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A PROPERTY OF SERIES OF QUASI-DIFFUSION PROCESSES

It is well known that the sum (finite or infinite) of diffusion processes may not be a diffusion process or more general the sum of Markov processes may not be a Markov process.

In [1], [2] conditions have been given under which the class of quasi-diffusion processes is closed with respect to the operation of the addition of a finite number of terms. In this paper we are going to give conditions under which the series of quasi-diffusion processes is as well a quasi-diffusion process.

Let X_t, where $t \in T_0 = (0,T)$, be a stochastic process with values in R^1. Let $0 \le t_1 < \ldots < t_{n+1} < T$, $\mathbf{t}_n = (t_1,\ldots,t_n)$, $\mathbf{x}_n = (x_1,\ldots,x_n)$, $\mathbf{x}_n \in R^n$ and A is a Borel set in R^1.

We denote

$$P(X_{t_{n+1}} \in A | X_{t_1},\ldots,X_{t_n}) = P^{(n)}(\mathbf{t}_n, X_{t_1},\ldots,X_{t_n}; t_{n+1}, A).$$

We say that X_t is a quasi-diffusion process if for every n, $\mathbf{t}_n$, $\mathbf{x}_n$ and $\varepsilon > 0$ we have

$$(1) \qquad \lim_{\Delta_n \to 0} \frac{1}{\Delta_n} P^{(n)}(\mathbf{t}_n, \mathbf{x}_n; t_{n+1}, V_\varepsilon(x_n)) = 0$$

(2) $$\lim_{\Delta_n \to 0} \frac{1}{\Delta_n} \int_{|x_{n+1}-x_n|<\varepsilon} (x_{n+1} - x_n)^k \, P^{(n)}(t_n, x_n; t_{n+1}, dx_{n+1}) =$$

$$= a_k(t_n, x_n), \quad k = 1,2,$$

where $\Delta_n = t_{n+1} - t_n$,

$$V_\varepsilon(x_n) = \left\{ x_{n+1} : |x_n - x_{n+1}| \geq \varepsilon \right\}$$

and $\lim\limits_{\Delta_n \to 0}$ denotes $\lim\limits_{t_{n+1} \to t_n}$ (i.e. t_n is fixed and $t_{n+1} \to t_n$).

In this paper, instead of (1), we shall consider a stronger condition, namely we assume that

(3) $$\lim_{\Delta_n \to 0} \frac{1}{\Delta_n} \int_{V_\varepsilon(x_n)} (x_{n+1} - x_n)^2 \, P^{(n)}(t_n, x_n; t_{n+1}, dx_{n+1}) = 0.$$

Now we shall assume that

$$E\, X_t = 0,$$

and that the following expectations exist

$$K(t_1, t_2) = E(X_{t_1} X_{t_2})$$

(4) $$\mu_{n+1} = \mu(t_{n+1}, X_{t_1}, \ldots, X_{t_n}) = E(X_{t_{n+1}} \mid X_{t_1}, \ldots, X_{t_n}),$$

(5) $$\sigma^2_{n+1} = \sigma^2(t_{n+1}, X_{t_1}, \ldots, X_{t_n}) = E\left[(X_{t_{n+1}} - \mu_{n+1})^2 \,\middle|\, X_{t_1}, \ldots, X_{t_n} \right].$$

We admit also that the matrix $\left[K(t_i, t_j)\right]_{1 \leq i, j \leq n}$ is positive definite.

If we admit (3) and the existence of (4), (5) then it is easy to see that (2) take the following form

(6) $$\lim_{\Delta_n \to 0} \frac{1}{\Delta_n} \left[\mu(t_{n+1}, x_n) - x_n\right] = a_1(t_n, x_n)$$

and

(7) $$\lim_{\Delta_n \to 0} \frac{1}{\Delta_n} \sigma^2(t_{n+1}, x_n) = a_2(t_n, x_n).$$

We shall use the following fact.

P r o p e r t y 1 [3]. Let X_t be a gaussian process. Process X_t is a quasi-diffusion process if the following limits exist

(8) $$\lim_{\Delta_n \to 0} \frac{1}{\Delta_n} \left[K(t_i, t_{n+1}) - K(t_i, t_n)\right] = \varkappa_{in}, \quad 1 \leqslant i \leqslant n$$

and

(9) $$\lim_{\Delta_n \to 0} \frac{1}{\Delta_n} \left[K(t_{n+1}, t_{n+1}) - K(t_n, t_{n+1})\right] = \varkappa_n.$$

We shall write

(10) $$Y_t^{(N)} \xrightarrow{\ 1\ } Y_t$$

if the sequence of the r-dimensional (r=1,2,...) distribution functions of $Y_t^{(N)}$ converges, when $N \to \infty$, to the r-dimensional distribution function of Y_t.

