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GENERAL TRANSMISSION AND BOUNDARY VALUE PROBLEMS FOR FIRST-ORDER ELLIPTIC EQUATIONS IN MULTIPLY-CONNECTED PLANE DOMAINS

1. Introduction

The paper deals with boundary value problems of the type $\mathrm{Re}\left\{\overline{a(t)}w(t)\right\} = \gamma(t)$, $w^+(t) = g(t)w^-(t) + h(t)$ for the general elliptic linear system of two first-order real equations.

Let $L = L_0 + L_1 + ,\dots, + L_m$ be the boundary contours of an $m+1$-connected Liapounoff [11] region D, where L_0 contains all contours L_k, $k \geq 1$. Consider the equation

$$(1.1)\quad w_{\bar{z}} - q_1(z)w_z - q_2(z)\overline{w}_z + A(z)w + B(z)\overline{w} = F(z), \quad (\text{in } D)$$

$w = w(z) = u(x,y) + i\,v(x,y)$, $z = x + i\,y$, which is the well-known complex form of the general elliptic linear system of two first-order real equations for the unknown functions $u(x,y)$ and $v(x,y)$ of two independent variables x, y. Let w satisfy the following boundary conditions

$$(1.2)\qquad \mathrm{Re}\left\{\overline{a(t)}\,w(t)\right\} = \gamma(t), \quad (t \in L).$$

The boundary value problem (1.1)-(1.2) for the unknown function w will be briefly called Problem I.

The simplest case is the boundary value problem I for a complex function $w(z)$ - i.e. when $q_1 = q_2 = A = B = F = 0$ and (1.1) reduces to the familiar complex form of the Cauchy-Riemann system for analytic functions.

If $q_1 = q_2 = 0$. Problem I is reduced to the Riemann-Hilbert problem of generalized analytic functions. To this problem are devoted the papers of L.Bers, B.Bojarski, I.N.Vekua, V.S.Vinogradov, W.Wendland and many others *). If one uses the regularity-proposition of the similarity-principle which was proved by L.Bers and I.N.Vekua, then the Green function method works for the Riemann-Hilbert problem of generalized analytic functions in a simply-connected domain. W.Wendland [14] developed the latter problem for multiply-connected case by application of the modified Green's function, which leads to a system of Fredholm integral equations with compact operators. In this case important results were elaborated by B.Bojarski [2] and [3]. When $q_1 = q_2 = 0$, another method was established by Vekua [12] using Noether's and Muschelischwili's theory of singular integral equations. This method works for multiply-connected domains.

When D is a simply-connected domain, V.S.Vinogradov [13] solved Problem I through the utilization of a conformal mapping onto the unit disc.

We have proposed not only a method different from the above, but one which can be used for the general boundary value problem I in multiply-connected domains.

2. Green's function and Dirichlet integral

Following Courant [6], let $g(z,\zeta)$ be the Green function for the domain D of the Dirichlet problem, where D is the connected domain bounded by a finite number of simple smooth non-intersecting contours L. It is known that this function can be represented in the form

$$g(z,\zeta) = \log \frac{1}{|z-\zeta|} + h(z,\zeta) \tag{2.1}$$

$z = x + iy$, $\zeta = \xi + i\eta$, $z \in L$, $\zeta \in D$, where ζ is a fixed point in D and $h(z,\zeta)$ is a harmonic function of the varia-

*) For a great many special references see [7], [12].

ble z which has the boundary value $h(z,\zeta) = \log|z-\zeta|$, $z \in L$, $\zeta \in D$. The dependence of the Green function on the parameter point ζ is clarified by application of the Green identity which proves the symmetry law

$$g(z,\zeta) = g(\zeta,z). \tag{2.2}$$

Hence we may consider z as a parameter point; then we observe that $g(z,\zeta)$ as a function of ζ, will be the Green function. Since for $z \in L$, $g(z,\zeta) = 0$, for all $\zeta \in D$, we obtain

$$\frac{\partial g(\zeta,z)}{\partial \zeta} = 0, \quad z \in L, \quad \zeta \in D. \tag{2.3}$$

