DEMONSTRATIO MATHEMATICA
Vol. XII No 2 1979

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ORTHOGONALITY AND ORTHOCOMPLEMENTATION IN PARTIALLY ORDERED SETS

0. Introduction

An orthogonality can be defined on an arbitrary non-empty set in twofold ways: either as the primary notion by means of which we define the further structure of the set as, for example, a partial order and orthocomplementation, or as a secondary notion in partially ordered sets. An orthogonality is the basic notion in the quantum logic and has a natural physical interpretation (see e.g. [4], [7], [8], [9]).

In this paper we consider connections between an abstract orthogonality on a set an an orthocomplementation on a partially ordered set, as well connections between a Boolean orthogonality and an orthocomplementation on partially ordered sets. It is also shown what conditions must be satisfied in a lattice in order that this lattice be a pseudocomplemented one or a Boolean algebra. Finally, it is well known that some algebraic structures can be represented as partially ordered sets of real functions, for example, Boolean orthomodular partially ordered sets, [6]. Also partially ordered sets with complete weak orthogonality admitting a full set of states can be represented in this way.

D e f i n i t i o n 0.1. ([2]). Let $(P, \leq)$ be a partially ordered set. A mapping $'$: $P \to P$ is called a weak orthocomplementation if and only if it satisfies the following conditions:

(0.1) $$(\forall x \in P)(x \leqslant x''),$$

(0.2) $$(\forall x,y \in P) \text{ (if } x \leqslant y, \text{ then } y' \leqslant x').$$

If the condition (1) is replaced by the condition

(0.3) $$(\forall x \in P)\ (x = x'')$$

then the mapping $'$ is called a strong orthocomplementation. If P has the least element 0 and the mapping $'$ satisfies conditions (0.1), (0.2) and if

(0.4) $$N = \{0\}, \text{ where } N = \{x \in P : x \leqslant x'\},$$

then $'$ is called a weak non-degenerate orthocomplementation.

D e f i n i t i o n 0.2. ([2]). Let $(P, \leqslant)$ be a partially ordered set. A binary relation $\perp$ on P is said to be a weak orthogonality if and only if the following conditions are satisfied

(0.5) $$(\forall x,y \in P)\ (x \perp y \Longrightarrow y \perp x),$$

(0.6) $$(\forall x,y \in P)\ (x \leqslant y \Longrightarrow \{y\}^{\perp} \subseteq \{x\}^{\perp}),$$

where $\{z\}^{\perp} = \{p \in P : z \perp p\}$.

If $\perp$ satisfies conditions (0.5), (0.6) and if

(0.7) $$(\forall x \in P) \text{ (there exists } \sup \{x\}^{\perp} \text{ and } \sup \{x\}^{\perp} \in \{x\}^{\perp}),$$

then $\perp$ is called a weak complete orthogonality.

By a strong orthogonality we mean a relation $\perp$ such that the conditions (0.5), (0.6) and

(0,8) $$(\forall x,y \in P)(\{y\}^{\perp} \subseteq \{x\}^{\perp} \Longrightarrow x \leqslant y)$$

are satisfied.

Assume that a partially ordered set $(P, \leqslant)$ has a least element 0. We call $\perp$ defined on P a weak non-degenerate orthogonality, if the conditions (0.5), (0.6) and

(0.9) $\quad \operatorname{Ker} \perp = \{0\}$, where $\operatorname{Ker} \perp = \{x \in P : x \perp x\}$

are satisfied.

D e f i n i t i o n 0.3. ([5]). Let P be a set. A binary relation on P is said to be an abstract orthogonality if and only if it satisfies the following conditions:

$$(0.10) \qquad (\forall x,y \in P)(x \perp y \Rightarrow y \perp x),$$

$$(0.11) \qquad (\forall x \in P)\ (x \perp x \Rightarrow \{x\}^{\perp} = P),$$

$$(0.12) \qquad (\forall x,y \in P)\ (\{x\}^{\perp} = \{y\}^{\perp} \Rightarrow x = y).$$

It is easy to see that in the set P with an abstract orthogonality there exists at most one element which is orthogonal to itself. If there exists such an element, we call it 0 and it is the least element in P with respect to the partial order defined in the following way:

$$(\forall x,y, \in P)(x \leqslant y \quad \text{iff} \quad \{y\}^{\perp} \subseteq \{x\}^{\perp}).$$

Hence we have $P^{\perp} = \operatorname{Ker} \perp$ in the set P with an abstract orthogonality and $\operatorname{Ker} \perp$ has at most one element.

