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ON NULL GEODESIC COLLINEATION IN CONFORMALLY 2-RECURRENT RIEMANNIAN MANIFOLDS

1. Introduction

A non-flat n-dimensional $(n > 2)$ Riemannian manifold is said to be of recurrent curvature [11] (briefly, a recurrent manifold) if its curvature tensor satisfies the condition

$$R_{hijk,l} = c_l R_{hijk}$$

for some non-zero vector field c_j, where the comma indicates covariant differentiation with respect to the metric of the manifold.

As a generalization of the concept of a recurrent manifold, Lichnerowicz [10] initiated investigations of n-dimensional $(n > 2)$ Riemannian manifolds whose curvature tensors satisfy relation of the form

$$R_{hijk,lm} = a_{lm} R_{hijk}.$$

Non-flat manifolds of such type (i.e. satisfying the above relation for some tensor a_{ij}) are called second order recurrent or, briefly, 2-recurrent manifolds. Roter proved [7] that the recurrence tensor a_{ij} of a 2-recurrent manifold is necessarily symmetric.

According to Adati and Miyazawa [1] an n-dimensional $(n > 3)$ Riemannian manifold is said to be conformally recurrent if its Weyl conformal curvature tensor

$$(1)\qquad C^h{}_{ijk} = R^h{}_{ijk} - \frac{1}{n-2}\left(g_{ij}R^h{}_k - g_{ik}R^h{}_j + \delta^h_k R_{ij} - \delta^h_j R_{ik}\right) + \frac{R}{(n-1)(n-2)}\left(\delta^h_k\, g_{ij} - \delta^h_j\, g_{ik}\right)$$

satisfies the relation

$$(2)\qquad C^h{}_{ijk,l} = \sigma_l\, C^h{}_{ijk},$$

for some non-zero vector field σ_l.

In this paper we consider n-dimensional $(n > 3)$ analytic Riemannian manifolds M whose Weyl conformal curvature tensor satisfies the condition

$$(3)\qquad C^h{}_{ijk,lm} = \varphi_{lm}\, C^h{}_{ijk},$$

for some non-zero tensor field φ_{lm}.

The manifolds of this type are called conformally 2-recurrent.

We assume that

$$(4)\qquad \varphi_{lm} = \varphi_{ml},$$

$$(5)\qquad C^h{}_{ijk} \neq 0 \quad \text{(for at least one point of M).}$$

Grycak has proved [3] that every analytic Riemannian manifold of dimension $n > 4$ satisfies $C^h{}_{ijk,lm} = C^h{}_{ijk,ml}$ if and only if $R^h{}_{ijk,lm} = R^h{}_{ijk,ml}$.

It is immediate that every recurrent or 2-recurrent, as well as every conformally recurrent Riemannian manifold satisfies the condition (3) and if the recurrence vector in (2) is locally a gradient then the 2-recurrence tensor in (3) is symmetric.

According to Katzin and Levine, a vector field v on a Riemannian manifold is said to be a null geodesic collineation if

$$L_v \Gamma^h_{ij} = g^{hr} g_{ij} Q_{,r},$$

where Q is a certain function, and L_v denotes the Lie derivative with respect to v (for geometrical interpretation, see [9]).

The purpose of this paper is to prove that null geodesic collineations in analytic conformally 2-recurrent Riemannian manifolds satisfying (4), (5) are necessarily affine.

Throughout this paper, by a manifold we shall mean a connected and paracompact Hausdorff manifold. The Riemannian metrics involved need not be positive definite.

2. Preliminary results

We shall use two theorems due to Grycak.

Theorem A. ([4], Theorem 1). Let M be an n-dimensional $(n \geq 3)$ Riemannian manifold (not necessarily of definite metric). If B_{hijk} is a generalized curvature tensor on M satisfying

$$B_{hijk,lm} = B_{hijk,ml},$$

and P_i is a vector field on M having the property

$$v_r R^r{}_{ijk} = P_k g_{ij} - P_j g_{ik},$$

for a suitably chosen vector field v_i, then

$$P_h \left(B_{lijk} - \frac{1}{n(n-1)} S \left(g_{ij} g_{lk} - g_{ik} g_{lj}\right)\right) = 0,$$

where $S = B_{rijs} g^{rs} g^{ij}$.

Theorem B. ([5], Theorem 1). If B_{hijk} is a generalized curvature tensor on a Riemannian manifold M $(\dim M = n \geq 3)$ such that

$$B_{hijk,lm} = B_{hijk,ml},$$

and a_{ij}, b_{ij} are symmetric tensor fields satisfying

$$a_{ij,lm} - a_{ij,ml} = b_{im} g_{lj} + b_{jm} g_{il} - b_{il} g_{jm} - b_{jl} g_{im}$$

then

$$\left(b_{lm} - \frac{b}{n} g_{lm}\right)\left(B_{hijk} - \frac{S}{n(n-1)} (g_{hk} g_{ij} - g_{hj} g_{ik})\right) = 0,$$

where $S = g^{rs} g^{pq} B_{rpqs}$, $b = b_{rs} g^{rs}$.