The following combinations of the derivative are of particular interest for future consideration.

$$\begin{aligned} L(z,\zeta) &= 2\,\frac{\partial^2 g(z,\zeta)}{\partial z\,\partial \zeta}, \\ l(z,\zeta) &= L(z,\zeta) + \frac{1}{(z-\zeta)^2}, \\ K(z,\bar{\zeta}) &= 2\,\frac{\partial^2 g(z,\zeta)}{\partial z\,\partial \bar{\zeta}}. \end{aligned} \tag{2.4}$$

Obviously, $l(z,\zeta)$ is a regular analytic function of z and ζ in D, therefore the function $L(z,\zeta)$ has at $z = \zeta$ a double pole; $K(z,\bar{\zeta})$ is a regular analytic function of z and $\bar{\zeta}$ over the whole of the domain D and continuous for $z \in D + L$ as long as $\zeta \in D$. As a direct consequence of (2.4), we obtain

$$\begin{aligned} L(z,\zeta) &= L(\zeta,z), \quad l(z,\zeta) = l(\zeta,z). \\ K(z,\bar{\zeta}) &= \overline{K(\zeta,\bar{z})}. \end{aligned} \tag{2.5}$$

Ir connection with the symmetry law (2.5), by differentiation from (2.3), we have

$$\frac{\partial^2 g(\zeta,z)}{\partial\zeta\,\partial z}\, z' + \frac{\partial^2 g(\zeta,z)}{\partial\zeta\,\partial\bar{z}}\,\bar{z}' = 0,$$

$z \in L$, $\zeta \in D$, $z' = dz/ds$; s-the length parameter of the curve system L. In other words

(2.6) $$L(z,\zeta)z' = \overline{-K(z,\bar{\zeta})\bar{z}'} \quad z \in L, \quad \zeta \in D.$$

Of fundamental importance for methods used in this paper will be the following Dirichlet integral

(2.7) $$\iint_D \overline{L(z,\zeta)}\Phi(z)\,d\sigma_z = 2\iint_D \frac{\partial^2 g(z,\zeta)}{\partial\bar{z}\;\partial\bar{\zeta}}\,\Phi(z)\,d\sigma_z,$$

$d\sigma_z$ - the area element, the function $\Phi(z)$ is analytic in the domain D. Bearing in mind the definitions (2.4), we observe that the function $L(z,\zeta)$ has a double pole at $z=\zeta$. However the integral may be understood as the Cauchy principle value. Making use of the Green formula

(2.8) $$\lim_{\varepsilon\to 0}\iint_{D(\zeta,\varepsilon)} \overline{L(z,\zeta)}\Phi(z)\,d\sigma_z = \lim_{\varepsilon\to 0}\int_{L+L_\varepsilon} \frac{\partial g(\zeta,z)}{i\,\partial\bar{\zeta}}\,\Phi(z)\,dz = 0,$$

where $D(\zeta,\varepsilon) = D \cap (|z-\zeta|>\varepsilon)$ which makes sense for every interior point ζ of D and sufficiently small ε; L_ε is the circumference $|z-\zeta| = \varepsilon$.
Since $l(z,\zeta)$ is a regular analytic function with respect to z and ζ inside D, in view of the definitions (2.4), we obtain the following Dirichlet integral

(2.9) $$\iint_D \overline{l(t,\zeta)}\,l(t,z)\,d\sigma_t = \iint_D \frac{l(t,z)}{(\bar{t}-\bar{\zeta})^2}\,d\sigma_t.$$

We remark that the above representation contains an improper integral on the right-hand side, which will be understood in the same sense as the integral (2.8) i.e. in the sense of the