1. Abstract orthogonality and orthocomplementation

To every set with an abstract orthogonality we can associate a partially ordered set with an orthocomplementation. The following theorem holds:

T h e o r e m 1.1. Let $(P, \perp)$ be a set with an abstract orthogonality which is a partially ordered set with respect to the partial order: $x \leqslant y$ iff $\{y\}^{\perp} \subseteq \{x\}^{\perp}$. Moreover, let P have a least element 0 and assume that there exists $\sup \{x\}^{\perp}$ in $\{x\}^{\perp}$ for each $x \in P$. Then the mapping $' : P \to P$ such that $x \mapsto x' = \sup \{x\}^{\perp}$ is a strong non-degenerate orthocomplementation and $x \perp y$ iff $x \leqslant y'$.

First, we note that if the assumptions of Theorem 1.1 are satisfied, then:

R e m a r k 1.2. $(\forall x \in P)(x \perp x')$.

P r o o f . We have $x' = \sup \{x\}^{\perp} \in \{x\}^{\perp}$, so $x \perp x'$.

R e m a r k 1.3. $(\forall x,z \in P)\ (x \perp z \Longrightarrow z \leqslant x)$.

P r o o f . By the definition of a partial order we need to show that if $x \perp z$ and $p \in \{x\}^{\perp}$, then $p \in \{z\}^{\perp}$. Note that if $p \in \{x\}^{\perp}$ then $p \leqslant \sup \{x\}^{\perp} = x'$, therefore $\{x'\}^{\perp} \subseteq \{p\}^{\perp}$. Suppose that $p \in \{x\}^{\perp}$ and $x' \perp z$. Then $z \in \{x'\}^{\perp} \subseteq \{p\}^{\perp}$, and so $z \perp p$, thus $p \in \{z\}^{\perp}$.

P r o o f o f T h e o r e m 1.1. Suppose that $x \leqslant y$. We know that $x \leqslant y$ iff $\{y\}^{\perp} \subseteq \{x\}^{\perp}$ also $\sup \{y\}^{\perp} \leqslant \sup \{x\}^{\perp}$. But by the definition of the mapping $'$ $y' = \sup \{y\}^{\perp} \leqslant \sup \{x\}^{\perp} = x'$, so $y' \leqslant x'$.

Now we are going to show that $x = x''$ for each $x \in P$. Let $x \in P$. By Remark 1.2 we have $x \in \{x'\}^{\perp}$ and so we infer that $x \leqslant \sup \{x'\} = (x')' = x''$. On the other hand we have $x'' = (x')' = \sup \{x'\}^{\perp}$, therefore $x' \perp x''$. Set $z = x''$ in Remark 1.3. We get $x'' \leqslant x$.

Now assume that $x \leqslant y'$. By the definition of a partial order we have $\{y'\}^{\perp} \subseteq \{x\}^{\perp}$. But by Remark 1.2 $y' \perp y$. Thus we get $y \in \{y'\}^{\perp} \subseteq \{x\}^{\perp}$ and so $x \perp y$, and conversely if $x \perp y$, then $x \in \{y\}^{\perp}$. But $x \leqslant \sup \{y\}^{\perp} = y'$. Thus $x \leqslant y'$.

By the definition of the least element in P we have $\operatorname{Ker} \perp = \{0\}$. Set $x = y$ in the property "$x \leqslant y'$ iff $x \perp y$". Then $N = \{x \in P : x \leqslant x'\} = \operatorname{Ker}\perp = \{0\}$.

On the other hand we have the following theorem.

T h e o r e m 1.4. Let $(P, \leqslant, 0, ')$ be a partially ordered set with the least element 0 and a strong non-degenerate orthocomplementation. If we define an orthogonality $\perp$ on P by $x \perp y$ iff $x \leqslant y'$ then $\perp$ is an abstract complete orthogonality and $x \leqslant y$ iff $\{y\}^{\perp} \subseteq \{x\}^{\perp}$.

