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ON SOME SYSTEM OF INTEGRAL EQUATIONS WITH DEVIATED ARGUMENT

Introduction

Let $\alpha \in \langle 0;1)$. Let us consider the set of all continuous and increasing functions $\varphi : \langle 0;\infty) \to \langle 0;\infty)$ for which $\varphi(0) = 0$ holds. Furthermore, let the functions satisfy the following inequalities:

(1) $$\bigvee_{\nu\geq 1} \bigvee_{\sigma\geq 0} \varphi(\nu\sigma) \leq \nu\varphi(\sigma),$$

(2) $$\exists_{A_1>0} \int_0^{\infty} \frac{\varphi(\sigma)d\sigma}{\sigma(1+\sigma)^{1-\alpha}} \leq A_1,$$

(3) $$\exists_{A_2>0} \bigvee_{s\geq 0} \int_0^{s} \frac{\varphi(\sigma)}{\sigma}\, d\sigma \leq A_2\varphi(s),$$

(4) $$\exists_{A_3>0} \bigvee_{0<a\leq b} \int_a^{b} \frac{\varphi(\sigma)}{\sigma^2}\, d\sigma \leq A_3 \frac{\varphi(a)}{a}.$$

This set is denoted by Φ^*.

Let S and S_s be singular integral operators which are defined by the integral in the sense of Cauchy's principal value, respectively:

(5) $$(Sh)(t) = \int_L \frac{h(\tau)}{\tau - t}\, d\tau,$$

(6) $$(S_s h)(t) = \int_L \frac{h(\tau)}{\tau - s(t)}\, d\tau,$$

where L is a smooth arc with different end-points a and b. Further, $s : L \rightarrowtail\!\!\!\rightarrow L$, $s(a) = a$, $s(b) = b$ and

(7) $$\underset{t,t_1 \in L}{\forall} \quad 0 < m_s \leqslant \left| \frac{s(t)-s(t_1)}{t-t_1} \right| \leqslant M_s.$$

Let $M \in (0;\infty)$. In the papers [1], [2] and [3] on $L_0 = L - \{a,b\}$ there was defined the class $\mathcal{H}(M,\alpha,\varphi)$. There was also shown that the conditions $\varphi \in \Phi^*$ and $\alpha \in (0;1)$ implied the relation

(8) $$S : \mathcal{H}(M,\alpha,\varphi) \to \mathcal{H}(C_1 M,\alpha,\varphi),$$

where C_1 is some positive constant. On the other hand in [4] there was proved that if $\varphi \in \Phi^*$ and $\alpha \in (0;1)$, then

(9) $$S_s : \mathcal{H}(M,\alpha,\varphi) \to \mathcal{H}(C_2 M,\alpha,\varphi)$$

where C_2 is some positive constant.

The Problem

Let us consider the system of singular integral equations the cardinality is the same as that of a set B

(10) $$f_b(t) = h_b(t) + \lambda \int_L \frac{K_b\left[\tau, \{f_b(\tau)\}_{b\in B}, \{f_b[s_{\alpha_1(b)}(\tau)]\}_{b\in B}\right]}{\tau - t}\, d\tau + \mu \int_L \frac{K_b^*\left[\tau, \{f_b(\tau)\}_{b\in B}, \{f_b[s_{\alpha_2(b)}(\tau)]\}_{b\in B}\right]}{\tau - s_{\alpha_3(b)}(t)}\, d\tau, \quad b \in B,$$

there $\{f_b\}_{b\in B}$ is a system of unknown functions.

We make the following assumptions:

1. $\varphi \in \Phi^*$ and $\alpha \in (0;1)$.
2. Functions $h_b : L_0 \to C$ belong to the class $\mathcal{H}(M_h, \alpha, \varphi)$ for each $b \in B$, where constant $M_h > 0$.
3. Let $C_b = C$ for each $b \in B$. For every $b \in B$ functions $K_b : L_0 \times \{C_b\}_{b\in B} \times \{C_b\}_{b\in B} \to C$ fulfil over their domains the following inequalities:

$$(11)\qquad \left|K_b\left[t, \{x_b\}_{b\in B}\, \{y_b\}_{b\in B}\right]\right| \leq \frac{M_k}{|t-t^*|^{\alpha}} + $$

