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ON A RANDOM LINEAR PARABOLIC EQUATION OF ANY EVEN ORDER

Introduction

In this paper we consider the random linear parabolic equation

$$(0.1) \qquad Lu \equiv \sum_{|k|\leqslant 2b} a_k(x,t)D_x^k u - D_t u = 0 \text{ [1]},$$

where b is a positive integer and a_k are complex random functions defined in the strip $H = \{(x,t): x\in R^n, t\in \langle 0,T\rangle\}$. Applying the same parametrix method as in [2], we construct a fundamental solution of the equation (0.1). This enables us to prove the existence of a solution of the Cauchy problem

$$(0.2) \qquad Lu = f(x,t),\ (x,t)\in H_0 = R^n \times (0,T\rangle ,$$

$$(0.3) \qquad u(x,0) = g(x),\ x\in R^n,$$

where f and g are given complex random functions.

In paper [4] there were treated random linear parabolic equations of the second order and there were obtained results similar to the above-mentioned ones. We applied in [4] the same parametrix method as in [3] (chapter 1) which required

[1] Notation and definitions will be stated in Section 1.

stronger assumptions concerning coefficients than those made in the present paper specialized to the case $b = 1$.

1. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probabilistic space, where P is a complete probability measure. By $L_p(\Omega)$ $(1 \leq p \leq \infty)$ we denote the Banach space of all complex random variables $f(\omega)$ defined on $(\Omega, \mathcal{F}, P)$ with finite norm

$$\|f\|_p = \left[\int_\Omega |f(\omega)|^p P(d\omega)\right]^{\frac{1}{p}}, \text{ if } 1 \leq p < \infty \text{ ; } \|f\| = \operatorname*{ess\ sup}_{\omega\in\Omega} |f(\omega)|.$$

The limit, continuity and partial derivatives of a random function $u : A \to L_p(\Omega)$, $A \subset R^n$, are understood in the strong sense and they are called respectively the L_p-limit, L_p-continuity and L_p-derivatives of u. By $C^k(A, L_p(\Omega))$, $k \in N_0 = \{0,1,2,\ldots\}$, we denote the set of all random functions $u: A \to L_p(\Omega)$ which are L_p-continuous in A together with all their L_p-derivatives $D_x^\alpha u$, $|\alpha| \leq k$, where

(1.1)
$$\alpha = (\alpha_1,\ldots,\alpha_n) \in N_0^n, \quad |\alpha| = \alpha_1 + \ldots + \alpha_n,$$
$$D_x^\alpha = D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} .$$

We abbreviate $C^0(A, L_p(\Omega)) = C(A, L_p(\Omega))$. Let us denote $C^\infty(A, L_p(\Omega)) = \bigcap_{k=1}^{\infty} C^k(A, L_p(\Omega))$. This enables us to introduce the set $S(R^n, L_p(\Omega))$ of all functions $u \in C^\infty(R^n, L_p(\Omega))$ such that for any multi-indexes α, $\beta \in N_0^n$ we have

$$\sup_{x\in R^n} \left\| x^\alpha D_x^\beta u(x) \right\|_p < \infty , \quad x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n} .$$

For a function $u \in S(R^n, L_p(\Omega))$ there is defined the Fourier transform

$$\hat{u}(\xi) = \int_{R^n} e^{-i(x,\xi)} u(x)dx \;^{2)}, \quad \xi \in R^n, \quad (x,\xi) = \sum_{j=1}^{n} x_j \xi_j.$$

The following lemmas will be needed.

L e m m a 1.1. If $u \in S(R^n, L_p(\Omega))$, then $\hat{u} \in S(R^n, L_p(\Omega))$ and

$$u(x) = (2\pi)^{-n} \int_{R^n} \hat{u}(\xi) e^{i(x,\xi)} d\xi .$$

The proof is the same as that for nonrandom functions.

L e m m a 1.2. If $h \in C^1(\langle\alpha, \beta\rangle, L_p(\Omega))$, then there exists a number $\gamma \in \langle\alpha, \beta\rangle$ such that $\|h(\beta)-h(\alpha)\|_p \leqslant \leqslant \|h'(\gamma)\|_p(\beta-\alpha)$.

P r o o f . We have

$$h(\beta) - h(\alpha) = \int_{\alpha}^{\beta} h'(x)dx.$$

Hence it follows from the integral mean value theorem that

$$\|h(\beta)-h(\alpha)\|_p \leqslant \int_{\alpha}^{\beta} \|h'(x)\|_p dx = (\beta-\alpha)\,\|h'(\gamma)\|_p .$$

L e m m a 1.3. Let $A \subset R^n$ be a convex domain. If $u \in C^1(A, L_p(\Omega))$, then for any points $\alpha, \beta \in A$, $\alpha \neq \beta$, there exists a point γ lying on the segment $\alpha\beta$ such that

$$\|u(\beta)-u(\alpha)\|_p \leqslant \sum_{j=1}^{n} (\beta_j-\alpha_j)\|u_{x_j}(\gamma)\|_p. \tag{1.2}$$

2) Throughout this paper the proper and improper integrals of L_p-continuous random functions are taken in the strong Riemann sense (L_p-integrals). Definitions and lemmas of paper [4] concerning L_p-integrals of real random functions hold true also for complex ones. Moreover, the measurability of random functions, assumed in [4], is superfluous.

P r o o f . The random function $v(\tau) = u((1-\tau)\alpha+\tau\beta)$, $\tau \in <0,1>$, belongs to $C^1(<0,1>, L_p(\Omega))$ and its L_p-derivative is defined by formula

$$(1.3) \qquad v'(\tau) = \sum_{j=1}^{n} (\beta_j-\alpha_j) u_{x_j}(\alpha+\tau(\beta-\alpha)).$$

In view of Lemma 1.2, there is a number $\tau_0 \in <0,1>$ such that $\|v(1)-v(0)\|_p \leqslant \|v'(\tau_0)\|_p$ which implies, by (1.3), the inequality (1.2) with $\gamma = \alpha+\tau_0(\beta-\alpha)$.

Finally we introduce the following definition.

D e f i n i t i o n 1.1. The operator L (defined by (0.1)) is called uniformly parabolic if

$$(1.3) \qquad \mathrm{Re} \sum_{|k|=2b} a_k(x,t)(i\sigma)^k \leqslant -\delta|\sigma|^{2b}, \quad (x,t) \in H, \quad \sigma \in R^n \text{ [3)]}$$

$\delta > 0$ being a constant and $|\sigma| = \left[\sum_{j=1}^{n} |\sigma_j|^2\right]^{\frac{1}{2}}$.

2. An equation with random coefficients depending only on t

We consider the equation

$$(2.1) \qquad L'u \equiv \sum_{|k|\leqslant 2b} a_k(t) D_x^k u - D_t u = 0.$$

T h e o r e m 2.1. If $a_k \in C(<0,T>, L_\infty(\Omega))$ and the operator L' is uniformly parabolic (Definition 1.1), then there exists a complex random function $Z(x,t,\xi,\tau)$ defined in the set

$$(2.2) \qquad A_0 = \{(x,t,\xi,\tau):\ x,\xi \in R^n,\ 0 \leqslant \tau < t \leqslant T\}$$

3) Note that $a_k(x,t)$ are random functions. Therefore inequality (1.3) holds for $(x,t) \in H$, $\sigma \in R^n$, $\omega \in \Omega_{x,t}$, where $P(\Omega_{x,t}) = 1$.

and possessing the following properties:

1^o $Z \in C(A_0, L_\infty(\Omega))$ and we have

(2.3) $$\|Z(x,t,\xi,\tau)\|_\infty \leqslant B(t-\tau)^{-\frac{n}{2b}} \exp\left[-\frac{\delta_0|x-\xi|^{b'}}{(t-\tau)^{b''}}\right],$$

where $b' = \frac{2b}{2b-1}$, $b'' = \frac{1}{2b-1}$ and $B, \delta_0 > 0$ are some constants;