Roter [8] proved that null geodesic collineations in general Riemannian manifolds satisfy the following relations

(6) $a_{ri} R^{r}{}_{hjk} + a_{rj} R^{r}{}_{hki} + a_{rk} R^{r}{}_{hij} = 0,$

(7) $a_{ri} R^{r}{}_{k} = a_{rk} R^{r}{}_{i},$

(8) $a_{hi,j} = A_h g_{ij} + A_i g_{hj},$

(9) $a_{hi,jk} - a_{hi,kj} = A_{h,k} g_{ij} + A_{i,k} g_{hj} - A_{h,j} g_{ik} - A_{i,j} g_{hk},$

where $A^h = g^{hr} Q_{,r}$, $a_{ij} = L_v g_{ij}$.

Furthermore Roter [8] has proved the following

T h e o r e m C. If an ARS_n-space with $R_{ij} \neq 0$ admits a null geodesic collineation, then this collineation is necessarily an affine one.
(A manifold is called almost Ricci-symmetric space or, briefly, ARS_n-space if its Ricci tensor satisfies the relation $R_{ij,k} = R_{ik,j}$).

L e m m a 1. If an analytic non-conformally flat Riemannian manifold admits a null geodesic collineation and condition $C^{h}{}_{ijk,lm} = C^{h}{}_{ijk,ml}$ holds, then the vector A_j satisfies the relations

(10) $$A_{l,m} = \frac{B}{n} g_{lm},$$

(11) $$B_{,k} = 0,$$

(12) $$A_r R^r{}_{lmk} = 0, \quad A_r R^r{}_k = 0,$$

(13) $$A_r R^r{}_{ijk,l} = -\frac{B}{n} R_{lijk}, \quad A_r R^r{}_{k,l} = -\frac{B}{n} R_{lk},$$

(14) $$A^r R_{hijk,r} = -\frac{2B}{n} R_{hijk}, \quad A^r R_{hk,r} = -\frac{2B}{n} R_{hk},$$

$$A^r R_{,r} = -\frac{2B}{n} R, \quad A^r C_{hijk,r} = -\frac{2B}{n} C_{hijk},$$

where $B = A_{r,s} g^{rs}$.

P r o o f . Applying Theorem B for $b_{ij} = A_{i,j}$, $a_{ij} = L_v g_{ij}$, $B_{hijk} = C_{hijk}$ (the hypothesis are satisfied in view of (9)), we obtain

$$\left(A_{l,m} - \frac{B}{n} g_{lm}\right) \left(C_{hijk} - \frac{S}{n(n-1)} (g_{hk} g_{ij} - g_{hj} g_{ik})\right) = 0$$

and, since $S = C_{rpqs} g^{rs} g^{pq} = 0$ and (5) holds, we have (10).

Differentiating (10) covariantly, we have

$$A_{l,mk} = \frac{B_{,k}}{n} g_{lm}, \quad A_{l,km} = \frac{B_{,m}}{n} g_{lk}.$$

Making use of Ricci identity, we obtain

$$A_{l,mk} - A_{l,km} = -R^r{}_{lmk} A_r = \frac{1}{n} (B_{,k} g_{lm} - B_{,m} g_{lk})$$

whence

$$nA_r R^r{}_{lmk} = -B_{,k} g_{lm} + B_{,m} g_{lk}.$$

Applying now Theorem A for $v_r = \frac{A_r}{n}$, $P_k = -B_{,k}$, $B_{hijk} = C_{hijk}$, we obtain (11) and immediately (using the last equation) (12). Differentiating (12) covariantly and substituting (10) we easily obtain (13). Transvecting now the second Bianchi identity

$$R_{hijk,l} + R_{hikl,j} + R_{hilj,k} = 0$$

with A^l and using (13), we immediately obtain (14). This completes the proof.

Grycak has proved [5] that null geodesic collineations in a Riemannian manifold satisfy the relation

$$(15) \qquad a_{ri}\, C^r{}_{hjk} + a_{rj}\, C^r{}_{hki} + a_{rk}\, C^r{}_{hij} = 0.$$

With help of this equation and Lemma 1, we prove the following lemma.