Cauchy principal value. Making use of formal rules on integration by parts and the Green formula, we establish the following formula:

$$(2.10)\qquad \iint_{D(\zeta,\varepsilon)} \frac{l(t,z)}{(\bar{t}-\bar{\zeta})^2}\, d\sigma_t = -\frac{1}{2i}\int_L \frac{l(t,z)}{\bar{t}-\bar{\zeta}}\, dt + O(\varepsilon),$$

where $D(\zeta,\varepsilon) = D \cap (|t-\zeta| > \varepsilon)$ (see (2.8)). Passing to limit as $\varepsilon \to 0$, we have

$$(2.11)\qquad \iint_D \frac{l(t,z)}{(\bar{t}-\bar{\zeta})^2}\, d\sigma_t = -\frac{1}{2i}\int_L \frac{l(t,z)}{\bar{t}-\bar{\zeta}}\, dt .$$

On the other hand, bearing in mind the boundary condition (2.6), we can write

$$(2.12)\qquad \frac{1}{2\pi i}\int_L \frac{L(t,z)}{\bar{t}-\bar{\zeta}}\, dt = \frac{1}{2\pi i}\int_L \frac{L(t,z)t'(s)}{\bar{t}-\bar{\zeta}}\, ds =$$

$$= -\frac{1}{2\pi i}\int_L \frac{\overline{K(t,\bar{z})}\;\overline{t'(s)}}{\bar{t}-\bar{\zeta}}\, ds = \overline{\frac{1}{2\pi i}\int_L \frac{K(t,\bar{z})}{t-\zeta}\, dt},$$

$t'(s) = dt/ds$. Therefore in view of the residue theorem we obtain

$$(2.13)\qquad \frac{1}{2\pi i}\int_L \frac{L(t,z)}{\bar{t}-\bar{\zeta}}\, dt = \overline{K(\zeta,\bar{z})} = K(z,\bar{\zeta}).$$

More can be said about the function $l(z,\zeta)$; by simple computation, it can be observed that in the case when the domain D is the closed disc $|z| \leqslant R$, then $l(z,\zeta) = 0$. Moreover, if $D \in C^3 (L \in C^3)$, then the function $l(z,\zeta)$ is continuous in both variables in the closed domain $D + L$ (multiply-connected case) (see [6]).

3. Integral equations and general boundary value problem I

We consider the Riemann-Hilbert boundary value problem (1.2) for equation (1.1) which is not reduced to the canonical form in a multiply-connected domain.

Concerning equation (1.1) and boundary conditions (1.2), we make the following assumptions.

H y p o t h e s i s I. 1) The coefficients $q_1(z)$ and $q_2(z)$ are measurable function; satisfying the condition of uniform ellipticity

$$(3.1) \qquad |q_1(z)| + |q_2(z)| \leqslant q_0 < 1; \quad (\text{in } D);$$

2) A, B and F are given complex functions of the point z belonging to the class $L_p(D)$ for some $p > 2$.

3) The complex function $a(t)$ and real function $\gamma(t)$ ($t = t(s) \in L$, s-the length parameter), satisfying the Hölder condition ($a, \gamma \in C_\alpha(L)$, $0 < \alpha < 1$) on L ($a \neq 0$, on L).

4) The index n corresponding to the boundary value problem I is equal to zero ($n = 0$).

Moreover, the solution w of the problem I will be sought in the class of functions, continuous in the closed domain $D + L$ and belonging to the class $W_p^1(D)$ for some $p > 2$ (see [4]). We recall that the index n corresponding to the boundary value problem I is the integer

$$n = n_0 + n_1 + ,\dots, + n_m \equiv \frac{1}{2\pi}\Delta_L \arg a(t),$$

where

$$n_i = \frac{1}{2\pi}\Delta_{L_i} \arg a(t) \quad (i = 0,1,2,\dots,m).$$

We shall prove the following theorem.

T h e o r e m 1. Under Hypothesis I, the general boundary value problem (1.1) - (1.2) is equivalent to a two-dimensional integral equation of Fredholm type with compact operator.

