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Anna Dymitruk, Jerzy Muszyński

DAMPED VIBRATIONS OF STRINGS CONNECTED IN A NODE

In this work, which is a continuation of work [1], we shall consider a system of N strings - lying in a common plane in their equilibrium position - with one end of each connected to a single common point called node, and the other end built - in (see Fig.1). Assume the strings are identical - all their lengths l are equal and their masses per unit length ϱ are equal and also the tensions T. We introduce the following assumptions:

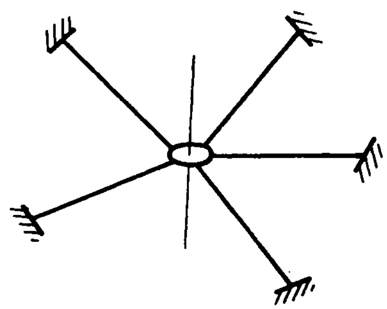

Fig.1

1) the node is massless;

2) both points of strings and the node can move only in the direction perpendicular to the plane in which whole the system lies in its equilibrium state (the node moves in this direction without any drag);

3) the node does not transmit any forces acting in the plane of strings. We assume moreover that the second ends of each strings, i.e. oposite to the node, are built-in.

We shall be interested in vibrations of the strings caused by their initial shape, initial impulses and external forces different for each string. Gravity forces of string will be treated as external ones. In this work we assume that the strings vibrate in an environment in which drag is proportional to velocity. To simplyfy the calculations we take $\sqrt{\frac{T}{\varrho}} = 1$. For each string we introduce axis Ox directed from its build-in end (let here $x = 0$) along the string in its equilibrium position to the node (let here $x = 1$).

Let $u^i(t,x)$ denote the displacement of the point x from its equilibrium position at a time t for i-th string. The movement of each string is described by the equation

$$(1) \quad u^i_{tt}(t,x) - u^i_{xx}(t,x) + ku^i_t(t,x) = p^i(t,x) \text{ for } x \in [0,1], \quad t \geq 0,\ i = 1,2,\dots,N,$$

where $p^i(t,x)$ describes an external force acting on i-th string and $k > 0$ is a damping coefficient. We take initial conditions in the form

$$(2) \quad \begin{aligned} u^i(0,x) &= f^i(x) \\ u^i_t(0,x) &= F^i(x) \quad \text{for } x \in [0,1], \quad i = 1,2,\dots,N. \end{aligned}$$

The built-in end of each string for which $x = 0$ leads to boundary conditions

$$(3) \quad u^i(t,0) = 0 \quad \text{for } t \geq 0, \quad i = 1,2,\dots,N.$$

For the node we have the following conditions:

- equality of displacements of each string connected to the node

$$(3') \quad u^1(t,1) = u^2(t,1) = \dots = u^N(t,1) \quad \text{for } t \geq 0$$

- equilibrium of force components acting along the direction perpendicular to the plane assigned by the system in its equilibrium position

$$(3'') \quad \sum_{i=1}^{N} u^i_x(t,1) = 0.$$

Solution of the problem

We introduce additional functions $u(t,x)$ and $p(t,x)$ defined for $t \geqslant 0$ and $x \in [0,l]$ in the form

$$u(t,x) = \sum_{i=1}^{N} u^i(t,x), \quad p(t,x) = \sum_{i=1}^{N} p^i(t,x).$$

The conditions of our problem imply that the function $u(t,x)$ satisfies equation

$$(4) \qquad u_{tt}(t,x) - u_{xx}(t,x) + ku_t(t,x) = p(t,x) \quad \text{for} \quad x \in [0,l], \; t \geqslant 0$$

and initial conditions

$$(5) \qquad u(0,x) = f(x), \quad u_t(0,x) = F(x) \quad \text{for } x \in [0,l],$$

where $f(x) = \sum_{i=1}^{N} f^i(x), \quad F(x) = \sum_{i=1}^{N} F^i(x);$

and boundary conditions

$$(6) \qquad u(t,0) = 0, \quad u_x(t,l) = 0 \quad \text{for} \quad t \geqslant 0.$$