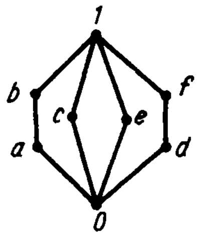
P r o o f . Note that $(\forall x \in P)(x \perp x')$, because $x \leqslant x'' = (x')'$. The relation $\perp$ has the completeness property, since $x' \in \{x\}^{\perp}$ and if $z \in \{x\}^{\perp}$, then $z \leqslant x'$, so $x' = \sup \{x\}^{\perp}$.

Now if $x \perp x$, then $x \leq x'$, therefore $x \in N = \{0\}$. Thus we have $\mathrm{Ker}\perp = N = \{0\}$. Moreover $0 \leq x'$, so $0 \perp x$ for each $x \in P$.

To conclude the proof we need to show that $x \leq y$ iff $\{y\}^{\perp} \subseteq \{x\}^{\perp}$. Assume that $\{y\}^{\perp} \subseteq \{x\}^{\perp}$. We have $y' \in \{y\}^{\perp}$, then $x \perp y'$ and so $x \leq y'' = y$. Now let $x \leq y$. If $z \in \{y\}^{\perp}$, then $y'' \leq z'$. Therefore $x \leq z'$, so $x \perp z$. Thus $z \in \{x\}^{\perp}$.

Observe that every abstract orthogonality in a partially ordered set P with 0 in which a partial order is defined according to Definition 0.3 is a strong non-degenerate orthogonality, but the converse in general fails.

For example, if we consider the following partially ordered set



in which the orthogonality $\perp$ is defined in such way, that $\{a\}^{\perp} = \{0,c,d\} \supseteq \{0,c\} = \{b\}^{\perp}$, $\{d\}^{\perp} = \{0,a,e\} \supseteq \{0,e\} = \{f\}^{\perp}$, $\{c\}^{\perp} = \{0,a,b\}$, $\{e\}^{\perp} = \{0,d,f\}$, $\{0\}^{\perp} = P - \{1\}$, $\{1\}^{\perp} = \emptyset$. Then we get a strong non-degenerate orthogonality which does not have the property "$0 \perp x$ for each $x \in P$".

If a strong non-degenerate orthogonality $\perp$ on a partially ordered set P with 0 satisfies the condition "$0 \perp x$ for each $x \in P$", then this orthogonality satisfies all the conditions of an abstract orthogonality.

2. Boolean orthogonality

It is easy to see that in a partially ordered set P with a least element 0 and with a non-degenerate weak orthogonality $\perp$ if $x \perp y$ then $x \wedge y$ exists in P and $x \wedge y = 0$

for all $x,y \in P$. The converse of this implication in general does not hold. For example, if $X = \{1,2,3,4\}$, then all subsets of X such that each of them contains even number of elements form a partially ordered set with respect to inclusion. If we define an orthogonality by $A \perp B$ iff $A \subseteq B' = X-B$ for all $A,B \subseteq X$, then we have $\{1,3\} \wedge \{1,4\} = \emptyset$ and $\{1,3\} \not\subseteq \{2,3\}^{\perp} = \{1,4\}$. This orthogonality is weak, non-degenerate and complete and the condition $(\forall x,y \in P)$ $(x \wedge y = 0$ implies $x \perp y)$ fails in P.

Now we define the Boolean orthogonality.

D e f i n i t i o n 2.1. Let $(P, \leq, 0)$ be a partially ordered set with the least element 0. An orthogonality $\perp$ on P is called Boolean, if we have

$$(\forall x,y \in P)\ (x \perp y \text{ iff } x \wedge y \text{ exists in } P \text{ and } x \wedge y = 0).$$

R e m a r k 2.2. Every Boolean orthogonality on a partially ordered set with 0 is a non-degenerate weak orthogonality.

R e m a r k 2.3. If we have $x,y \neq 0$, $x \perp y$ in a partially ordered set with a Boolean orthogonality $\perp$, then $x \neq y$ and x and y are incomparable.

The theorems analogous to Theorem 1.1 and Theorem 1.4 hold for Boolean orthogonality.