2^o there exist L_∞-derivatives $D_x^m Z \in C(A_0, L_\infty(\Omega))$, $m \in N_0^n$ and we have the estimate

(2.4) $$\| D_x^m Z(x,t,\xi,\tau) \|_\infty \leqslant B(|m|)(t-\tau)^{-\frac{n+|m|}{2b}} \exp\left[-\frac{\delta_0|x-\xi|^{b'}}{(t-\tau)^{b''}}\right],$$

$B(|m|)$ being a certain constant depending on $|m|$;

3^o there exists an L_∞-derivative $D_t Z \in C(A_0, L_\infty(\Omega))$ which fulfils the inequality

(2.5) $$\|D_t Z(x,t,\xi,\tau)\|_\infty \leqslant B(t-\tau)^{-\frac{n}{2b}-1} \exp\left[-\frac{\delta_0|x-\xi|^{b'}}{(t-\tau)^{b''}}\right];$$

4^o for fixed $\xi \in R^n$, $\tau \in \langle 0,T)$ the function $Z(x,t,\xi,\tau)$ satisfies with respect to the variables $x \in R^n$, $t \in (\tau, T\rangle$ the equation (2.1);

5^o for any function $g \in C(R^n, L_q(\Omega))$, $1 \leqslant q \leqslant \infty$, which is L_q-bounded (i.e. $\|g(x)\|_q \leqslant$ const., $x \in R^n$) we have

$$\lim_{t \to \tau+0} \left\| \int_{R^n} Z(x,t,\xi,\tau) g(\xi) d\xi - g(x) \right\|_q = 0,$$

where the convergence is uniform with respect to $x \in A$ ($A \subset R^n$ being a bounded domain).

The above-mentioned function $Z(x,t,\xi,\tau)$ is called a fundamental solution of the equation (2.1)[4].

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[4] In the definition of the fundamental solution it suffices to assume that the condition 3^o holds for $|m| \leqslant 2b$.

P r o o f . We use the same method as in [2] (p.15-20, 34). Namely let us introduce the function

$$(2.6)\qquad Q(t,\tau,\sigma) = \exp\left[\sum_{|k|\leq 2b} (i\sigma)^k \int_\tau^t a_k(\beta)d\beta\right]$$

defined in the set $A_1 = \{(t,\tau,\sigma):\ 0 \leq \tau \leq t \leq T,\quad \sigma \in R^n\}$. Obviously $Q \in C(A_1, L_\infty(\Omega))$ and there exists L_∞-derivative $Q_t \in C(A_1, L_\infty(\Omega))$ given by formula

$$(2.7)\qquad Q_t(t,\tau,\sigma) = \left[\sum_{|k|\leq 2b} (i\sigma)^k a_k(t)\right] Q(t,\tau,\sigma).$$

Note that

$$(2.8)\qquad \|a_k(t)\|_\infty \leq B_1,\ t \in \langle 0,T\rangle ,$$

$B_1 > 0$ being some constant. In view of (2.6), (2.8) and (1.3), we have

$$(2.9)\quad \|Q(t,\tau,\sigma)\|_\infty \leq B_2 \exp\left[-\delta_1 |\sigma|^{2b}(t-\tau)\right] \quad \text{in } A_1 \quad (\delta_1 \in (0,\delta)).$$

It follows from (2.7)-(2.9) that

$$(2.10)\quad \|Q_t(t,\tau,\sigma)\|_\infty \leq B_3(|\sigma|^{2b}+1)\exp\left[-\delta_1 |\sigma|^{2b}(t-\tau)\right] \quad \text{in } A_1.$$

Now let us introduce the function

$$(2.11)\qquad G(x,t,\tau) = (2\pi)^{-n} \int_{R^n} e^{i(x,\sigma)} Q(t,\tau,\sigma)d\sigma$$

defined in the set

$$(2.12)\qquad A_2 = \left\{(x,t,\tau):\ x \in R^n,\ 0 \leq \tau < t \leq T\right\}.$$

In virtue of the estimate (2.9) and of Lemma 4.6 of [4], the integral (2.11) is uniformly L_∞-convergent for $x \in A$, $t-\tau \geq h$, where $A \subset R^n$ is a bounded domain and $h \in (0,T)$ is a constant. Thus, by Lemma 4.7 of [4], $G \in C(A_2, L_\infty(\Omega))$. Similarly, the estimates (2.8), (2.9) and Lemma 4.9 of [4] imply the existence of L_∞-derivatives $D_x^m G, D_t G \in (A_2, L_\infty(\Omega))$, $m \in N_0^n$ given by formulas

$$(2.13) \qquad D_x^m G(x,t,\tau) = (2\pi)^{-n} \int_{R^n} (i\sigma)^m e^{i(x,\sigma)} Q(t,\tau,\sigma)d\sigma,$$

$$(2.14) \qquad D_t G(x,t,\tau) = (2\pi)^{-n} \int_{R^n} e^{i(x,\sigma)} Q_t(t,\tau,\sigma)d\sigma.$$

We have proved that for the function

$$(2.15) \qquad Z(x,t,\xi,\tau) = G(x-\xi,t,\tau), (x,t,\xi,\tau) \in A_0,$$

hold the assertions 1°-3° except the estimates (2.3)-(2.5). The relations (2.11), (2.13)-(2.15) and (2.7) imply the assertion 4°.

. In order to derive the estimate (2.3) we use the inequality

$$(2.16) \quad \| Q(t,\tau,\sigma+i\gamma)\|_\infty \leq B_4 \exp\left[(-\delta_1 |\sigma|^{2b} + \alpha_0 |\gamma|^{2b})(t-\tau)\right],$$

$0 \leq \tau < t \leq T$, $\sigma, \gamma \in R^n$, $\alpha_0 > 0$ being some constant. Introducing the variables

$$\sigma_j = \beta_j (t-\tau)^{-\frac{1}{2b}}, \quad j = 1,\dots,n,$$

in the integral (2.11) we obtain

(2.17) $G(x,t,\tau) =$

$$= \left[2\pi(t-\tau)^{\frac{1}{2b}}\right]^{-n} \int_{R^n} \tilde{G}\left(t,\tau,\frac{\beta}{(t-\tau)^{1/2b}}\right) \exp\left[i\left(\frac{x}{(t-\tau)^{1/2b}},\beta\right)\right] d\beta.$$

The integrand of the last integral is, under fixed x,t,τ, an L_∞-analytic function of the complex variables $\beta_1+i\eta_1,\dots,\beta_n+i\eta_n$. Let $\eta=(\eta_1,\dots,\eta_n)\in R^n$ be a fixed point. Changing the integral in (2.17) into the iterated one[5], applying successively the Cauchy integral theorem (see [1], p.226) and using the estimate (2.16) we obtain

$$G(x,t,\tau) = \left[2\pi(t-\tau)^{\frac{1}{2b}}\right]^{-n} \int_{R^n} \tilde{G}\left(t,\tau,\frac{\beta+i\eta}{(t-\tau)^{1/2b}}\right) \exp\left[i\left(\frac{x}{(t-\tau)^{1/2b}},\beta+i\eta\right)\right] d\beta.$$

Hence it follows from (2.16) that

$$(2.18)\quad \|G(x,t,\tau)\|_\infty \leq B_5(t-\tau)^{-\frac{n}{2b}} \exp\left[-\left(\frac{x}{(t-\tau)^{1/2b}},\eta\right)+\alpha_0|\eta|^{2b}\right].$$

Setting

$$\eta_j = \eta_0(\operatorname{sgn} x_j)^{b'}\left[x_j(t-\tau)^{-\frac{1}{2b}}\right]^{b'-1},\quad \eta_0>0,\quad j=1,\dots,n,$$

we find that

$$(2.19)\qquad -\left(\frac{x}{(t-\tau)^{1/2b}},\eta\right) + \alpha_0|\eta|^{2b} \leq -\delta_2 \sum_{j=1}^{n} \frac{|x_j|^{b'}}{(t-\tau)^{b''}},$$

where $\delta_2 = \eta_0 - \alpha_0\eta_0^{2b}\eta^{b+\frac{b'}{2}} > 0$ for sufficiently small $\eta_0>0$. Now, the estimate (2.3) results from (2.18), (2.19) and (2.15).