We are seeking a solution of (4) under conditions (5) and (6) in the form of Fourier series. Further investigations will be limited to the case for which $k < \frac{\pi}{l}$; if no, the Fourier series obtained below will possess few additional terms which have no influence neither on convergence of series or on behavior of solutions. The obtained solution is of the form

$$(7) \quad u(t,x) = \sum_{n=0}^{\infty} \frac{2l}{\omega_n} e^{-\frac{k}{2}t} \int_0^t e^{\frac{k}{2}\tau} p_n(\tau) \sin \frac{\omega_n}{2l}(t-\tau) \sin \frac{(2n+1)\pi}{2l} x \, d\tau +$$

$$+ \sum_{n=0}^{\infty} e^{\frac{k}{2}t} \left[f_n \cos \frac{\omega_n t}{2l} + \frac{2l}{\omega_n} (F_n + \frac{k}{2} f_n) \sin \frac{\omega_n t}{2l} \right] \sin \frac{(2n+1)\pi}{2l} x,$$

where $p_n(t)$, f_n and F_n are Fourier coefficients and are described by

$$p_n(t) = \frac{2}{l}\int_0^l p(t,\xi)\sin\frac{(2n+1)\pi}{2l}\xi\, d\xi, \quad f_n = \frac{2}{l}\int_0^l f(\xi)\sin\frac{(2n+1)\pi}{2l}\xi\, d\xi,$$

$$F_n = \frac{2}{l}\int_0^l F(\xi)\sin\frac{(2n+1)\pi}{2l}\xi\, d\xi \quad \text{and} \quad \omega_n = \sqrt{(2n+1)^2\pi^2 - k^2l^2}.$$

Coming back to the starting point of our problem, the equality of displacement condition for the node and the definition of the function $u(t,x)$ lead to formulae for vibration desplacements of each string in this point as follows

$$(8)\quad u^i(t,l) = \frac{1}{N}u(t,l) = \frac{1}{N}e^{-\frac{k}{2}t}\sum_{n=0}^{\infty}(-1)^n\frac{2l}{\omega_n}\int_0^t e^{\frac{k}{2}\tau}p_n(\tau)\sin\frac{\omega_n}{2l}(t-\tau)d\tau +$$

$$+\frac{1}{N}e^{-\frac{k}{2}t}\sum_{n=0}^{\infty}(-1)^n\left[f_n\cos\frac{\omega_n t}{2l} + \frac{2l}{\omega_n}\left(F_n + \frac{k}{2}f_n\right)\sin\frac{\omega_n t}{2l}\right] =$$

$$= \varphi(t)$$

for $i = 1,2,\ldots,N$.

Let us investigate the vibrations of each string. Let us take the i-th string. We are looking for the function $u^i(t,x)$ which satisfies equation (1), initial conditions (2) and boundary conditions

$$(9)\qquad u^i(t,0) = 0, \quad u^i(t,l) = \varphi(t),$$

where $\varphi(t)$ is defined as above.

We see that the function $w(t,x) = \frac{1}{N}u(t,x)$ satisfies the equation

$$w_{tt}(t,x) - w_{xx}(t,x) + kw_t(t,x) = \frac{1}{N}\, p(t,x), \tag{10}$$

and the initial conditions

$$w(0,x) = \frac{1}{N}\, f(x), \qquad w_t(0,x) = \frac{1}{N}\, F(x) \tag{11}$$

and the boundary conditions (9).

The function $u^i(t,x)$ can be written in the form

$$u^i(t,x) = w(t,x) + z^i(t,x), \tag{12}$$

where $z^i(t,x)$ satisfies the equation

$$z^i_{tt}(t,x) - z^i_{xx}(t,x) + kz^i_t(t,x) = p^i(t,x) - \frac{1}{N}\, p(t,x), \tag{13}$$

and the initial conditions

$$z^i(0,x) = f^i(x) - \frac{1}{N}\, f(x), \quad z^i_t(0,x) = F^i(x) - \frac{1}{N}\, F(x) \tag{14}$$

and the zero boundary conditions

$$z^i(t,0) = 0, \quad z^i(t,1) = 0. \tag{15}$$

We see that the form of external force in equation (13) and the initial conditions (14) imply that the condition (3") is satisfied by the sum of functions $z^i(t,x)$, namely $\sum_{i=1}^{N} z^i_x(t,x) = 0$. We denote

$$h^i(x) = f^i(x) - \frac{1}{N}\, f(x), \quad H^i(x) = F^i(x) - \frac{1}{N}\, F(x),$$

$$q^i(t,x) = p^i(t,x) - \frac{1}{N}\, p(t,x).$$

The solution of the equation (13) under conditions (14), (15) is

$$(16)\quad z^i(t,x) = e^{-\frac{k}{2}t} \sum_{n=1}^{\infty} \frac{2l}{\alpha_n} \int_0^t e^{\frac{k}{2}\tau} q_n^i(\tau) \sin \frac{\alpha_n}{2l}(t-\tau) \sin \frac{n\pi}{l} x\, d\tau +$$

$$+ e^{-\frac{k}{2}t} \sum_{n=1}^{\infty} \left[h_n^i \cos \frac{\alpha_n}{2l} t + \left(H_n^i + \frac{k}{2} h_n^i\right) \frac{2l}{\alpha_n} \sin \frac{\alpha_n}{2l} t \right] \sin \frac{n\pi}{l} x,$$

where $q_n^i(t)$, h_n^i and H_n^i are Fourier coefficients and are described by

$$q_n^i(t) = \frac{2}{l} \int_0^l q^i(t,\xi) \sin \frac{n\pi}{l} \xi\, d\xi, \quad h_n^i = \frac{2}{l} \int_0^l h_h^i(\xi) \sin \frac{n\pi}{l} \xi\, d\xi,$$

$$H_n^i = \frac{2}{l} \int_0^l H_n^i(\xi) \sin \frac{n\pi}{l} \xi\, d\xi \quad \text{and} \quad \alpha_n = \sqrt{4n^2\pi^2 - k^2 l^2}.$$