$$+ N_k \left(\sup_{b\in B} |x_b| + \sup_{b\in B} |y_b|\right),$$

$$(12)\quad \left|K_b\left[t, \{x_b\}_{b\in B}\, \{y_b\}_{b\in B}\right] - K_b\left[t_1, \{x'_b\}_{b\in B}\, \{y'_b\}_{b\in B}\right]\right| \leq$$

$$\leq \frac{M_k}{|t-t^*|^{\alpha}}\, \varphi\left(\frac{|t-t_1|}{|t-t^*|}\right) + N_k \left(\sup_{b\in B} |x_b - x'_b| + \sup_{b\in B} |y_b - y'_b|\right),$$

where t^* denotes this end-point, either a or b, of the arc L, for which the condition le $\overset{\frown}{tt^*} = \min(\text{le } \overset{\frown}{at}, \text{le } \overset{\frown}{tb})$ (symbol le means length) holds.

4. For each $b \in B$ functions $K_b^* : L_0 \times \{C_b\}_{b\in B} \times \{C_b\}_{b\in B} \to C$ satisfy over their domains the following inequalities:

$$(13)\qquad \left|K^*_b\left[t, \{x_b\}_{b\in B}\, \{y_b\}_{b\in B}\right]\right| \leq \frac{M^*_k}{|t-t^*|^{\alpha}} + $$

$$+ N^*_k \left(\sup_{b\in B} |x_b| + \sup_{b\in B} |y_b|\right),$$

$$(14)\quad \left|K^*_b\left[t, \{x_b\}_{b\in B}\, \{y_b\}_{b\in B}\right] - K^*_b\left[t_1, \{x'_b\}_{b\in B}\, \{y'_b\}_{b\in B}\right]\right. \leq$$

$$\leq \frac{M^*_k}{|t-t^*|^{\alpha}}\, \varphi\left(\frac{|t-t_1|}{|t-t^*|}\right) + N^*_k \left(\sup_{b\in B} |x_b - x'_b| + \sup_{b\in B} |y_b - y'_b|\right).$$

5. For each $b \in B$ functions $s_b : L \rightarrowtail\!\!\!\twoheadrightarrow L$ fulfil the conditions $s_b(a) = a$, $s_b(b) = b$ and the following inequality

$$\forall_{t, t_1 \in L} \quad 0 < m_s \leqslant \left| \frac{s_b(t) - s_b(t_1)}{t - t_1} \right| \leqslant M_s . \tag{15}$$

6. Functions α_i map B into B, $i = 1,2,3$.

Now, we shall show the existence of a solution of the system (10) in the class $\mathfrak{K}(M,\alpha,\varphi)$.

The problem stated above is a generalization of results obtained in the paper [5] and it will be solved by means of topological methods. Let, for each $b \in A$, Λ_b be the linear space whose points are complex functions $f_b : L_0 \to C$. These functions are continuous and on L_0 they satisfy the following inequalities:

$$\sup_{t \in L_0} \left[|t - t^*|^{1+\alpha} |f_b(t)| \right] < \infty . \tag{16}$$

The norm $\|f_b\|$ we define by the formula

$$\|f_b\| = \sup_{t \in L_0} \left[|t - t^*|^{1+\alpha} |f_b(t)| \right]. \tag{17}$$

It is the known fact [6], that every space Λ_b defined as above is a locally convex Hausdorff space. Furthermore, it is evident [6] that the product $\Lambda = \prod_{b \in B} \Lambda_b$ with the Tichonov topology is also a locally convex Hausdorff space. In the space Λ we consider the set

$$Z(\varkappa) = \left\{ \{f_b\}_{b \in B} \in \Lambda : \forall_{b \in B} \; f_b \in \mathfrak{K}(\varkappa, \alpha, \varphi) \right\}, \tag{18}$$

where $\varkappa$ is some positive constant. It will be evaluated further. The set $Z(\varkappa)$ is nonempty, convex and compact ([1], [2], [5]).