T h e o r e m 2.4. Let $(P, \leq, 0, \perp)$ be a partially ordered set with 0 and with a complete Boolean orthogonality. Then the mapping $'$: $P \to P$ defined by $x \mapsto x' = \sup \{x\}^{\perp}$ is a non-degenerate weak orthocomplementation. Moreover

$$(2.1) \qquad (\forall x,y \in P)\ (x \perp y \text{ iff } x \leq y'),$$

$$(2.2) \qquad 1 \text{ exists in } P \text{ and } 1 = 0'.$$

P r o o f . By Remark 2.2 we can apply Corollary 1B of Theorem 3.1 of [2].

T h e o r e m 2.5. Let $(P, \leqslant, 0, ')$ be a partially ordered set with 0 and with a weak orthocomplementation. Moreover assume that the following condition holds

(*) $(\forall x, y \in P)$ $(x \leqslant y'$ iff $x \wedge y$ exists in P and $x \wedge y = 0)$.

Then the orthogonality $\perp$ on P defined by $x \perp y$ iff $x \leqslant y'$ is a complete Boolean orthogonality and

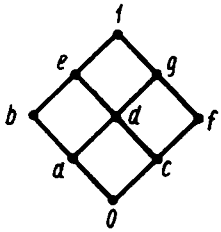
$$(2.3) \qquad (\forall x \in P)\ (\sup \{x\}^{\perp} = x'),$$

$$(2.4) \qquad \{0\}^{\perp} = P \text{ and } P^{\perp} = \{0\}.$$

P r o o f . From the condition (*) it follows that $N = \{x \in P : x \leqslant x'\} = \{0\}$, thus the orthocomplementation is non-degenerate. So by Corollary 1A of Theorem 3.1 of [2] we infer that $\perp$ is a complete non-degenerate weak orthogonality such that (2.3) and (2.4) are satisfied and it is easy to see that this is a Boolean orthogonality.

R e m a r k 2.6. A Boolean orthogonality on a partially ordered set need be neither strong, nor complete.

For example, if we consider a Boolean orthogonality on the following distributive lattice



then we have: $\{a\}^{\perp} = \{0, c, f\} = \{b\}^{\perp}$, $\{c\}^{\perp} = \{0, a, b\} = \{f\}^{\perp}$ and $\{d\}^{\perp} = \{e\}^{\perp} = \{g\}^{\perp} = \{1\}^{\perp} = \{0\}$. This orthogonality is not strong.

In the second case let us consider again the example given at the beginning of the section 2, but now with a Boolean orthogonality. Then we have:

$\{\{1,2\}\}^{\perp} = \{\{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \emptyset\}$ and of course $\sup \{\{1,2\}\}^{\perp} \notin \{\{1,2\}\}^{\perp}$.

R e m a r k 2.7. Every Boolean orthoposet, [3], is defined as a partially ordered set with 0 and with a Boolean orthogonality.

3. Complete Boolean and complete non-degenerate strong orthogonality in lattices

Here we show when a lattice with some kind of an orthogonality is a pseudocomplemented lattice or a Boolean algebra.

D e f i n i t i o n 3.1. A lattice $(B,\vee,\wedge,0)$ with 0 is called a pseudocomplemented lattice if it has the following property:

$$(3.1)\qquad (\forall a \in B)\ (\exists a^* \in B)\ (\forall x \in B)\ (a \wedge x = 0 \text{ iff } x \leq a^*).$$

a^* is called a pseudocomplement of a.

If B is a distributive lattice with 0 and if it satisfies the condition 3.1, then B is called a pseudocomplemented lattice.

T h e o r e m 3.2. If $(B,\vee,\wedge,0,1,\perp)$ is a finite distributive lattice with 0 and 1 and a Boolean orthogonality, then this orthogonality is complete.

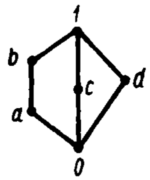
P r o o f . Since the orthogonality is Boolean we conclude from $x \wedge 0 = 0$ for every $x \in B$ that $x \perp 0$, $\{0\}^{\perp} = B$ and $\{x\}^{\perp} \neq \emptyset$. Suppose that there is an element $b \in B$ such that $\sup\{b\}^{\perp}$ does not exist in $\{b\}^{\perp}$. Since $\{0\}^{\perp} = B$ and $\sup\{0\}^{\perp} = 1$ we have $b \neq 0$. If we have $\{b\}^{\perp} = \{0,a\}$ and $0 < a$, then we have $\sup\{b\}^{\perp} = a$ which is a contradiction.