Some remarks are necessary before the proof of the theorem. Let $q_1 = q_2 = A = B = F = 0$. Consider the following particular case of the boundary value problem I:

Find a function $\Phi(z)$, holomorphic in D, continuous in the closed domain $D + L$, satisfying the boundary conditions

$$\text{(3.2)} \qquad \mathrm{Re}\left\{\Phi(t)\right\} = \gamma(t) \quad t \in L.$$

Taking for $\gamma(t)$ a function of the form $\gamma(t) = \gamma_0(t) + c_j$ on L_j $(j = 0,1,2,\ldots,m)$, where c_j $(j = 1,2,\ldots,m)$ are unknown real constants, $c_0 = 0$ and $\gamma_0(t)$ is a given function continuous on L, we obtain the modified formulation of the Dirichlet problem which was investigated by Muskhelishvili [11]. A somewhat different method of solution of this problem is presented by Vekua [12].

The boundary value problem (3.2) for the right-hand side of the form $\gamma = \gamma_0 + c_j$ where c_j are determind constants, has always a solution, moreover, a unique solution for a defined constant c_j. Clearly, if γ is Hölder continuous on L, then the solution Φ is Hölder continuous on the closed domain $D + L$. As a direct consequence of (3.2), we note the following representation formula.

Every function $\eta(t)$ belonging to the class $C_\beta(L)$, $0 < \beta < 1$ can be represented in the form

$$\text{(3.3)} \qquad \eta(t) = \Psi(t) - i\, q(t) + c(t) \quad \text{(on L)},$$

where q is the imaginary part of the function Ψ; $\Psi(z)$ is a function holomorphic in D, continuous in the Hölder sense in the closed domain $D + L$ and satisfies the boundary condition (3.2) with $\gamma(t) = \eta(t) + c(t)$, $c(t)$ - a piecewise constant function on L; $c_0 = 0$, on L_0 and $c = c_j = \text{const.}$ on L_j $(j = 1,2,\ldots,m)$. Ψ and c are uniquely expressible by $\eta(t)$. In connection with the problem I and Hypothesis I, we observe that, by subsitution of the form (see [12] also [8])

$$w(z)e^{\chi(z)},$$

the boundary condition (1.2) can be reduced to the canonical form (since $n = 0$)

$$\mathrm{Re}\{w(t)\} = \gamma_1(t),$$

where $\gamma_1(t)$ is continuous in the Hölder sense on $L(t \in L)$. In view of the boundary value problem (3.2), with no loss of generality we shall assume that $\gamma_1(t) = 0$. These substitutions leave invariant the form of the equation (1.1) and do not violate the condition (3.1). Therefore we shall consider the equation (1.1) with the boundary condition

$$\mathrm{Re}\{w(t)\} = 0 \qquad (t \in L). \tag{3.4}$$

The remaining assumptions concerning the data of the boundary value problem I are preserved.

We shall prove Theorem 1 through the following lemma:

L e m m a 1. If $D \in C^3$. Then, any function $w(z)$, continuous in the closed $D + L$, belonging to the class $W_p^1(D)$, $p > 2$ and satisfying the boundary condition (3.2) can be represented by the formula

$$w(z) = T(\omega) = \frac{1}{\pi} \iint_D \left\{ 2\,\frac{\partial g(\zeta,z)}{\partial \zeta} + \overline{A(\zeta,z)}\ \omega(\zeta) - A(\zeta,z)\omega(\zeta) \right\} d\sigma_\zeta + ip(z), \tag{3.5}$$

$z \in D + L$, $\zeta \in D$ where $g(z,\zeta)$ is the Green function for the domain D of the Dirichlet problem (see (2.1)); The function A is introduced as the following

$$A(\zeta,z) = \frac{1}{2\pi i} \int_L \frac{dt}{(\bar{t}-\bar{\zeta})(t-z)} .$$

L is the boundary of the domain D; ω is an arbitrary function of the class $L_p(D)$, $p > 2$ and $p(z)$ is a continuous constant function of the point z.

P r o o f . The function $w(z)$ is continuous in the closed domain $D + L$ (for instance see [12] or [10]). We shall prove that w belongs to the class $W_p^1(D)$, $p > 2$.