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[5] We use the estimate (2.9) and Fubini's theorem [1] (p.193).

Proceeding in a similar way and using (2.13), we obtain the estimate (2.4). The estimate (2.5) is an immediate consequence of (2.4) and of the equality

$$_{t}Z(x,t,\xi,\tau) = \sum_{|m|\leqslant 2b} a_m(t) D_x^m Z(x,t,\xi,\tau).$$

In order to prove the assertion 5° let us denote

(2.20) $K_a = \left\{\xi : |\xi - x| \leqslant a\right\},\ K_a' = \left\{\xi : |\xi - x| > a\right\},\quad a > 0.$

We have

(2.21) $$\int_{R^n} Z(x,t,\xi,\tau)\, g(\xi) d\xi - g(x) = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{K_a} Z(x,t,\xi,\tau) \left[g(\xi) - g(x)\right] d\xi,$$

$$I_2 = \int_{K_a'} Z(x,t,\xi,\tau) \left[g(\xi) - g(x)\right] d\xi,$$

$$I_3 = g(x)\left[\int_{R^n} Z(x,t,\xi,\tau)\, d\xi - 1\right].$$

Let $A \subset R^n$ be an arbitrary bounded domain and let $\varepsilon > 0$. According to the uniform L_q-continuity of g in any bounded domain and the estimate (2.3), there exists an $a > 0$ such that for any $x \in A$, $0 < \tau < t \leqslant T$ we have

(2.22) $$\|I_1\|_q \leqslant \int_{K_a} \|Z(x,t,\xi,\tau)\|_\infty \|g(\xi) - g(x)\|_q d\xi < \frac{\varepsilon}{3}.$$

For the above a there exists, by (2.3), an h_0 such that for $x \in A$, $0 \leqslant \tau < t \leqslant \tau + h_0$ holds the inequality

$$(2.23)\qquad \|I_2\|_q \leqslant \int_{K_a'} \|Z(x,t,\xi,\tau)\|_\infty \left[\|g(\xi)\|_q + \|g(x)\|_q\right] d\xi < \frac{\varepsilon}{3}\,.$$

The estimate (2.9) guarantees the existence of the Fourier transform

$$\hat{Q}(t,\tau,\xi) = \int_{R^n} e^{-i(x,\xi)} Q(t,\tau,x)\,dx,\quad 0 \leqslant \tau < t \leqslant T,\quad \xi \in R^n,$$

whence, by Lemma 1.1, we have

$$Q(t,\tau,x) = (2\pi)^{-n} \int_{R^n} e^{i(x,\xi)} \hat{Q}(t,\tau,\xi)\,d\xi,\quad 0 \leqslant \tau < t \leqslant T,\ x \in R^n.$$

The above equality implies that

$$(2.24)\qquad Q(t,\tau,0) = (2\pi)^{-n} \int_{R^n} \hat{Q}(t,\tau,\xi)\,d\xi =$$

$$= (2\pi)^{-n} \int_{R^n} \hat{Q}(t,\tau,\xi - x)\,d\xi = \int_{R^n} Z(x,t,\xi,\tau)\,d\xi.$$

Taking into considerations the relations

$$Q(t,\tau,0) = \exp\left[\int_\tau^t a_0(\beta)\,d\beta\right],\quad \mathrm{Re}\int_\tau^t a_0(\beta)\,d\beta \leqslant B_1 T,$$

$$(2.25)\qquad \left|e^{z_1} - e^{z_2}\right| \leqslant 2e^{\alpha}\left|z_1 - z_2\right|,\quad \mathrm{Re}\,z_1,\ \mathrm{Re}\,z_2 \leqslant \alpha,$$

we conclude that

$$(2.26)\qquad \|Q(t,\tau,0) - 1\|_\infty \leqslant 2e^{B_1 T}(t-\tau).$$

It follows from (2.24) and (2.26) the inequality

$$(2.27)\qquad \|I_3\|_q < \frac{\varepsilon}{3}\,,\quad x \in R^n,\quad 0 < t-\tau < h_1$$

provided $h_1 > 0$ is sufficiently small. Now the assertion 5^o is a consequence of relations (2.21)-(2.23) and (2.27). This completes the proof.

3. An equation with parameter

In this section we shall consider the equation

$$(3.1) \qquad L_0 u \equiv \sum_{|k|=2b} a_k(y,t) D_x^k u - D_t u = 0$$

with the parameter $y \in R^n$. The following assumptions will be needed.

(3.I) The coefficients a_k, $|k| = 2b$, are complex random functions L_∞-bounded in H and L_∞-continuous in $t \in \langle 0,T \rangle$ uniformly with respect to $y \in R^n$ [6].

(3.II) There are constants $\alpha \in (0,1)$ and $M_1 > 0$ such that

$$\|a_k(y,t) - a_k(y',t)\|_\infty \leq M_1 |y-y'|^\alpha, \quad t \in \langle 0,T \rangle, \; y,y' \in R^n.$$

(3.III) The operator L_0 is uniformly parabolic (Definition 1.1).

As in Section 2, we introduce the function

$$(3.2) \qquad Q(t,\tau,y,\sigma) = \exp\left\{ \sum_{|k|=2b} (i\sigma)^k \int_\tau^t a_k(y,\beta)\,d\beta \right\},$$

defined in the set $A_3 = \{(t,\tau,y,\sigma)\colon y, \sigma \in R^n, \; 0 \leq \tau \leq t \leq T\}$. Clearly $Q \in C(A_3, L_\infty(\Omega))$ and there exists L_∞-derivative $Q_t \in C(A_3, L_\infty(\Omega))$ given by formula

$$(3.3) \qquad Q_t(t,\tau,y,\sigma) = \left[\sum_{|k|=2b} (i\sigma)^k a_k(y,\beta) \right] Q(t,\tau,y,\sigma).$$

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[6] I.e. for any $t \in \langle 0,T \rangle$ we have

$$\lim_{\Delta t \to 0} \|a_k(y,t+\Delta t) - a_k(y,t)\|_\infty = 0,$$

where the convergence is uniform with respect to $y \in R^n$.

For the function (3.2) remain valid the estimates (2.9), (2.10) and (2.16). The function

$$(3.4)\qquad G_0(x,t,y,\tau) = (2\pi)^{-n}\int_{R^n} e^{i(x,\sigma)}Q(t,\tau,y,\sigma)d\sigma$$

is L_∞-continuous in A_0 (defined by (2.2)). Moreover, there exist L_∞-derivatives $D_x^m G_0$, $D_t G_0 \in C(A_0, L_\infty(\Omega))$, $m \in N_0^n$, given by formulas

$$(3.5)\qquad D_x^m G_0(x,t,y,\tau) = (2\pi)^{-n}\int_{R^n} (i\sigma)^m e^{i(x,\sigma)}Q(t,\tau,y,\sigma)d\sigma,$$

$$(3.6)\qquad D_t G_0(x,t,y,\tau) = (2\pi)^{-n}\int_{R^n} e^{i(x,\sigma)}Q_t(t,\tau,y,\sigma)d\sigma.$$

We have the estimates

$$(3.7)\qquad \left\| D_x^m G_0(x,t,y,\tau)\right\|_\infty \leqslant M(|m|)(t-\tau)^{-\frac{n+|m|}{2b}} \exp\left[-\frac{\delta'|x|^{b'}}{(t-\tau)^{b''}}\right],$$

$$(3.8)\qquad \left\| D_t G_0(x,t,y,\tau)\right\|_\infty \leqslant M_2(t-\tau)^{-\frac{n}{2b}-1} \exp\left[-\frac{\delta'|x|^{b'}}{(t-\tau)^{b''}}\right].$$