Finally, the solutions of problem considered in this work are functions of the form

$$(17)\quad u^i(t,x) = \frac{1}{N} u(t,x) + z^i(t,x),$$

where $u(t,x)$ is given by (17) and $z^i(t,x)$ by (16).
We see that when the functions $f^i(x)$, $F^i(x)$ and $p^i(t,x)$ are identical for every $i = 1,2,\ldots,N$ we have $z^i(t,x) \equiv 0$. The above considerations (see formulae (12)) lead to the conclusion that the vibrations of i-th string may be treated as composed of two components: vibrations of the whole system caused by averaged external forces and under averaged initial conditions and vibrations peculiar to the i-th string caused by corrected external forces and under corrected initial conditions.

Taking into account the form of ω_n and α_n, we see that the functions $u^i(t,x)$ described by (17) may be rewritten by the following formulae

$$(18)\quad u^i(t,x) = e^{-\frac{k}{2}t} \sum_{n=1}^{\infty} \frac{2l}{\gamma_n} \int_0^t e^{\frac{k}{2}\tau}\, Q_n^i(\tau) \sin \frac{\gamma_n}{2l}(t-\tau) \sin \frac{n\pi}{2l}\, x\, d\tau +$$

$$+\, e^{-\frac{k}{2}t} \sum_{n=1}^{\infty} \left[A_n^i \cos \frac{\gamma_n t}{2l} + \frac{2l}{n} \left(B_n^i + \frac{k}{2} A_n^i\right) \sin \frac{\gamma_n t}{2l} \right] \sin \frac{n\pi}{2l}\, x,$$

where

$$(19)\quad Q_n^i(t) = \begin{cases} q^i_{\frac{n}{2}}(t) & \text{for even } n \\ \frac{1}{N}\, P_{\frac{n-1}{2}} & \text{for odd } n \end{cases}, \qquad \gamma_n = \sqrt{n^2\pi^2 - k^2 l^2}$$

$$A_n^i = \begin{cases} h^i_{\frac{n}{2}} & \text{for even } n \\ \frac{1}{N}\, f_{\frac{n-1}{2}} & \text{for odd } n, \end{cases} \qquad B_n^i = \begin{cases} H^i_{\frac{n}{2}} & \text{for even } n \\ \frac{1}{N}\, F_{\frac{n-1}{2}} & \text{for odd } n. \end{cases}$$

Particular cases

1. Let us assume that there are no external forces acting on a system; it means that all the functions $p^i(t,x) \equiv 0$. In this case the solutions of problem considered are of the form

$$(20)\quad u^i(t,x) = e^{-\frac{k}{2}t} \sum_{n=1}^{\infty} \left[A_n^i \cos \frac{\gamma_n t}{2l} + \frac{2l}{\gamma_n} \left(B_n^i + \frac{k}{2} A_n^i\right) \sin \frac{\gamma_n t}{2l} \right] \sin \frac{n\pi}{2l} x,$$

where A_n^i, B_n^i, γ_n are described by (19).

As is seen from the above formulae - due to presense of decreasing exponential factor $e^{-\frac{k}{2}t}$ - the free vibrations of each string tend to die down.

2. Let us assume that the damping of environment is so small, that it can be neglected, i.e. let $k = 0$. Under this assumption the solutions of problem considered are of the form

$$u^i(t,x) = \sum_{n=1}^{\infty} \frac{2l}{\pi n} \int_0^t Q_n^i(\tau) \sin \frac{\pi n}{2l} (t-\tau) \sin \frac{\pi n}{2l} x \, d\tau + \tag{21}$$

$$+ \sum_{n=1}^{\infty} \left(A_n^i \cos \frac{\pi n}{2l} t + \frac{2l}{\pi n} B_n^i \sin \frac{\pi n}{2l} t\right) \sin \frac{\pi n}{2l} x,$$

where $Q_n^i(t)$, A_n^i and B_n^i are described by formulae (19). This particular result is identical to that obtained as the general result in paper [1], while the working methods are different; in paper [1] the d'Alembert method was used.

BIBLIOGRAPHY

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INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW
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