We shall use the following theorems arising from the well known theorems for random variables with values in R^r [4], [5]:

Theorem 1. Let $X_t^{(1)}, X_t^{(2)}, \ldots$ be independent stochastic processes with identical r-dimensional distributions ($r = 1,2,\ldots$). If $E\,X_t^{(k)} = 0$, $E\left(X_t^{(k)}\right)^2 < \infty$, $k \geq 1$, and

$$(11) \qquad Y_t^{(N)} = \frac{1}{N}\sum_{i=1}^{N} X_t^{(i)}$$

then Y_t, given by (10), is a gaussian process with the covariance function

$$(12) \qquad K_Y(t_1,t_2) = K(t_1,t_2) = E\left(X_{t_1}^{(1)}\,X_{t_2}^{(1)}\right).$$

Theorem 2. Let $X_t^{(1)}, X_t^{(2)}, \ldots$ be independent stochastic processes with $E\,X_t^{(k)} = 0$, $k \geq 1$ and with the covariance functions

$$K^{(i)}(t_1,t_2) = E\left(X_{t_1}^{(i)}\,X_{t_2}^{(i)}\right)$$

such that there exist limits

$$(13) \qquad \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N} K^{(i)}(t_1,t_2) = K(t_1,t_2)$$

and K is positive definite. If, for every $r \geq 1$, the cumulative distribution functions

$$F^{(i)}(\mathbf{x}_r) = P\left(X_{t_1}^{(i)} < x_1, \ldots, X_{t_r}^{(i)} < x_r\right)$$

are such that for every $\varepsilon > 0$ we have

$$(14) \quad \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\int_{x_1^2+\ldots+x_2^2>\varepsilon N}(x_1^2+\ldots+x_r^2)\,dF^{(i)}(\mathbf{x}_r) = 0,$$

then Y_t given by (10), as a limit process of (11), is a gaussian process with covariance function K.

The following two properties follows from Theorem 1, Theorem 2 and Property 1.

P r o p e r t y 2. If $X_t^{(1)}, X_t^{(2)}, \ldots$ are independent quasi-diffusion processes with identical r-dimensional distributions ($r = 1,2,\ldots$) and with the covariance function K satisfying (8) and (9), then Y_t given by (10), as a limit of (11) is a quasi-diffusion process.

P r o p e r t y 3. If $X_t^{(1)}, X_t^{(2)}, \ldots$ are independent quasi-diffusion processes satisfying (13), (14) and if K given by (13), is positive definite and satisfies (8), (9), then Y_t given by (10), as a limit of (11) is a quasi-diffusion process.

In examples we shall use the following

P r o p e r t y 4. If X_t is a quasi-diffusion process $Y_t = g(X_t)$, $g \in C^2$, there exist a set $(x_0,\ldots,x_M)$ such that $-\infty = x_0 < x_1 < \ldots < x_M = \infty$ and g is strictly monotone in (x_i, x_{i+1}) for $i = 0,\ldots,M-1$, then Y_t is a quasi-diffusion process.

Now we shall give examples of series such that the terms and the sum of the series are quasi-diffusion processes.

E x a m p l e 1. If $V_t^{(1)}, V_t^{(2)}, \ldots$ are independent gaussian quasi-diffusion processes with identical r-dimensional distributions ($r = 1,2,\ldots$)

$$X_t^{(i)} = \left(V_t^{(i)}\right)^l ,$$

where l is a positive integer, then Y_t given by (10) and (11) is a quasi-diffusion process.

In virtue of Property 4 $X_t^{(i)}$ are quasi-diffusion processes. Thus, taking into account Property 1 and Theorem 1 it suffices to show that the covariance function K_X of $X_t^{(i)}$ satisfies (8) and (9).

Let us denote

$$k_{ij} = E\left(V_{t_i}^{(1)} V_{t_j}^{(1)}\right) = R(t_i, t_j),$$

$$m_k = \frac{1}{\sqrt{2\pi}} \int_{R^1} x^k \exp\left(-\frac{x^2}{2}\right) dx.$$

Then we have

$$(15)\quad K_X(t_i,t_j) = \frac{1}{2\pi\sqrt{k_{ii}k_{jj} - k_{ij}^2}} \times$$

$$\times \int_{R^2} x_i^l x_j^l \exp\left[-\frac{1}{2(k_{ii}k_{jj}-k_{ij}^2)}\left(x_i^2 k_{jj} - 2k_{ij}x_i x_j + x_j^2 k_{ii}\right)\right] dx_i dx_j =$$

$$= \sum_{s=0}^{l} \binom{l}{s} (k_{ii}k_{jj} - k_{ij}^2)^{\frac{s}{2}} k_{ij}^{l-s} m_s \, m_{2l-s},$$

where evidently $m_{2n+1} = 0$ for every positive integer n.