Therefore we can choose elements $a,c \neq 0$, $a,c \in \{b\}^{\perp}$ such that a and c are incomparable. If the elements of $\{b\}^{\perp}$ are parwise comparable, then $\{b\}^{\perp}$ is a finite chain and $\sup\{b\}^{\perp}$ exists in $\{b\}^{\perp}$ which is a contradiction. So if a and c are incomparable then they are also incomparable with b by Remark 2.3.

Thus $\{0,a,b,c,1\}$ forms a sublattice of B which is not distributive, a contradiction. Therefore $\sup\{b\}^{\perp}$ exists in $\{b\}^{\perp}$ for every $b \in B$ and the orthogonality is complete.

R e m a r k 3.3. If the lattice B in Theorem 3.2 is infinite or not distributive, then the thesis of Theorem 3.2 fails. For example, the set of all positive integers in which we define $x \vee y = \text{l.c.m.}(x,y)$ and $x \wedge y = \text{g.c.d.}(x,y)$ forms a complete distributive lattice in which a Boolean orthogonality does not have the completeness property.

In the second case we consider the following lattice B



with a Boolean orthogonality. Then $\{d\}^{\perp} = \{0,a,b,c\}$ and $\sup \{d\}^{\perp} \notin \{d\}^{\perp}$.

T h e o r e m 3.4. If $(B,\vee,\wedge,0,\perp)$ is a lattice with 0 and a complete Boolean orthogonality $\perp$, then $(B,\vee,\wedge,0,{}^*)$ is a pseudocomplemented lattice in which $x^* = \sup \{x\}^{\perp}$ for every $x \in B$.

P r o o f . Observe that $\sup \{x\}^{\perp} = \sup \{p \in B : x \perp p\} = \sup \{p \in B : x \wedge p = 0\} \in \{x\}^{\perp}$, so $x^* = \sup \{x\}^{\perp}$ for each $x \in B$.

T h e o r e m 3.5. If $(B,\vee,\wedge,0,{}^*)$ is a pseudocomplemented lattice then the orthogonality defined on B by

$$(\forall x,y \in B)\ (x \perp y \ \text{ iff } \ x \leqslant y^*)$$

is a complete Boolean orthogonality.

P r o o f . We have $x \perp y$ iff $x \leqslant y^*$ and iff $x \wedge y = 0$. But by the definition of * x^* is the greatest element disjoint with x. So we get: $x^* = \sup \{p \in B : p \wedge x = 0\} = \sup \{p \in B : p \leqslant x^*\} = \sup \{p \in B : p \perp x\} = \sup \{x\}^{\perp} \in \{x\}^{\perp}$.

C o r o l l a r y 3.6. If $(B,\vee,\wedge,0,1,\perp)$ is a finite distributive lattice with 0 and 1 and with a Boolean orthogonality, then $(B,\vee,\wedge,0,1,{}^*)$ is a pseudocomplemented distributive lattice in which $x^* = \sup \{x\}^{\perp}$ for all $x \in B$.

The theorems analogous to the theorems 3.4 and 3.5 hold also for Boolean algebras.

T h e o r e m 3.7. If $(B,\vee,\wedge,0,\perp)$ is a distributive lattice with 0 and with a complete non-degenerate strong orthogonality, then there exists a greatest element 1 in B and $(B,\vee,\wedge,0,1,')$ is a Boolean algebra in which $x' = \sup\{x\}^{\perp}$ for each $x \in B$.

P r o o f . We only need to show that 1 exists in B and $(\forall x \in B)\ (\exists x' \in B)\ (x \vee x' = 1$ and $x \wedge x' = 0)$. But this follows from Corollary 2B of Theorem 3.1 [2].

T h e o r e m 3.8. If $(B,\vee,\wedge,0,1,')$ is a Boolean algebra, then the orthogonality $\perp$ defined on B by

$$(\forall x,y \in B)\ (x \perp y \text{ iff } x \leqslant y')$$

is a complete non-degenerate strong orthogonality.

P r o o f . In a Boolean algebra B we have: $x = x''$ for each $x \in B$ and $x \leqslant y$ iff $y' \leqslant x'$ for all $x,y \in B$. Moreover if $x \leqslant x'$, then $x = x \wedge x'$. But $x \wedge x' = 0$, so $x = 0$ and $N = \{0\}$. Now we can apply Corollary 2A of Theorem 3.1 [2], and we get the thesis.