The formulae for the generalized derivatives of w (3.5) with respect to $\bar{z}$ and z take the form

$$(3.6)\qquad \frac{\partial w}{\partial \bar{z}} = \frac{\partial T(\omega)}{\partial \bar{z}} = \omega + \frac{1}{\pi}\iint_D M(\zeta,z)\omega(\zeta)d\sigma_\zeta ,$$

$$(3.6a)\qquad \frac{\partial w}{\partial z} = \frac{\partial T(\omega)}{\partial z} = \frac{1}{\pi}\iint_D \left\{ \frac{-\omega(\zeta)}{(\zeta-z)^2} + l(\zeta,z)\omega(\zeta) - \frac{\partial A(\zeta,z)}{\partial z}\overline{\omega(\zeta)} \right\} d\sigma_\zeta$$

where

$$l(z,\zeta) = 2\,\frac{\partial^2 h(z,\zeta)}{\partial z\,\partial \zeta}$$

$$M(\zeta,z) = 2\,\frac{\partial^2 h(\zeta,z)}{\partial \zeta\,\partial \bar{z}} + \overline{\frac{\partial A(\zeta,z)}{\partial z}} ,$$

$h(z,\zeta)$ is defined by (2.1). Let us prove that $M(\zeta,z)$ is continuous in both variables on the closed domain D. In view of the definitions (2.4) and symmetry law (2.5), we obtain

$$(3.7)\qquad \overline{M(\zeta,z)} = K(z,\bar{\zeta}) + \frac{\partial A(\zeta,z)}{\partial z} .$$

By means of the identities (2.9), (2.13)

$$(3.8)\qquad \overline{M(\zeta,z)} = -\frac{1}{\pi}\iint_D \overline{l(t,\zeta)}\,l(t,z)\,d\sigma_t .$$

With regard to our observation in Sec. 2, we have proved the continuity of $M(\zeta,z)$ with respect to ζ and z in the closed domain $D + L$. Here, the derivatives are understood in the generalized Sobolev-Schwartz sense, which in this case is equal to the Pompeiu sense and the existence of the singular integral (3.6a) for any function $\omega \in L_p(D)$, $p > 2$ follows from the Calderon-Zygmund Theorem (for instance see [4] or [12]). Hence w given by representation formula (3.5) belong to the class $W_p^1(D)$, $p > 2$.

On the other hand, making use of the identity (2.3), it can be readily verified that, the real part of w defined by (3.3) equals zero on the boundary L.

Introducing the expression (3.5) into the equation (1.1), we obtain for the following integral equation

$$\omega - q_1 S(\omega) - q_2 \overline{S(\omega)} + A\, T(\omega) + B\, \overline{T(\omega)} + \tag{3.9}$$

$$+ \frac{1}{\pi} \iint_D M(\zeta, z)\omega(\zeta)\, d\sigma_\zeta = F,$$

where

$$S(\omega) = \frac{\partial w}{\partial z} = \frac{\partial T(\omega)}{\partial z},$$

and $M(\zeta,z)$ is defined by the formula (3.8). We shall prove the equality

$$\|S(\omega)\|_{L_2(D)} = \|\omega\|_{L_2(D)}. \tag{3.10}$$

It suffices to assume that,for instance, ω belongs to the set of the linear manifold $\overset{0}{D}_\infty(D)$. Then we have

$$\|S(\omega)\|^2 = \iint_D \overline{S(\omega)}\, S(\omega) d\sigma_z = \iint_D \frac{\partial T(\omega)}{\partial z} \overline{S(\omega)} d\sigma_z =$$

$$= \iint_D \frac{\partial \left(T(\omega)\, \overline{S(\omega)}\right)}{\partial z} d\sigma_z - \iint_D T(\omega) \frac{\partial \overline{S(\omega)}}{\partial z} d\sigma_z =$$

$$= -\frac{1}{2i} \int_L T(\omega)\, \overline{S(\omega)}\, dz - \iint_D T(\omega) \frac{\partial \bar{\omega}}{\partial \bar{z}} d\sigma_z,$$

where we have used Green's formula.