Now let us consider the difference

$$\Delta Q = Q(t,\tau,y,\sigma+i\gamma)-Q(t,\tau,z,\sigma+i\gamma),\quad y,z,\sigma,\gamma\in R^n,\quad 0\leqslant\tau<t\leqslant T.$$

In view of assumptions (3.I) and (3.III), we have

$$(3.9)\qquad \operatorname{Re}\sum_{|k|=2b}(is)^k\int_\tau^t a_k(y,\beta)d\beta \leqslant (-\delta_1|\sigma|^{2b}+\alpha_0|\gamma|^{2b})(t-\tau)$$

for $s = \sigma+i\gamma$, $y,\sigma,\gamma \in R^n$, $0 \leqslant\tau\leqslant t\leqslant T$ ($\delta_1 \in (0,\delta)$, $\alpha_0 > 0$). Assumption (3.II) implies that

$$(3.10)\qquad \Big\| \sum_{|k|=2b} (is)^k \int_{\tau}^{t} \left[a_k(y,\beta)-a_k(z,\beta)\right] d\beta \Big\|_{\infty} \leqslant$$

$$\leqslant M_3(|\sigma|^{2b}+|\gamma|^{2b})|y-z|^{\alpha}(t-\tau).$$

It follows from (3.9), (3.10) and (2.25) that

$$\|\Delta Q\|_{\infty} \leqslant M_4 |y-z|^{\alpha} \exp\left[(-\delta_1' |\sigma|^{2b}+\alpha_0' |\gamma|^{2b})(t-\tau)\right] ,$$

where $0<\delta_1'<\delta_1$, $\alpha_0'>\alpha_0$. Hence, arguing as in the proof of (2.3), we obtain the estimate

$$(3.11)\qquad \|D_x^m G_0(x,t,y,\tau) - D_x^m G_0(x,t,z,\tau)\|_{\infty} \leqslant$$
$$\leqslant M'(|m|)|y-z|^{\alpha}(t-\tau)^{-\frac{n+|m|}{2b}} \exp\left[-\frac{\delta_2' |x|^{b'}}{(t-\tau)^{b''}}\right] .$$

We introduce the following assumption.

(3.IV) The random function $f \in C(H,L_q(\Omega))$, $1 \leqslant q \leqslant \infty$, is L_q-bounded and satisfies for $x \in R^n$ a local Hölder condition in the L_q-sense with exponent α, uniform with respect to $t \in \langle 0,T\rangle$, i.e. for any bounded domain $\Lambda \subset R^n$ there is a constant $M>0$ such that

$$(3.12)\quad \|f(x,t)-f(x',t)\|_q \leqslant M|x-x'|^{\alpha}, \quad x,x' \in \Lambda,\ t \in \langle 0,T\rangle .$$

T h e o r e m 3.1. If assumptions (3.I)-(3.IV) are satisfied, then the function

$$(3.13)\qquad W(x,t) = \int_0^t \int_{R^n} G_0(x-\xi,t,\xi,\tau)f(\xi,\tau)d\xi\, d\tau$$

has the following properties:

1° it is L_q-continuous in H_0 and

$$(3.14)\qquad \lim_{t\to 0} \|W(x,t)\|_q = 0$$

uniformly with respect to $x \in A$ $(A \subset R^n$ being a bounded domain);

2° there exist L_q-derivates $D_x^m W \in C(H_0, L_q(\Omega))$, $|m| \leqslant 2b-1$, defined by formula

$$(3.15)\qquad D_x^m W(x,t) = \int_0^t \int_{R^n} D_x^m G_0(x-\xi,t,\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau;$$

3° there exist L_q-derivates $D_x^m W$, $D_t W \in C(H_0, L_q(\Omega))$, $|m| = 2b$, given by formulas

$$(3.16)\qquad D_x^m W(x,t) = \int_0^t d\tau \int_{R^n} D_x^m G_0(x-\xi,t,\xi,\tau) f(\xi,\tau)\, d\xi,$$

$$(3.17)\qquad D_t W(x,t) = \int_0^t d\tau \int_{R^n} D_t G_0(x-\xi,t,\xi,\tau) f(\xi,\tau)\, d\xi + f(x,t).$$

P r o o f . The assertions 1° and 2° follow from (3.7) and Remark 4.9 of [4]. In order to prove 3° we introduce the function

$$(3.18)\qquad J(x,t,\tau) = \int_{R^n} G_0(x-\xi,t,\xi,\tau) f(\xi,\tau)\, d\xi.$$

Taking into considerations the estimate (3.7) with $|m| = 0$ and Lemmas 4.6 and 4.7 of [4], we find that $J \in C(A_2, L_q(\Omega))$, A_2 being defined by (2.12). The estimates (3.7), (3.8) and Lemma 4.8 of [4] imply the existence of L_q-derivatives $D_x^m J$, $D_t J \in C(A_2, L_q(\Omega))$ given by formulas

$$(3.19)\qquad D_x^m J(x,t,\tau) = \int_{R^n} D_x^m G_0(x-\xi,t,\xi,\tau) f(\xi,\tau)\, d\xi,$$

$$(3.20)\qquad D_tJ(x,t,\tau) = \int_{R^n} D_tG_0(x-\xi,t,\xi,\tau)f(\xi,\tau)d\xi .$$

Write the function (3.19) with $|m| = 2b$ in the form

$$(3.21)\qquad D_x^mJ(x,t,\tau) = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{K_1} D_x^mG_0(x-\xi,t,\xi,\tau)\left[f(\xi,\tau)-f(x,\tau)\right]d\xi ,$$

$$I_2 = \int_{K_1'} D_x^mG_0(x-\xi,t,\xi,\tau)\left[f(\xi,\tau)-f(x,\tau)\right]d\xi ,$$

$$I_3 = f(x,\tau)\int_{R^n} D_x^mG_0(x-\xi,t,\xi,\tau)d\xi$$

and K_1, K_1' are defined by (2.20).
Let $A \subset R^n$ be an arbitrary bounded domain. The estimate (3.7) and the inequality (3.12) yield

$$(3.22)\qquad \|I_1\|_q \leqslant M_5(t-\tau)^{-1+\frac{\alpha}{2b}}, \quad x \in A,\ 0 \leqslant \tau < t \leqslant T.$$

The L_q-boundedness of f and the estimate (3.7) imply that

$$(3.23)\qquad \|I_2\|_q \leqslant M_6(t-\tau)^{-1+\frac{1}{2b}} \text{ in } A_2.$$

Applying Green's theorem for functions with values in Banach space and the estimate (3.7), we obtain

$$(3.24)\qquad \int_{R^n} D_x^mG_0(x-\xi,t,y,\tau)d\xi = D_x^{m'}\int_{R^n} D_{x_j}G_0(x-\xi,t,y,\tau)d\xi =$$

$$= -D_x^{m'}\int_{R^n} D_{\xi_j}G_0(x-\xi,t,y,\tau)d\xi = 0,$$

where $m' = (m_1', \dots, m_n')$, $m = (m_1, \dots, m_n)$, $m_k' \neq m_k$ $(k \neq j)$, $m_j' = m_j - 1$, $1 \leqslant j,k \leqslant n$ and $D_x^{m'}$, D_x^m are defined by (1.1). So we have

$$I_3 = f(x,\tau) \int_{R^n} \left[D_x^m G_0(x-\xi,t,\xi,\tau) - D_x^m G_0(x-\xi,t,y,\tau)\Big|_{y=x} \right] d\xi ,$$

which implies, by (3.11), the inequality

$$\|I_3\|_q \leqslant M_7 (t-\tau)^{-1+\frac{\alpha}{2b}} \quad \text{in } A_2. \tag{3.25}$$

It follows from relations (3.21)-(3.23) and (3.25) that

$$\| D_x^m J(x,t,\tau) \|_q \leqslant M_8 (t-\tau)^{-1+\frac{\alpha}{2b}} , \quad x \in A, \ 0 \leqslant \tau < t \leqslant T, \tag{3.26}$$

$M_8 > 0$ being a constant depending on the domain A. Hence, by Remark 4.9 of [4], there exist L_q-derivatives $D_x^m W \in$ $\in C(H_0, L_q(\Omega))$ given by formula

$$D_x^m W(x,t) = \int_0^t D_x^m J(x,t,\tau) d\tau, \quad |m| = 2b, \tag{3.27}$$

which implies (3.16).