We define the following operation

(19) $$\Psi : Z(\varkappa) \rightarrow \Lambda$$

on the set $Z(\varkappa)$. This operation maps every point of the mentioned set $Z(\varkappa)$ to a point of the space Λ i.e.

$$Z(\varkappa) \ni \{f_b\}_{b\in B} \rightarrow \Psi\left[\{f_b\}_{b\in B}\right] = \{\Psi_b\}_{b\in B} \in \Lambda,$$

where

(20) $$\Psi_b = h_b + \lambda S\left[K_b \circ (\mathrm{id}, \{f_b\}_{b\in B}, \{f_b \circ s_{\alpha_1(b)}\}_{b\in B})\right] + \\ + \mu S_{s_{\alpha_3(b)}}\left[K_b^* \circ (\mathrm{id}, \{f_b\}_{b\in B}, \{f_b \circ s_{\alpha_2(b)}\}_{b\in B})\right]$$

for every $b \in B$.

Now, we shall find a sufficient condition for the range of the operation Ψ to satisfy the following inclusion: $R\Psi \subset Z(\varkappa)$. It is easy to show that there exists a positive constant H such that for every $b \in B$ and $i = 1,2$ the implication

(21) $$f_b \in \mathcal{K}(\varkappa,\alpha,\varphi) \Longrightarrow f_b \circ s_{\alpha_i(b)} \in \mathcal{K}(H\varkappa,\alpha,\varphi)$$

holds. Further, according to (8) and (9) and using the assumptions 1 - 6 we obtain the validity of the following implication

(22) $$f_b \in \mathcal{K}(\varkappa,\alpha,\varphi) \Longrightarrow \Psi_b \in \mathcal{K}(M_h + |\lambda|\, C_1\left[M_k + N_k(\varkappa + H\varkappa)\right] + \\ + |\mu|\, C_2\left[M_k^* + N_k^*(\varkappa + H\varkappa)\right], \alpha, \varphi)$$

for each $b \in B$. Then, if the inequality

(23) $$M_h + |\lambda| \; C_1\left[M_k+N_k(\varkappa+H\varkappa)\right] + |\mu| \; C_2\left[M_k^*+N_k^*(\varkappa+ H\varkappa)\right] \leq \varkappa$$

holds, then the operation described by (19) maps the set $Z(\varkappa)$ into itself. Let, for the constants N_k, and N_k^* the following condition

(24) $$\left(|\lambda| C_1 N_k + |\mu| C_2 N_k^*\right) (1+H) < 1$$

holds. Now, if

(25) $$\varkappa = \frac{M_h + |\lambda| \; C_1 M_k + |\mu| C_2 M_k^*}{1 - \left(|\lambda| \; C_1 N_k + |\mu| C_2 N_k^*\right)(1+H)},$$

it is easy to see that the operation (19) maps the set $Z(\varkappa)$ into itself.

Making use of the method described in [5] one can prove that the operation (19) is continuous. Further, due to the Schauder - Tichonov theorem [6], we can formulate the following theorem one.

T h e o r e m. If the assumptions 1 - 6 are fulfilled, the inequality (24) holds and $\varkappa$ is given by the formula (25), then the system of integral equations (10) has at least one solution in the class $\mathcal{K}(\varkappa,\alpha,\psi)$.

REFERENCES

[1] W. L e k s i ń s k i : On some class $\mathcal{K}(M,\alpha,\psi)$ of functions defined on a non-closed arc, Demonstratio Math. 4 (1972) 51-61.

[2] W. L e k s i ń s k i, W. Ż a k o w s k i : On some classes of non-bounded functions, Demonstratio Math. 4 (1972) 125-131.

[3] W. L e k s i ń s k i , W. Ż a k o w s k i : A theorem on preserving the class $\mathcal{K}(M,\alpha,\psi)$ by a singular in-

tegral in the sense of Cauchy's principal value. Demonstratio Math. 5 (1973) 139-145.

[4] G. D e c e w i c z , W. L e k s i ń s k i : On some class preserved by a singular integral with a deviated argument, Demonstratio Math. 11 (1978) 1065-1070.

[5] W. L e k s i ń s k i , W. Ż a k o w s k i : Investigation of a certain system of singular integral equations of arbitrary power. Annales Polonici Mathematici 29 (1975) 373-378.

[6] R.E. E d w a r d s : Functional Analysis. Theory and Applications, New York 1965.

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