Since

$$\frac{\partial \overline{S(\omega)}}{\partial z} = \frac{\partial}{\partial z}\left(\overline{\frac{\partial T(\omega)}{\partial z}}\right) = \frac{\partial}{\partial \bar z}\left(\overline{\frac{\partial T(\omega)}{\partial z}}\right) =$$

$$= \frac{\partial}{\partial \bar z}\left(\overline{\frac{\partial T(\omega)}{\partial \bar z}}\right) = \frac{\partial}{\partial \bar z}\left(\overline{\omega + \frac{1}{\pi}\iint_D M(\zeta,z)\,\omega(\zeta)d\sigma_\zeta}\right) = \frac{\partial \bar\omega}{\partial \bar z},$$

we obtain

$$(3.11)\qquad ||S(\omega)||^2 = \overline{\frac{1}{2i}\int_L T(\omega)\,\frac{\partial T(\omega)}{\partial z}\,dz} - \frac{1}{2i}\int_L \bar\omega T(\omega)dz + \iint_D \omega\bar\omega\, d\sigma_z .$$

Taking into account that the first and second integral on the right-hand side of the last relation (3.11) are zero, we have proved that the norm Λ_2 of the operator S in the space $L_2(D)$ is equal to 1. The above results have been performed by a few application of the integrations by parts, making use of the Green identity.

We represent the equation (3.9) in the form

$$(3.12)\qquad \omega - S_1(\omega) + T_1(\omega) = F,$$

where

$$S_1(\omega) = q_1\, S(\omega) + q_2\, \overline{S(\omega)}$$

$$T_1(\omega) = A\, T(\omega) + B\, \overline{T(\omega)} + \frac{1}{\pi}\iint_D M(\zeta,z)\,\omega(\zeta)d\sigma_\zeta .$$

In view of (3.1) and (3.12), since $||S||_{L_2(D)} = 1$ and $q_0 < 1$, we can choose the **number** $p > 2$ such that $q_0\Lambda_p < 1$, Λ_p - the norm of S in L_p, i.e. $\Lambda_p = ||S||_{L_p(D)}$.

Consequently (3.9) is reducible to the following integral equation of Fredholm type with compact operator:

$$\omega - (I - S_1)^{-1} T_I(\omega) = (I - S_1)^{-1}F.$$

4. Mixed Riemann-Hilbert and Hilbert type boundary value problem for the general elliptic linear system of two first--order real equations.

Consider the m+1-connected Liapounoff region D (see Sec.1) with boundary L. Let L' be a system of finite non-intersecting oriented contours in the domain D. Assume that the systems L and L' have no points in common. Then the system L' decomposes the domain D into a finite number of connected subsets (Figure 1). The union of all these region D will be called domain G.

We consider here the general form of the equation (see also (1.1))

$$(4.1) \qquad w_{\bar z} - q_1(z)\, w_z - q_2(z)\, \overline{w}_z + A(z)\, w + B(z)\, \overline{w} = F(z)$$

in the domain G.

For Eq. (4.1) we investigate the following boundary value problem. Problem II: Find a solution $w = w(z)$ of (4.1), belonging to the class W_p^1, $p > 2$; continuous in the domain G, having continuous extension up to the boundary of every connected components of G, and satisfying the boundary conditions

$$(4.2) \qquad \mathrm{Re}\left\{\overline{a(t)}\, w(t)\right\} = \gamma(t), \quad (t \in L),$$

$$(4.3) \qquad w^+(t) = g(t)\, w^-(t) + h(t) \quad (t \in L').$$

The symbols w^+ and w^- are understood in the sense of the theory of the Hilbert boundary value problem.