In order to prove (3.17) observe that

$$D_t J(x,t,\tau) = \sum_{|m|=2b} \int_{R^n} \left[a_m(\xi,\tau) f(\xi,\tau) \right] D_x^m G_0(x-\xi,t,\xi,\tau) d\xi .$$

Since the functions $a_m(\xi,\tau) f(\xi,\tau)$, $|m| = 2b$, satisfy assumption (3.IV), therefore it follows from the proof of (3.26) that

$$\| D_t J(x,t,\tau) \|_q \leqslant M_9 (t-\tau)^{-1+\frac{\alpha}{2b}}, \quad x \in A, \ 0 \leqslant \tau < t \leqslant T. \tag{3.28}$$

Hence, by Remark 4.9 of [4], the integral

$$(3.29) \qquad \int_0^t D_t J(x,t,\tau)d\tau$$

is uniformly L_q-convergent for $x \in A$, $t \in \langle h_0, T\rangle$ ($h_0 \in (0,T)$) and consequently the function (3.29) is L_q-continuous in H_0.

Now let us introduce the function

$$W_h(x,t) = \int_0^{t-h} J(x,t,\tau)d\tau, \quad x \in A, \quad t \in \langle h,T\rangle \qquad (0 < h < T).$$

There exists L_q-derivative

$$D_t W_h(x,t) = \int_0^{t-h} D_t J(x,t,\tau)d\tau + J(x,t,t-h).$$

Take an arbitrary sequence $\{h_m\}$, $0 < h_m < h_0$ ($h_0 \in (0,T)$ being an arbitrary fixed number) such that $h_m \to h_0$ as $m \to \infty$. Then the sequence

$$\left\{ \int_0^{t-h_m} D_t J(x,t,\tau)d\tau \right\}$$

is uniformly L_q-convergent to the integral (3.29) for $x \in A$, $t \in \langle h_0, T\rangle$. Further we have

$$(3.30) \qquad J(x,t,t-h_m) - f(x,t) = I_1' + I_2' + I_3' + I_4',$$

where

$$I_1' = \int_{K_1} G_0(x-\xi,t,\xi,t-h_m)\left[f(\xi,t-h_m) - f(x,t-h_m)\right] d\xi,$$

$$I_2' = \int_{K_1'} G_0(x-\xi,t,\xi,t-h_m)\left[f(\xi,t-h_m) - f(x,t-h_m)\right] d\xi,$$

$$I_3' = f(x,t-h_m)\left[\int_{R^n} G_o(x-\xi,t,\xi,t-h_m)d\xi - 1\right],$$

$$I_4' = f(x,t-h_m) - f(x,t)$$

and K_1, K_1' are defined by (2.20). Evaluating the norms $\|I_j'\|_q$ (j=1,2) like as $\|I_j\|_q$ (see (3.22) and (3.23)), we conclude that for j = 1,2

(3.31) $$\lim_{m\to\infty}\|I_j'\|_q = 0$$

uniformly with respect to $x \in A$, $t \in \langle h_o,T\rangle$. Obviously (3.31) holds for j = 4, too. The validity of (3.31) for j = 3 is a consequence of the relation

(3.32) $$\lim_{m\to\infty}\left\|\int_{R^n} G_o(x-\xi,t,\xi,t-h_m)d\xi - 1\right\|_\infty = 0^{7)}.$$

So we have

$$\lim_{m\to\infty}\|J(x,t,t-h_m) - f(x,t)\|_q = 0$$

uniformly with respect to $x \in A$, $t \in \langle h_o,T\rangle$ and consequently the sequence $\left\{D_t W_{h_m}(x,t)\right\}$ is uniformly L_q-convergent to the function

$$\int_0^t D_t J(x,t,\tau)d\tau + f(x,t), \quad x \in A, \quad t \in \langle h_o,T\rangle.$$

---

7) This relation can be proved by the similar considerations as those for relations (2.24), (2.26) and (2.27).

Hence, in virtue of the uniform L_q-convergence of the sequence $\left\{W_{h_m}(x,t)\right\}$ to the function $W(x,t)$ in $A\times\langle h_o,T\rangle$, there exists L_q-derivative

$$(3.33)\qquad D_tW(x,t) = \int_0^t D_tJ(x,t,\tau)d\tau + f(x,t), \quad (x,t)\in H_o,$$

which is L_q-continuous in H_o. This completes the proof.

4. A fundamental solution

In this section we discuss the existence of a fundamental solution of equation (0.1). The following assumption will be additionally needed.

(4.I) The coefficients a_k, $|k|\leqslant 2b-1$ satisfy assumptions (3.I) and (3.II), where the norm $\|\cdot\|_\infty$ in (3.II) is replaced by $\|\cdot\|_p$, $(1\leqslant p\leqslant\infty)$.

T h e o r e m 4.1. If assumptions (3.I)-(3.III) and (4.I) are fulfilled, then there exists a complex random function $Z(x,t,\xi,\tau)$ such that:

1^o it has the properties 1^o, 2^o for $|m|\leqslant 2b-1$ and 5^o from Theorem 2.1;

2^o there exist L_p-derivatives D_x^mZ, $D_tZ\in C(A_o,L_p(\Omega))$, $|m| = 2b$, which fulfil the inequalities

$$(4.1)\qquad \left\|D_x^mZ\right\|_p,\ \left\|D_tZ\right\|_p \leqslant B(t-\tau)^{-\frac{n}{2b}-1}\exp\left[-\frac{\delta_c|x-\xi|^{b'}}{(t-\tau)^{b''}}\right];$$

3^o $Z(x,t,\xi,\tau)$ has the property 4^o from Theorem 2.1 with respect to the equation (0.1).

The above-mentioned function $Z(x,t,\xi,\tau)$ is called a fundamental solution of equation (0.1).

P r o o f . We shall prove that

$$(4.2)\qquad Z(x,t,\xi,\tau) = G_o(x-\xi,t,\xi,\tau) + \int_\tau^t\int_{R^n} G_o(x-y,t,y,\beta)\varphi(y,\beta,\xi,\tau)dyd\beta,$$

where G_o is defined by (3.4) and φ is a solution of the integral equation

$$(4.3)\quad \varphi(x,t,\xi,\tau) = K(x,t,\xi,\tau) + \int_\tau^t \int_{R^n} K(x,t,y,\beta)\varphi(y,\beta,\xi,\tau)dyd\beta$$

with

$$(4.4)\quad K(x,t,\xi,\tau) = \sum_{|k|\leqslant 2b} D_x^k G_o(x-\xi,t,\xi,\tau) - D_t G_o(x-\xi,t,\xi,\tau).$$

We have

$$K(x,t,\xi,\tau) = \sum_{|k|=2b} \left[a_k(x,t) - a_k(\xi,\tau)\right] D_x^k G_o(x-\xi,t,\xi,\tau) +$$

$$+ \sum_{|k|\leqslant 2b-1} a_k(x,t) D_x^k G_o(x-\xi,t,\xi,\tau).$$

This implies, by (3.7) and (3.II), the estimate

$$(4.5)\quad \|K(x,t,\xi,\tau)\|_\infty \leqslant C_1(t-\tau)^{-\frac{n+2b-\alpha}{2b}} \exp\left[-\frac{\delta'|x-\xi|^{b'}}{(t-\tau)^{b''}}\right].$$