Now let us notice, that from (8) and (9), it follows that there the following limits exist

$$(16)\qquad \lim_{\Delta_n \to 0} \frac{1}{\Delta_n} \left[K(t_{n+1}, t_{n+1}) - K(t_n, t_n)\right],$$

$$(17)\qquad \lim_{\Delta_n \to 0} \frac{1}{\Delta_n} \left[K(t_n, t_n)\, K(t_{n+1}, t_{n+1}) - K^2(t_n, t_{n+1})\right].$$

Taking into account (15) we can express the left-hand sides of (8) and (9) in the following form

$$\lim_{\Delta_n \to 0} \frac{1}{\Delta_n} \left[K_X(t_i,t_{n+1}) - K_X(t_i,t_n)\right] =$$

$$= \lim_{\Delta_n \to 0} \frac{1}{\Delta_n} \sum_{s=0}^{l} \binom{l}{s} m_s \, m_{2l-s} \times$$

$$\times\left[(k_{ii}k_{n+1,n+1}-k_{i,n+1}^2)^{\frac{s}{2}}\,k_{i,n+1}^{l-s}-(k_{ii}k_{nn}-k_{in}^2)^{\frac{s}{2}}\,k_{in}^{l-s}\right]=$$

$$=\lim_{\Delta_n\to 0}\sum_{s=0}^{l}\binom{l}{s} m_s\, m_{2l-s}\,\times$$

$$\times\left\{(k_{ii}k_{n+1,n+1}-k_{i,n+1}^2)^{\frac{s}{2}}\,\frac{k_{i,n+1}^{l-s}-k_{in}^{l-s}}{\Delta_n}+\right.$$

$$\left.+\,k_{in}^{l-s}\,\frac{1}{\Delta_n}\left[(k_{ii}k_{n+1,n+1}-k_{i,n+1}^2)^{\frac{s}{2}}-(k_{ii}k_{nn}-k_{in}^2)^{\frac{s}{2}}\right]\right\},$$

$$\lim_{\Delta_n\to 0}\frac{1}{\Delta_n}\left[K_X(t_{n+1},t_{n+1})-K_X(t_n,t_{n+1})\right]=\lim_{\Delta_n\to 0}\frac{1}{\Delta_n}\left[m_{2l}k_{n+1,n+1}^{l}-\right.$$

$$\left.-\sum_{s=0}^{l}\binom{l}{s} m_s m_{2l-s}(k_{nn}k_{n+1,n+1}-k_{n,n+1}^2)^{\frac{s}{2}}\,k_{n,n+1}^{l-s}\right]=$$

$$=\lim_{\Delta_n\to 0}\frac{1}{\Delta_n}\left[m_{2l}k_{n+1,n+1}^{l}-m_{2l}(k_{nn}k_{n+1,n+1}-k_{n,n+1}^2)k_{n,n+1}^{l}\right]+$$

$$+\sum_{s=0}^{l}\binom{l}{s} m_s m_{2l-s}\lim_{\Delta_n\to 0}\frac{1}{\Delta_n}\;k_{n,n+1}^{l-s}(k_{nn}k_{n+1,n+1}-k_{n,n+1}^2)^{\frac{s}{2}}\,.$$

Taking into account the fact that the covariance function K satisfies (8) and (9) and consequently that there exist limits (16) and (17), it is easy to see that there exist limits of the right sides of (18) and (19). In other words, the relations (8) and (9) hold for K_X.

E x a m p l e 2. Let $V_t^{(1)}, V_t^{(2)},\dots$ be independent Wiener processes, $E\left(V_t^{(i)}\right)^2=\sigma_i^2 t$

$$X_t^{(i)}=\left(V_t^{(i)}\right)^l\,,$$

where l is a positive integer. Suppose that there exist a constant σ^2 such that

$$\sigma_i^2 \leq \sigma^2 \quad \text{for} \quad i = 1,2,\ldots$$

If the following limit exists

$$(18) \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \sigma_i^{2l} ,$$

then Y_t given by (10) and (11) is a quasi-diffusion process. It is sufficient to show that the conditions (13) and (14) hold.