In case when equation (4.1) has been reduced to the cannonical form, i.e. $q_1 = q_2 = 0$, Problem II is investigated in

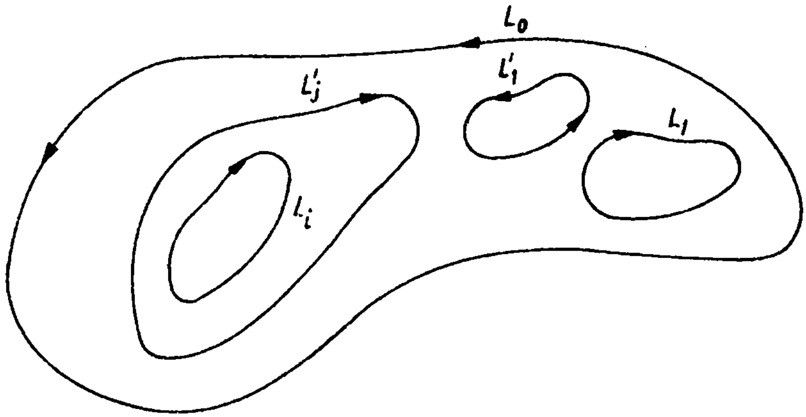

Fig.1

the author's paper [9]. Here we continue the study of the problem II for the general form of equation (4.1).

Concerning the coefficients of the equation (4.1) and boundary conditions (4.2) - (4.3), we make the following

H y p o t h e s i s 2.

1) q_1, q_2 are measurable functions satisfying the condition of uniform ellipticity (3.1) in G; 2) $L \in C^3$, $L' \in C^1$; 3) $a(t)$, $\gamma(t) \in C_\beta(L)$, $0 < \beta < 1$ $(t \in L,\ a \neq 0)$; 4) The complex functions $g(t)$ and $h(t) \in C_\nu(L')$, $\frac{1}{2} < \nu < 1 (t \in L'$, $g \neq 0)$; 5) A, B and $F \in L_p$, $p > 2$ in G.

The following lemma is of particular interest for future consideration.

L e m m a 4.1. Let L' be a finite number of smooth non-intersecting oriented contours $L'_1, L'_2, \ldots, L'_p$ in the finite part of the plane, for instance in the domain D, such that the domain bounded by the contours L' with respect to all plane is connected (Figure 2a) and let $g(t)$ be a non-vanishing function, continuous in the Hölder sense on L'. Then there exists a function, holomorphic inside the connected domain and outside ($f^+(z)$, $f^-(z)$ respectively), Hölder continuous from left and right on L' and vanishing nowhere in the finite part of the plane, including the boundary values $f^+(t)$, $f^-(t)$, $t \in L$ and satisfying the boundary condition

$$\frac{f^+(t)}{f^-(t)} = g(t), \quad t \in L'. \tag{4.4}$$

There exists such a function, for explicit form of this function see [8] or [11].
We recall an important property of the Cauchy type integral

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\mu(t)}{t-z}\,dt, \tag{4.5}$$

which, we have already used in Sec.3. This will be of our great use from latter on. Consider the comain D with boundary L. Let $D \in C_\beta^{k+1}$, and $\mu \in C^k(L)$, $0 < \beta < 1$, $k > 0$. Then the Cauchy type integral (4.5) belongs to the class $C_\beta^k(D+L)$, i.e. $\Phi(z) \in C_\beta^k(D+L)$. When $k = 0$, the proof is obtained from Muskhelishvili's results [11], and making use of the property of boundary L, it can be generalized for an arbitrary finite integer k.

When $k = 0$, the requirement in respect of the domain can be weakened,in other words, if $D \in C^1$ and $\mu \in C_\beta(L)$, $0 < \beta < 1$, then $\Phi(z) \in C_\beta(D+L)$. More can be said about such a function, for instance the derivative $|\Phi'(z)| < M(\mu,\beta)(\delta)^{\beta-1}$, where δ is the distance of the point z from the boundary L of the domain D; M-const. This inequality implies that $\Phi'(z) \in L_p(D)$ where p is an arbitrary number satisfying the relation $1 < p < \frac{1}{1-\beta}$, obviously, $p > 2$, when $1 > \beta > \frac{1}{2}$.
We shall give below an investigation of the problem II.
An important stop is the introduction of the so-called total winding number corresponding to the boundary conditions (4.2) - (4.3), which we shall define in the sequel.

D e f i n i t i o n. Let $n = \frac{1}{2\pi}\Delta_L \arg a(t)$ and $r = \frac{1}{2\pi}\Delta_L \arg g(t)$, then the integer $n_1 = n + r$ will be called the total winding number corresponding to the boundary conditions (4.2) - (4.3).