Now consider the series

$$(4.6)\quad \varphi(x,t,\xi,\tau) = \sum_{m=1}^{\infty} K_m(x,t,\xi,\tau),$$

where

$$K_1(x,t,\xi,\tau) = K(x,t,\xi,\tau),$$

$$K_{m+1}(x,t,\xi,\tau) = \int_\tau^t \int_{R^n} K(x,t,y,\beta)K_m(y,\beta,\xi,\tau)dyd\beta, \quad m=1,2,\ldots$$

For the norms $\|K_m\|_\infty$ one can obtain the estimates analogous to those of [3] (p.254). Hence, it follows the uniform L_∞-convergence of the series (4.6) for $x, \xi \in R^n$, $t-\tau \geq h$ ($h \in (0,T)$ being a constant). Since $K_m \in C(A_0, L_\infty(\Omega))$, therefore also $\varphi \in C(A_0, L_\infty(\Omega))$. Moreover, we have

$$(4.7) \qquad \|\varphi(x,t,\xi,\tau)\|_\infty \leq C_2(t-\tau)^{-\frac{n+2b-\alpha}{2b}} \exp\left[-\frac{\delta_1'|x-\xi|^{b'}}{(t-\tau)^{b''}}\right].$$

Taking into account the estimates of the norms $\|K_m\|_\infty$, (4.5), (4.7) and the theorem on termwise integration of functional series, one can find that the function (4.6) is a solution of the equation (4.3).

Now we shall consider the difference

$$(4.8) \qquad \Delta\varphi = \varphi(x,t,\xi,\tau) - \varphi(x',t,\xi,\tau)$$

under the condition

$$(4.9) \qquad |x - x'|^{2b} \leq t-\tau.$$

For this purpose we first evaluate the difference

$$(4.10) \qquad \Delta K = K(x,t,\xi,\tau) - K(x',t,\xi,\tau) = F_1+F_2+F_3+F_4,$$

where

$$F_1 = \sum_{|k|=2b} \left[a_k(x,t)-a_k(x',t)\right] D_x^k G_0(x-\xi,t,\xi,\tau),$$

$$F_2 = \sum_{|k|=2b} \left[a_k(x,t)-a_k(\xi,t)\right] \left[D_x^k G_0(x-\xi,t,\xi,\tau) - D_x^k G_0(x'-\xi,t,\xi,\tau)\right],$$

$$F_3 = \sum_{|k|\leq 2b-1} \left[a_k(x,t)-a_k(x',t)\right] D_x^k G_o(x-\xi,t,\xi,\tau),$$

$$F_4 = \sum_{|k|\leq 2b-1} a_k(x',t)\left[D_x^k G_o(x-\xi,t,\xi,\tau)-D_x^k G_o(x'-\xi,t,\xi,\tau)\right].$$

Using Lemma 1.3, the estimate (3.7) and the condition (4.9), we get for some $\theta \in \langle -1,1\rangle$ the inequality

$$(4.11)\quad \left\|D_x^k G_o(x-\xi,t,\xi,\tau)-D_x^k G_o(x'-\xi,t,\xi,\tau)\right\|_\infty \leq$$

$$\leq C(|k|)\,|x-x'|\,(t-\tau)^{-\frac{n+|k|+1}{2b}} \exp\left[-\frac{\delta\,|x-\xi+\theta(x-x')|^{b'}}{(t-\tau)^{b''}}\right] \leq$$

$$\leq C'(|k|)\,|x-x'|\,(t-\tau)^{-\frac{n+|k|+1}{2b}} \exp\left[-\frac{\delta_2'|x-\xi|^{b'}}{(t-\tau)^{b''}}\right],$$

where $C(|k|)$, $C'(|k|)$ are positive constants depending on $|k|$ and $\delta_2' \in (0,\delta')$. Hence, by assumption (3.II), we get

$$(4.12)\quad \|F_2\|_\infty \leq C_3|x-x'|^{\alpha_1}(t-\tau)^{-\frac{n+2b-\alpha_2}{2b}} \exp\left[-\frac{\delta_3'|x-\xi|^{b'}}{(t-\tau)^{b''}}\right],$$

where $0<\alpha_1<\alpha$, $\alpha_2 = \alpha-\alpha_1$, $0<\delta_3'<\delta_2'$. Similarly, the estimates (4.11), (3.7) and assumptions (3.II), (4.I) imply the estimates of the norms $\|F_3\|_p$, $\|F_j\|_\infty$ $(j=1,4)$ of the form (4.12). Therefore, by (4.10), we obtain, under the condition (4.9), the estimate

$$(4.13)\quad \|\Delta K\|_p \leq C_4|x-x'|^{\alpha_1}(t-\tau)^{-\frac{n+2b-\alpha_2}{2b}} \exp\left[-\frac{\delta_3|x-\xi|^{b'}}{(t-\tau)^{b''}}\right].$$

In view of the estimates (4.7), (4.12) and by Lemma 7 ([3], p.253), the function

$$W(x,t,\xi,\tau) = \int_{\tau}^{t}\int_{R^n} K(x,t,y,\beta)\varphi(y,\beta,\xi,\tau)dyd\beta$$

satisfies, under the condition (4.9), the inequality of the form (4.13) with C_4 and δ_3' replaced by some constants $C_5 > 0$ and $\delta_4' \in (0,\delta_3')$, respectively. Hence, if (4.9) holds, then with the aid of (4.3), (4.8) and (4.13) we conclude that for the norm $\|\Delta\varphi\|_p$ remains valid the estimate (4.13) with C_4 and δ_3' replaced by $C_6 > 0$ and δ_4', respectively.

Now we shall consider the function

$$(4.14) \qquad V(x,t,\xi,\tau) = \int_{\tau}^{t}\int_{R^n} G_0(x-y,t,y,\beta)\varphi(y,\beta,\xi,\tau)dyd\beta.$$

According to the estimates (3.7) and (4.7), Lemma 7 of [3] and Remark 4.9 of [4], we have $V \in C(A_0,L_\infty(\Omega))$ and for $|m| \leq 2b-1$ there exist L_∞-derivatives $D_x^m V \in C(A_0,L_\infty(\Omega))$ given by formula

$$(4.15) \qquad D_x^m V(x,t,\xi,\tau) = \int_{\tau}^{t}\int_{R^n} D_x^m G_0(x-y,t,y,\beta)\varphi(y,\beta,\xi,\tau)dyd\beta.$$

Moreover, these derivatives satisfy the inequalities

$$(4.16) \qquad \|D_x^m V(x,t,\xi,\tau)\|_\infty \leq C_7(t-\tau)^{-\frac{n+|m|-\alpha}{2b}} \exp\left[-\frac{\delta_5'|x-\xi|^{b'}}{(t-\tau)^{b''}}\right].$$

In order to prove the existence of derivatives $D_x^m V$, $|m| = 2b$, and $D_t V$ we introduce the function

$$(4.17) \qquad J(x,t,\beta,\xi,\tau) = \int_{R^n} G_0(x-y,t,y,\beta)\varphi(y,\beta,\xi,\tau)dy$$

in the set $A_4 = \{(x,t,\beta,\xi,\tau): x, \xi \in R^n,\ 0 \leq \tau < \beta < t \leq T\}$. Like as for the function (4.14) we have $J \in C(A_4,L_\infty(\Omega))$ and there exist L_∞-derivatives $D_x^m J \in C(A_4, L_\infty(\Omega))$ defined by formula

$$(4.18) \qquad D_x^m J(x,t,\beta,\xi,\tau) = \int_{R^n} D_x^m G_0(x-y,t,y,\beta)\varphi(y,\beta,\xi,\tau)dy.$$

Moreover, for $|m| = 2b$ holds the inequality

$$(4.19) \qquad \| D_x^m J \|_\infty \leq C_8(\beta-\tau)^{-1+\frac{\alpha}{2b}}(t-\tau)^{-\frac{n}{2b}-1} \exp\left[-\frac{\delta_6'|x-\xi|^{b'}}{(t-\tau)^{b''}}\right]$$

for $x, \xi \in R^n$, $0 \leq \tau < t \leq T$, $\tau < \beta < \frac{t+\tau}{2} \equiv t_1$.