R e m a r k 3.9. $(L,\leqslant,')$ is a lattice with a strong orthocomplementation if and only if $(L,\leqslant,')$ is a polarity lattice, [11].

4. A characterization of partially ordered sets with a complete weak orthogonality and with a full set of states

A partially ordered set with a complete weak orthogonality admitting a full set of states can be represented as a set of functions satisfying some properties.

Let M be an arbitrary set and let L be a set of functions (not necessarily all) from M into $[0,1]$ such that

(i) $0 \in L$ (the zero function),

(ii) $(\forall f \in L)$ (there exists $\sup\{g \in L : f+g \leqslant 1\}$ in $\{g \in L : f+g \leqslant 1\}$).

L e m m a 4.1. L is a partially ordered set with respect to the natural order of functions; L contains the least element 0 and if we define an orthogonality $\perp$ on L by $f \perp g$ iff $f+g \leqslant 1$, then this orthogonality is weak and complete.

P r o o f . If we define $\leqslant$ on L by $f \leqslant g$ iff $(\forall x \in M)\ (f(x) \leqslant g(x))$ then it is easy to see that it is a partial order on L. Of course $\perp$ is symmetric and if $f \leqslant g$ and $g \perp h$, i.e. $g+h \leqslant 1$, then $f+h \leqslant 1$, so $f \perp h$. By the condition (ii) we have: $(\forall f \in L)$ (there exists $\sup \{g \in L : f+g \leqslant 1\} = \sup \{g \in L : f \perp g\} = \sup \{f\}^{\perp}$ in $\{f\}^{\perp}$), thus the orthogonality is complete.

D e f i n i t i o n 4.2. Let $(P, \leqslant, \perp, 0)$ be a partially ordered set with 0 and with a weak orthogonality. A mapping $m : P \to [0,1]$ is called a state on P, if

(4.1) $$(\forall a,b \in P)\ (a \leqslant b \text{ implies } m(a) \leqslant m(b)),$$

(4.2) $$(\forall a,b \in P)\ (a \perp b \text{ implies } m(a) + m(b) \leqslant 1),$$

(4.3) $$m(0) = 0.$$

D e f i n i t i o n 4.3. Let $(P, \leqslant, \perp, 0)$ be a partially ordered set with 0 and with a weak orthogonality. We say that a set M of states on P is full if

(4.4) $$(\forall a,b \in P)\left[\text{if } (\forall m \in M)\ (m(a) \leqslant m(b)), \text{ then } a \leqslant b\right],$$

(4.5) $$(\forall a,b \in P)\left[\text{if } (\forall m \in M)\ (m(a) + m(b) \leqslant 1), \text{ then } a \perp b\right].$$

D e f i n i t i o n 4.4. Let $(P, \leqslant, \perp)$ and $(P_1, \leqslant_1, \perp_1)$ be partially ordered sets with a weak orthogonality. We say that P and P_1 are isomorphic, if there exists a mapping $i : P \xrightarrow[\text{onto}]{1-1} P_1$ such that

(4.6) $$(\forall x,y \in P)\ (x \leqslant y \text{ iff } i(x) \leqslant_1 i(y)),$$

(4.7) $$(\forall x,y \in P)\ (x \perp y \text{ iff } i(x) \perp_1 i(y)).$$

D e f i n i t i o n 4.5. Let S be a set of functions from X into Y. and let $S' = \{\bar{x} : x \in X\}$ where $\bar{x}$ is a function from S into Y induced by $x \in X$, such that $\bar{x}(f) = f(x)$ for all $f \in S$. We call S' the dual of S.

T h e o r e m 4.6. Let $(P, \leqslant, \perp, 0)$ be a partially ordered set with 0 and with a complete weak orthogonality. Assume that P admits a full set of states M and let M' be the dual of M. Then M' is a partially ordered set with respect to the natural order of real functions

$$(\forall a,b \in P)\ \left[\bar{a} \leqslant_1 \bar{b} \ \text{ iff } \ (\forall m \in M)(\bar{a}(m) \leqslant \bar{b}(m))\right]$$

with a complete orthogonality $\perp_1$ defined by

$$(\forall a,b \in P)\ \left[\bar{a} \perp_1 \bar{b} \ \text{ iff } \ (\forall m \in M)(\bar{a}(m) + \bar{b}(m) \leqslant 1)\right] .$$

Moreover $(M', \leqslant_1, \perp_1, 0)$ and $(P, \leqslant, \perp, 0)$ are isomorphic.