It is evident by (15), that we have

$$K^{(i)}(t_1,t_2) = E\left(X_{t_1}^{(i)} X_{t_2}^{(i)}\right) =$$

$$= \sigma_i^{2l} \sum_{s=0}^{l} \binom{l}{s} m_s m_{2l-s} t_1^{2l-\frac{s}{2}} (t_2-t_1)^{\frac{s}{2}} .$$

Thus, in virtue of (18), we obtain that

$$\frac{1}{N} \sum_{i=1}^{N} K^{(i)}(t_1,t_2) =$$

$$= \sum_{s=0}^{l} \binom{l}{s} m_s m_{2l-s} t_1^{2l-\frac{s}{2}} (t_2-t_1)^{\frac{s}{2}} \frac{1}{N} \sum_{i=1}^{N} \sigma_i^{2l}$$

has a limit when $N \to \infty$ and this limit is a function positive definite.

Now we are going to show that (14) holds. Let l be a non-even number, $F^{(i)}$ be c.d.f. of $X^{(i)}$. From the properties of integrals it follows that

$$\frac{1}{N}\sum_{i=1}^{N} \int\limits_{x_1^2+\ldots+x_r^2>\varepsilon N} (x_1^2+\ldots+x_r^2)\, d\,F^{(i)}(\mathbf{x}_r) =$$

$$= \frac{1}{N}\sum_{i=1}^{N} \frac{1}{(2\pi)^{\frac{r}{2}}\,(t_1,\ldots,(t_r-t_{r-1}))^{\frac{1}{2}}\,6_i^r} \times$$

$$\times \int\limits_{x_1^{2l}+\ldots+x_r^{2l}>N} (x_1^{2l}+\ldots+x_r^{2l}) \exp\left[-\frac{x_1^2}{2t_1 6_i^2}-\ldots-\frac{(x_r-x_{r-1})^2}{2(t_r-t_{r-1})6_i^2}\right] d\mathbf{x}_r =$$

$$= \frac{1}{N(2\pi)^{\frac{r}{2}}\,(t_1\ldots(t_r-t_{r-1}))^{\frac{1}{2}}} \sum_{x=1}^{N} 6_i^{2l} \times$$

$$\times \int\limits_{x_1^{2l}+\ldots+x_r^{2l}>\frac{\varepsilon N}{6_i^{2l}}} (x_1^{2l}+\ldots+x_r^{2l}) \exp\left[-\frac{x_1^2}{2t_1}-\ldots-\frac{(x_r-x_{r-1})^2}{2(t_r-t_{r-1})}\right] d\mathbf{x}_r =$$

$$= \frac{6^{2l}}{(2\pi)^{\frac{r}{2}}(t_1\ldots(t_r-t_{r-1}))^{\frac{1}{2}}} \int\limits_{x_1^{2l}+\ldots+x_r^{2l}>\frac{\varepsilon N}{2l}} (x_1^{2l}+\ldots+x_r^{2l}) \exp\left(-\frac{x_1^2}{2t_1}-\ldots-\frac{(x_r-x_{r-1})^2}{2(t_r-t_{r-1})}\right) d\mathbf{x}_r \to 0$$

when $N\to\infty$.

In a similar way we can show that (14) holds in the case when l is even.

E x a m p l e 3. Let $X_t^{(1)}, X_t^{(2)},\ldots$ be independent quasi-diffusion processes with identical r-dimensional distributions $(r = 1,2,\ldots)$ with covariance function

$$K(t_1,t_2) = q_1^2 f_1(t_1) + q_2^2 f_2(t_2-t_1) + q_3^3 f_3(t_1,t_2), \tag{19}$$

where q_1^2, q_2^2, q_3^2 are some real numbers, $f_1 \in C^1$, $f_3 \in C^1$, f_2 has the right-hand side derivative and f_1, f_2, f_3 are covariance functions, then Y_t given by (10) is a quasi-diffusion process.

It is evident that K given by (19) is a covariance function and that K satisfies (8) and (9). Thus, in virtue of Property 2, Y_t is also a quasi-diffusion process.

The assertion of Example 3 is also true even if we omit the assumption that $X_t^{(1)}$, $X_t^{(2)}$,... are quasi-diffusion processes. Thus (19) can be treated as a condition which makes the limit of the sum (11), the sum of independent stochastic processes with identical r-dimensional distributions (r = = 1,2,...), a quasi-diffusion process.

Formula (19) is quite general. The special cases of (19) are the most often encountered covariance functions

$$6^2 \exp(-c^2(t_2-t_1)^2), \quad 6^2 \sum_{i=1}^{n} c_i |t_2-t_1|^i \exp(-c^2|t_2-t_1|),$$

$$\sum_{i=1}^{\infty} c_i^2 \cos \lambda_i(t_2-t_1), \quad \cos(t_2-t_1)\exp(-|t_2-t_1|), \quad c^2 t_1, \text{ etc.}$$

(evidently these functions are covariance functions if the coefficients c_i satisfy some additional conditions).

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