In virtue of (4.19) and Remark 4.9 of [4], the function

$$V_1(x,t,\xi,\tau) = \int_\tau^{t_1} J(x,t,\beta,\xi,\tau)d\beta$$

possesses L_∞-derivatives $D_x^m V_1 \in C(A_0, L_\infty(\Omega))$, $|m| = 2b$, given by formula

$$D_x^m V_1(x,t,\xi,\tau) = \int_\tau^{t_1} D_x^m J(x,t,\beta,\xi,\tau)d\beta$$

and there is satisfied the inequality

$$\| D_x^m V_1 \|_\infty \leq C_9(t-\tau)^{-\frac{n+2b-\alpha}{2b}} \exp\left[-\frac{\delta_6'|x-\xi|^{b'}}{(t-\tau)^{b''}}\right] .$$

It remains to prove the existence of derivatives $D_x^m V_2$, $|m| = 2b$, of the function

$$V_2(x,t,\xi,\tau) = \int_{t_1}^{t} J(x,t,\beta,\xi,\tau)d\beta.$$

For this purpose let us denote

$$A_5 = \left\{(x,t,\beta,\xi,\tau) : x, \xi \in R^n, \ 0 \leq \tau < t \leq T, \ t_1 < \beta < t\right\},$$

(4.20) $$E_a = \left\{ y : |y-x|^{2b} \leq a \right\}, \quad E'_a = \left\{ y : |y-x|^{2b} > a \right\}, \quad a > 0.$$

Now we write the derivatives $D^m_x J$, $|m| = 2b$, in the set Δ_T, in the following form

(4.21) $$D^m_x J = J_1 + J_2 + J_3 + J_4,$$

where

$$J_1 = \int_{E_a} D^m_x G_0(x-y,t,y,\beta)\left[\varphi(y,\beta,\xi,\tau) - \varphi(x,\beta,\xi,\tau)\right] dy,$$

$$J_2 = \int_{E'_a} D^m_x G_0(x-y,t,y,\beta)\varphi(y,\beta,\xi,\tau)dy,$$

$$J_3 = -\varphi(x,\beta,\xi,\tau) \int_{E'_a} D^m_x G_0(x-y,t,y,\beta)dy,$$

$$J_4 = \varphi(x,\beta,\xi,\tau) \int_{R^n} D^m_x G_0(x-y,t,y,\beta)dy$$

and $a = \beta - \tau$. In view of (3.24), we have

$$J_4 = \varphi(x,\beta,\xi,\tau) \int_{R^n} \left[D^m_x G_0(x-y,t,y,\beta) - D^m_x G_0(x-y,t,z,\beta)\Big|_{z=x} \right] dy.$$

Hence, using estimates (3.11) and (4.7), it follows that

(4.22) $$\left\| J_4 \right\|_\infty \leq C_{10}(t-\tau)^{-\frac{n+2b-\alpha}{2b}} (t-\beta)^{-1+\frac{\alpha}{2b}} \exp\left[- \frac{\delta_1 |x-\xi|^{b'}}{(t-\tau)^{b''}} \right].$$

The estimates (3.7) and (4.13) for φ imply, by Lemma 7 of [3], the inequality

(4.23) $$\left\| J_1 \right\|_p \leq C_{11}(t-\tau)^{-\frac{n+2b-\alpha}{2b}} (t-\beta)^{-1+\frac{\alpha}{2b}} \exp\left[- \frac{\delta'_7 |x-\xi|^{b'}}{(t-\tau)^{b''}} \right].$$

Taking into account the estimates (3.7), (4.7) and the above-mentioned Lemma 7 we find that

$$(4.24)\quad \|J_k\|_\infty \leqslant C_{12}(t-\beta)^{-1+\frac{\alpha}{2b}}(t-\tau)^{-\frac{n}{2b}-1}\exp\left[-\frac{\delta_8'|x-\xi|^{b'}}{(t-\tau)^{b''}}\right],$$

k=2,3.

Relations (4.21)-(4.24) immediately imply, for $|m| = 2b$, the estimate

$$\|D_x^m J\|_p \leqslant C_{13}(t-\beta)^{-1+\frac{\alpha}{2b}}(t-\tau)^{-\frac{n}{2b}-1}\exp\left[-\frac{\delta_9'|x-\xi|^{b'}}{(t-\tau)^{b''}}\right] \quad \text{in } A_5.$$

So, by Remark 4.9 of [4], there exist L_p-derivatives $D_x^m V_2 \in C(A_0, L_p(\Omega))$ given by formula

$$D_x^m V_2(x,t,\xi,\tau) = \int_{t_1}^{t} D_x^m J(x,t,\beta,\xi,\tau)d\beta$$

and there holds the estimate

$$(4.25)\quad \|D_x^m V_2\|_p \leqslant C_{14}(t-\tau)^{-\frac{n+2b-\alpha}{2b}}\exp\left[-\frac{\delta_9'|x-\xi|^{b'}}{(t-\tau)^{b''}}\right].$$

Thus we have proved the existence of L_p-derivatives $D_x^m V \in C(A_0, L_p(\Omega))$, $|m| = 2b$, defined by formula

$$(4.26)\quad D_x^m V(x,t,\xi,\tau) = \int_{\tau}^{t} D_x^m J(x,t,\beta,\xi,\tau)d\beta.$$

Moreover, for $\|D_x^m V\|_p$ remains valid the estimate (4.25) (with possibly other constants C_{14} and δ_9') and the integral (4.26) is uniformly L_p-convergent in the set

$$A_6 = \{(x,t,\xi,\tau) : x,\xi \in R^n,\ t-\tau \geqslant 3h > 0\}.$$

Similarly to $D_x^m J$ there exists L_p-derivative $D_t J \in C(A_0, L_p(\Omega))$. Observing that

$$D_t J(x,t,\beta,\xi,\tau) = \sum_{|m|=2b} \int_{R^n} D_x^m G_0(x-y,t,y,\beta)\left[a_m(y,\beta)\varphi(y,\beta,\xi,\tau)\right] dy$$

and using the above considerations,we conclude the uniform L_p-convergence of the integral

$$(4.27) \qquad \int_\tau^t D_t J(x,t,\beta,\xi,\tau)d\beta$$

in the set A_6. Moreover, for the norm $\|\cdot\|_p$ of the integral (4.27) holds the estimate (4.25) in the set A_0.

Like in the proof of Theorem 3.1 we introduce the function

$$V_h(x,t,\xi,\tau) = \int_{\tau+h}^{t-h} J(x,t,\beta,\xi,\tau)d\beta \quad \text{in } A_6.$$

This function possesses L_∞-derivative

$$D_t V_h(x,t,\xi,\tau) = \int_{\tau+h}^{t-h} D_t J(x,t,\beta,\xi,\tau)d\beta + J(x,t,t-h,\xi,\tau).$$

Take an arbitrary sequence $\{h_m\}$, $0 < h_m < h$, tending to zero as $m \to \infty$. Then the sequence

$$\left\{ \int_{\tau+h_m}^{t-h_m} D_t J(x,t,\beta,\xi,\tau)d\beta \right\}$$

is uniformly L_p-convergent to the integral (4.27) in the set A_6. We write the expression $J(x,t,t-h_m,\xi,\tau) - \varphi(x,t,\xi,\tau)$ in the form similar to (3.30) with K_1 and K_1' replaced by E_h and E_h' (defined by (4.20)), respectively. Next, using the estimates (3.7) (for $|m| = 0$), (4.13) (for φ), (4.7) and relation (3.32),we find that

$$\lim_{m\to\infty} \|J(x,t,t-h_m,\xi,\tau) - \varphi(x,t,\xi,\tau)\|_p = 0$$

uniformly with respect to $x, \xi \in A$ and $t-\tau \geqslant 3h$ ($A \subset R^n$ being a bounded domain). From the above considerations it follows the existence of L_p-derivative $D_tV \in C(A_0, L_p(\Omega))$ given by formula

$$(4.28) \qquad D_tV(x,t,\xi,\tau) = \int_\tau^t D_tJ(x,t,\beta,\xi,\tau)d\beta + \varphi(x,t,\xi,\tau).$$

This implies the estimate of the form (4.25) for the norm $\|D_tV\|_p$. So we have proved assertion 2^0. Assertion 3^0 follows immediately from relations (4.28), (4.26), (4.17), (4.18), (4.14), (4.15) and (4.2)-(4.4).