P r o o f . Since M is a full set of states on P, we obtain:

(a) $\bar{a} \leqslant_1 \bar{b} \Longleftrightarrow (\forall m \in M)\ (\bar{a}(m) \leqslant \bar{b}(m)) \Longleftrightarrow (\forall m \in M)(m(a) \leqslant m(b))$
$\Longleftrightarrow a \leqslant b$,

(b) $\bar{a} \perp_1 \bar{b} \Longleftrightarrow (\forall m \in M)\ (\bar{a}(m) + \bar{b}(m) \leqslant 1) \Longleftrightarrow$
$\Longleftrightarrow (\forall m \in M)(m(a) + m(b) \leqslant 1) \Longleftrightarrow a \perp b$.

By definition the relation $\leqslant_1$ is a partial order on M'. It is also easy to see that $\perp_1$ is symmetric and if $\bar{a} \leqslant_1 \bar{b}$ and $\bar{b} \perp_1 \bar{c}$ then by (a) and (b) we get $\bar{a} \perp_1 \bar{c}$. Now we will show that there exists $\sup \{\bar{a}\}^{\perp_1}$ in $\{\bar{a}\}^{\perp_1}$ for every $\bar{a} \in M'$. Let $\bar{a}^*$ be a function induced by $a^* = \sup \{a\}^{\perp}$ (as the orthogonality $\perp$ is complete, such a^* exists for each $a \in P$). Since $a \perp a^*$, so by (b) we obtain $\bar{a} \perp_1 \bar{a}^*$. If $\bar{a} \perp_1 \bar{b}$, then by (b) we have $a \perp b$. But by the completeness of the orthogonality $\perp$ we get $b \leqslant a^*$, so again we have by (b) $\bar{b} \leqslant_1 \bar{a}^*$. Thus there exists $\sup \{\bar{a}\}^{\perp_1}$ in $\{\bar{a}\}^{\perp_1}$ for every $\bar{a} \in M'$ and the orthogonality $\perp_1$ is complete.

Finally, it is easy to see that the mapping $i : P \to M'$ defined by $i(p) = \bar{p}$ for all $p \in P$ is a natural isomorphism between $(P, \leqslant, \perp, 0)$ and $(M', \leqslant_1, \perp_1, 0)$.

The following theorem provides a full characterization of partially ordered sets with a complete weak orthogonality and with a full set of states:

T h e o r e m 4.7. (1) If L is a set of functions satisfying the conditions (i) and (ii) at the beginning of this section, then L is a partially ordered set with 0 and with a complete weak orthogonality.

Moreover: each element $m \in M$ induces a state $\bar{m}$ on L such that $\bar{m}(f) = f(m)$ for every $f \in L$ and $M' = \{\bar{m} : m \in M\}$ is a full set of states on L.

(2) On the other hand, if $(P, \leq, 0, \perp)$ is a partially ordered set with 0 and with a complete weak orthogonality, and if P admits a full set of states M, then the dual of M satisfies (i) and (ii) and M' is isomorphic to P.

P r o o f (1). The first part of the theorem follows from Lemma 4.1. By the definition of $\bar{m}$ we see that each of $\bar{m}$ is a state on L and since

$$(\forall m \in M)(\bar{m}(f) \leq \bar{m}(g)) \Longleftrightarrow (\forall m \in M)(f(m) \leq f(g)) \Longleftrightarrow f \leq g,$$

$$(\forall m \in M)(\bar{m}(f) + \bar{m}(g) \leq 1) \Longleftrightarrow (\forall m \in M)(f(m)+g(m) \leq 1) \Longleftrightarrow f \perp g$$

we infer that M' is a full set of states on L.

(2) This follows from Theorem 4.6. In fact the condition (ii) is satisfied, because we have shown in Theorem 4.6 that the orthogonality $\perp_1$ is complete. The least element in M' is the zero function induced by the element $0 \in P$, i.e. such that $\bar{0}(m) = m(0) = 0$ for all $m \in M$. So the condition (i) also holds.

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Received December 9, 1978.