiom 1^0 is said to be the kernel of a measure μ and is denoted by $\ker \mu$. It can be shown that the family $(\ker \mu)^c$ is a closed subspace of the space $\mathfrak{M}_E^c$ with respect to the topology generated by the Hausdorff distance D and has some other properties [2].

Notice that the functions α and χ are measures of noncompactness such that $\ker\alpha = \ker \chi = \mathfrak{N}_E$ [6], [8].

3. The space of continuous tempered functions

Let $p(t)$ be a given function defined and continuous on the interval $\langle 0,+\infty)$ with real positive values. We shall denote by $C(\langle 0,+\infty), p(t)) = C$ the set of all real continuous functions $x(t)$ defined on the interval $\langle 0,+\infty)$ and such that

$$\sup\left[|x(t)|\, p(t):\ t \geq 0\right] < +\infty.$$

If we norm it by

$$\|x\| = \sup\left[|x(t)|\, p(t):\ t \geq 0\right]$$

then C is a Banach space. This space will be called space of continuous tempered functions.

It is worth mentioning that the space C was often used in a lot of applications ([3], [4], [9]).

In the space C the Arzelà criterion of compactness fails to work and we do not know the convenient description of the family of all relative compacts. We may give only a "good" description of the some subfamilies of $\mathfrak{N}_C$.

4. Some measures of noncompactness in the space C and their properties

Let $C = C(\langle 0,+\infty), p(t))$ be a fixed space and let $x \in C$, $X \in \mathfrak{M}_C$. For a given $T > 0$ and $\varepsilon > 0$ let us denote:

$$\omega^T(x,\varepsilon) = \sup\Big[|x(t)p(t) - x(s)p(s)| : t,s \in \langle 0,T\rangle, |t-s| \leqslant \varepsilon\Big],$$

$$\omega^T(X,\varepsilon) = \sup\big[\omega^T(x,\varepsilon) : x \in X\big],$$

$$\omega_0^T(X) = \lim_{\varepsilon \to 0} \omega^T(X,\varepsilon),$$

$$\omega_0(X) = \lim_{T \to \infty} \omega_0^T(X).$$

Further, we shall denote:

$$a(X) = \lim_{T \to +\infty} \sup_{x \in X} \Big\{\sup\Big[|x(t)p(t) - x(s)p(s)| : t,s \geqslant T\Big]\Big\},$$

$$b(X) = \lim_{T \to +\infty} \sup_{x \in X} \Big\{\sup\Big[|x(t)|\, p(t) : t \geqslant T\Big]\Big\},$$

$$c(X) = \lim_{T \to +\infty} \sup \operatorname{diam} (X(T)p(T)),$$

where $X(T)p(T) = \big[x(T)p(T) : x \in X\big]$ and $\operatorname{diam}(X(T)p(T))$ denotes the diameter of the set $X(T)p(T)$.

Now we have the following theorem.

Theorem. The functions

$$\mu_a(X) = \omega_0(X) + a(X),$$

$$\mu_b(X) = \omega_0(X) + b(X),$$

$$\mu_c(X) = \omega_0(X) + c(X),$$

are the measures of noncompactness in the space C. Moreover, the following inequalities hold

(1) $$\chi_C(X) \leqslant \mu_a(X),$$

(2) $$\chi_C(X) \leqslant \mu_c(X),$$

(3) $$\mu_a(X) \leqslant 2\mu_b(X), \ \mu_c(X) \leqslant 2\mu_b(X).$$

P r o o f . First we prove the inequality (1). Let us assume that $\mu_a(X) = r$ and let $\omega_0(X) = r_1$, $a(X) = r_2$, $r_1 + r_2 = r$. For an arbitrary given $\varepsilon > 0$ we may find $T > 0$ such that

$$(4) \qquad \sup\Big[\,|x(t)p(t) - x(s)p(s)| : t,s \geqslant T\Big] \leqslant r_2 + \varepsilon .$$

We consider now in the space $C_T = C(\langle 0,T\rangle, p(t))$ the set $X_T = \big[x(t)p(t)\chi_T(t): x \in X\big]$, where χ_T denotes the characteristic function of the interval $\langle 0,T\rangle$. By virtue of the formula for Hausdorff's measure of noncompactness in the space C_T [6] we may find the $\frac{r_1}{2} + \varepsilon$ - net $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_k$ of the set X_T in the space C_T, i.e. for any $x \in X$ there exists $m \in \{1,2,\dots,k\}$ such that

$$(5) \qquad |x(t)p(t) - \bar{x}_m(t)p(t)| \leqslant \frac{r_1}{2} + \varepsilon$$

for $t \in \langle 0,T\rangle$.

We consider now the extensions x_m of the functions $\bar{x}_m$ $(m = 1,\dots,k)$ on the whole interval $\langle 0,+\infty)$, given by the formula

$$x_m(t) = \begin{cases} \bar{x}_m(t) & \text{for} \quad t \in \langle 0,T\rangle , \\ \dfrac{\bar{x}_m(T)p(T)}{p(t)} & \text{for} \quad t \geqslant T. \end{cases}$$

Hence and with respect to (4) and (5), for any $t \geqslant T$ we have

$$|x(t)p(t) - x_m(t)p(t)| \leqslant |x(t)p(t) - x(T)p(T)| + |x(T)p(T) - $$
$$- x_m(t)p(t)| \leqslant r_2 + \varepsilon + |x(T)p(T) - \bar{x}_m(T)p(T)| \leqslant r_2 + \frac{r_1}{2} + 2\varepsilon.$$

Finally we get

$$|x(t)p(t) - x_m(t)p(t)| \leqslant r_2 + r_1 + 2\varepsilon = r + 2\varepsilon$$

for any $t \geqslant 0$, which means that the functions $x_m(t)p(t)$ $(m=1,\ldots,k)$ form $r+2\varepsilon$ - net of the set X in the space C. This completes the proof of the inequality (1). The proof of the inequality (2) is similar and the proof of the inequalities (3) follows directly from the definition of the functions μ_a, μ_b, μ_c.

We remark now that from (1), (2) and (3) it follows that our functions satisfy the condition 1^o of the definition of the measure of noncompactness. In view of the properties of Hausdorff's measure χ [6] it is easy to deduce that these functions satisfy the condition 6^o, too. The proof other conditions is easy and may be omitted.

It is worth mentioning that kernels of the measures μ_a, μ_b, μ_c are the families of all bounded sets X consisting of functions which are equicontinuous on each compact interval and additionally satisfying one of the following conditions, respectively: $a(X) = 0$, $b(X) = 0$, $c(X) = 0$. Obviously $\ker \mu_b \subset \ker \mu_a \cap \ker \mu_c$. Moreover it is easy to observe that $\ker \mu_a \neq \mathfrak{N}_C$, $\ker \mu_c \neq \mathfrak{N}_C$ and $\ker \mu_a \neq \ker \mu_c$.

Finally we notice that in our consideration we may replace the interval $\langle 0,+\infty)$ by any locally compact domain.

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