It remains to prove the property 5^0 of Theorem 2.1 with respect to the equation (0.1). For this purpose it suffices to observe that

$$\int_{R^n} Z(x,t,\xi,\tau)g(\xi)d\xi = \int_{R^n} G_0(x-\xi,t,\xi,\tau)g(\xi)d\xi + \\ + \int_{R^n} V(x,t,\xi,\tau)g(\xi)d\xi$$

and next to apply assertion 5^0 of Theorem 2.1 (for function $Z(x,t,\xi,\tau) = G_0(x-\xi,t,\xi,\tau)$) and the estimate (4.16) with $|m| = 0$.

5. The Cauchy problem

In this section the fundamental solution $Z(x,t,\xi,\tau)$ constructed in the previous section will be used in proving of the existence of a solution of the Cauchy problem (0.2), (0.3).

T h e o r e m 5.1. Let assumptions (3.I)-(3.IV), (4.I) be satisfied and suppose that a function $g \in C(R^n, L_q(\Omega))$ is L_q-bounded, where $r^{-1} = p^{-1} + q^{-1} \leqslant 1$. Then the function

$$(5.1)\quad u(x,t) = \int_{R^n} Z(x,t,\xi,0)g(\xi)d\xi - \int_0^t \int_{R^n} Z(x,t,\xi,\tau)f(\xi,\tau)d\xi d\tau$$

has the following properties:

1^o $u \in C(H,L_q(\Omega))$ and for $|k| \leqslant 2b-1$ there exist L_q-derivatives $D_x^k u \in C(H_o,L_q(\Omega))$;

2^o there exist L_r-derivatives $D_x^k u$, $|k| = 2b$ and $D_t u$ which are L_r-continuous in H_o;

3^o $u(x,t)$ is a solution of the problem (0.2), (0.3).

P r o o f . The method of proving is similar to that applied in the proof of Theorem 3.1 of [4]. In view of the estimate (2.3) the function

$$(5.2)\qquad u_1(x,t) = \int_{R^n} Z(x,t,\xi,0)g(\xi)d\xi$$

is L_q-continuous in H_o. Using assertion 5^o of Theorem 2.1 (with respect to the equation (0.1)) and setting additionally

$$(5.3)\qquad u_1(x,0) = g(x),\ x \in R^n,$$

we conclude that $u_1 \in C(H,L_q(\Omega))$. By (2.4) for $|m| \leqslant 2b-1$, there exist L_q-derivatives $D_x^m u_1 \in C(H_o,L_q(\Omega))$ given by formula

$$(5.4)\qquad D_x^m u_1(x,t) = \int_{R^n} D_x^m Z(x,t,\xi,0)g(\xi)d\xi,\quad |m| \leqslant 2b-1.$$

Similarly, the estimates (4.1) imply the existence of L_r-derivatives $D_x^m u_1$, $D_t u_1 \in C(H_o,L_r(\Omega))$, $|m| = 2b$, where $D_x^m u_1$ are defined by (5.4) and

$$(5.5)\qquad D_t u_1(x,t) = \int_{R^n} D_t Z(x,t,\xi,0)g(\xi)d\xi.$$

Taking into considerations formulas (5.2), (5.4), (5.5) and assertion 3^{o} of Theorem 4.1, we find that

$$Lu_1(x,t) = 0, \quad (x,t) \in H_0. \tag{5.6}$$

Now let us denote

$$u_2(x,t) = \int_0^t \int_{R^n} Z(x,t,\xi,\tau)f(\xi,\tau)d\xi d\tau. \tag{5.7}$$

The estimate (2.3) implies that $u_2 \in C(H_0, L_q(\Omega))$ and $\lim_{t\to 0} \|u_2(x,t)\|_q = 0$ uniformly with respect to $x \in R^n$. Hence, setting additionally

$$u_2(x,0) = 0, \quad x \in R^n, \tag{5.8}$$

we conclude that $u_2 \in C(H, L_q(\Omega))$. By (2.3) with $|m| \leqslant 2b-1$, there exist L_q-derivatives $D_x^m u_2 \in C(H_0, L_q(\Omega))$ given by formula

$$D_x^m u_2(x,t) = \int_0^t \int_{R^n} D_x^m Z(x,t,\xi,\tau)f(\xi,\tau)d\xi \quad , \quad |m| \leqslant 2b-1. \tag{5.9}$$

In order to show the existence of derivatives $D_x^m u_2$ for $|m| = 2b$ and $D_t u_2$ we write

$$u_2(x,t) = W(x,t) + W_1(x,t), \tag{5.10}$$

where

$$W_1(x,t) = \int_0^t \int_{R^n} G_0(x-\xi,t,\xi,\tau)f_1(\xi,\tau)d\xi d\tau, \tag{5.11}$$

$$f_1(x,t) = \int_0^t \int_{R^n} \varphi(x,t,\xi,\tau)f(\xi,\tau)d\xi d\tau \tag{5.12}$$

and $W(x,t)$ is defined by (3.13). It results from (4.7) that $f_1 \in C(H, L_q(\Omega))$. Taking into account (3.IV), (4.7) and the estimate

$$\|\Delta\varphi\|_p \leqslant C' |x-x'|^{\alpha_1} (t-\tau)^{-\frac{n+2b-\alpha_2}{2b}} \left\{ \exp\left[- \frac{\delta''|x-\xi|^{b'}}{(t-\tau)^{b''}} \right] + \exp\left[- \frac{\delta''|x'-\xi|^{b'}}{(t-\tau)^{b''}} \right] \right\}$$

following from (4.7) and (4.13) (with $K = \varphi$), we obtain

$$\| f_1(x,t) - f_1(x',t) \|_r \leqslant C''|x-x'|^{\alpha_1}, \quad x,x' \in A, \quad t \in \langle 0,T \rangle,$$

$A \subset R^n$ being any bounded domain.

Hence, by (5.10), (5.11) and by Theorem 2.1, there exist L_r-derivatives $D_x^m u_2$, $D_t u_2 \in C(H_0, L_r(\Omega))$, $|m| = 2b$, given by formulas

$$(5.13) \qquad D_x^m u_2 = \int_0^t d\tau \int_{R^n} D_t G_0(x-\xi,t,\xi,\tau) \left[f(\xi,\tau) + f_1(\xi,\tau) \right] d\xi,$$

$$(5.14) \qquad D_t u_2 = \int_0^t d\tau \int_{R^n} D_t G_0(x-\xi,t,\xi,\tau) \left[f(\xi,\tau) + f_1(\xi,\tau) \right] d\xi \; +$$

$$+ f(x,t) + f_1(x,t).$$

Combining (5.7), (5.9), (5.13), (5.14), (4.2)-(4.4) and (5.12) we conclude that

$$(5.15) \qquad L u_2(x,t) = -f(x,t), \quad (x,t) \in H_0.$$

Relations (5.6), (5.15), (5.2), (5.7), (5.1), (5.3) and (5.8) immediately imply assertion 3°. This completes the proof.

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