Then, concerning the general mixed boundary value problem II, we have the following theorem.

T h e o r e m 2. Under Hypothesis 2., if the total winding number n_1 corresponding to the boundary conditions (4.2) - (4.3) equals zero ($n_1 = 0$), then the boundary value problem II is equivalent to a two-dimensional integral equation of Fredholm type with compact operator.

The proof is carried out with the help of the result proposed in Sec.3 and the above considerations.

At first, we shall restrict ourselves to the case when the domain bounded by the contours L' with respect to all plane be connected. This domain was chosen merely for the sake of convenience. As a matter of fact, our method permits us to treat the general case of the domain G (Figure 1).
We make a subsitution of the form

$$(4.6) \qquad w(z) = \frac{W(z)}{f(z)} - \frac{p(z)}{f(z)},$$

where $f(z)$ is a sectionally holomorphic function inside the connected domain and outside, vanishing nowhere in the finite part of the plane, satisfying the boundary condition (4.4), $g(t)$ being the coefficient of the boundary condition (4.3) and

$$(4.7) \qquad p(z) = \frac{-1}{2\pi i} \int_{L'} \frac{h(t)\, f^-(t)}{t-z}\, dt,$$

$h(t)$ is the free term of the boundary condition (4.3). In view of the properties of the Cauchy type integral (4.5) and Hy-

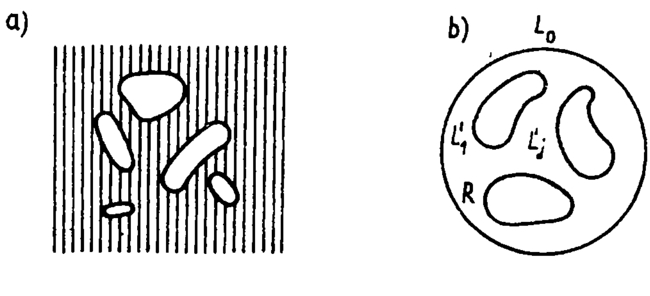

Fig.2

pothesis 2, it can be verified that the boundary value problem II is reduced to the general form of Problem I in the domain D for the unknown function W. By simple computation, we observe that the condition of the uniform ellipticity (3.1) holds in the domain D. On the other hand W is continuous on the closed domain D + L and belongs to the class $W_p^1(D)$, $p > 2$. The index corresponding to this boundary value problem is n_1. If we assume that $n_1 = 0$, it can be verified that all the requirements of Hypothesis 1 are satisfied. Therefore, problem II leads to an equivalent integral equation of Fredholm type with compact operator.

In accordance with the above considerations, Theorem 2 is also valid for the case where the domain bounded by L' is not connected. Substitutions of the form (4.6) leave invariant the form of the equation (1.1) and boundary condition (1.2), also they do not violate the conditions in Hypothesis 1.

R e m a r k . In particular, if $m = 0$, i.e. D is a simply-connected domain, the requirement in respect to the boundary L may be somewhat weakend. We make the following

H y p o t h e s i s 3. Let $m = 0$ and $L \in C_\beta^1$, $0 < \beta < 1$. The remaining assumptions concern the data of Problem II (Hypothesis 2) are preserved.

Then we have:

T h e o r e m 3. Under Hypothesis 3, if the total winding number $n_1 > 0$, the non-homogeneous Problem II is always solvable and the corresponding homogeneous problem of the Problem II has exactly $2n_1 + 1$ linearly independent solutions (over the field of real numbers).

C o r o l l a r y 1. Under Hypothesis 3, if n_1 is a negative number, the corresponding homogeneous Problem II has no non-trivial solutions.

In view of substitutions of the form (4.6) and the results established by Vinogradov [13], Theorem 3 and Corollary 1 can be easily proved.

When n_1 is a negative number, the conditions of solvability of the latter non-homogeneous problem II can be derived, but we shall not bring here the conditions, since it would involve extremely lengthy expressions.

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