L e m m a 2. Every null geodesic collineation in conformally 2-recurrent Riemannian manifold with symmetric 2-recurrence tensor satisfies the condition

$$(16) \qquad B\left[\frac{R}{n-1}\,(A_k\, g_{jh} - A_j\, g_{kh}) - (A_k\, R_{jh} - A_j\, R_{kh})\right] = 0.$$

P r o o f . Differentiating (15) twice covariantly and using (10), (8) and (3), (15) we find

$$(17) \qquad g_{il}\, A_r\, C^r{}_{hjk,m} + g_{im}\, A_r\, C^r{}_{hjk,l} + g_{jl}\, A_r\, C^r{}_{hki,m} +$$

$$+ g_{jm}\, A_r\, C^r{}_{hki,l} + g_{kl}\, A_r\, C^r{}_{hij,m} + g_{km}\, A_r\, C^r{}_{hij,l} +$$

$$+ A_i(C_{lhjk,m} + C_{mhjk,l}) + A_j(C_{lhki,m} + C_{mhki,l}) +$$

$$+ A_k(C_{lhij,m} + C_{mhij,l}) + \frac{B}{n}\,(g_{il}\, C_{mhjk} + g_{im}\, C_{lhjk} +$$

$$+ g_{jl}\, C_{mhki} + g_{jm}\, C_{lhki} + g_{kl}\, C_{mhij} + g_{km}\, C_{lhij}) = 0.$$

The contraction of this equation with g^{il} gives

$$n A_r C^r{}_{hjk,m} + g_{jm} A^r C^s{}_{khr,s} - g_{km} A^r C^s{}_{jhr,s} + A^r C_{mhjk,r} + + A_j C^s{}_{khm,s} - A_k C^s{}_{jhm,s} + \frac{B(n-1)}{n} C_{mhjk} = 0.$$

This relation together with (14) and the equation ([2], page 91)

$$C^s{}_{ijk,s} = \frac{n-3}{n-2}\left[(R_{ij,k} - R_{ik,j}) - \frac{1}{2(n-1)}(R_{,k}\, g_{ij} - R_{,j}\, g_{ik})\right]$$

leads immediately to the condition

$$n A_r C^r{}_{hjk,m} + \frac{B(n-3)}{n(n-2)}(g_{km} R_{jh} - g_{jm} R_{kh}) + \frac{B\cdot R(n-3)}{n(n-1)(n-2)}(g_{kh}\, g_{jm} - - g_{km}\, g_{jh}) + A_j \frac{n-3}{n-2}(R_{kh,m} - R_{km,h}) - A_k \frac{n-3}{n-2}(R_{jh,m} - R_{jm,h}) + + \frac{n-3}{2(n-1)(n-2)} R_{,m} (A_k\, g_{hj} - A_j\, g_{hk}) + \frac{B(n-3)}{n} C_{mhjk} = 0.$$

It is easy to see that (1) and (13) give

$$A_r C^r{}_{hjk,m} = -\frac{B}{n} C_{mhjk} - \frac{B}{n(n-2)}(g_{mk} R_{hj} - g_{mj} R_{hk}) + + \frac{B\,R}{n(n-1)(n-2)}(g_{mk}\, g_{hj} - g_{mj}\, g_{hk}) - \frac{1}{n-2}(A_k R_{hj,m} - A_j R_{hk,m}) + + \frac{R_{,m}}{(n-1)(n-2)}(A_k\, g_{hj} - A_j\, g_{hk}).$$

The last two equations lead to the relation

$$-\frac{3B}{n} C_{mhjk} - \frac{3B}{n(n-2)}(g_{mk} R_{hj} - g_{mj} R_{hk}) - \frac{3BR}{n(n-1)(n-2)}(g_{mj}\, g_{hk} - - g_{mk}\, g_{hj}) + \frac{2n-3}{n-2}(A_j R_{hk,m} - A_k R_{hj,m}) + \frac{3R_{,m}}{2(n-2)}(A_k\, g_{hj} - A_j\, g_{hk}) + + \frac{n-3}{n-2}(A_k R_{jm,h} - A_j R_{km,h}) = 0.$$

Transvecting now this equation with A^m and using (1), (12), (13) and (14), we obtain (16).

3. Main result

First we prove two lemmas. Using (1) and (12) we find

$$B\, A_r\, C^r{}_{hjk} = \frac{B}{n-2}\left[\frac{R}{n-1}\,(A_k\, g_{hj} - A_j\, g_{hk}) - (A_k\, R_{hj} - A_j\, R_{hk})\right].$$

This relation together with (16) gives

$$B\, A_r\, C^r{}_{hjk} = 0.$$

Differentiating this equation twice covariantly and using (10), (11), (3) and the above relation, we obtain

(18) $$B(C_{lhjk,m} + C_{mhjk,l}) = 0.$$

Suppose that $B \neq 0$ ($B = \text{const}$). Then, differentiating the equation

$$C_{lhjk,m} + C_{mhjk,l} = 0$$

and substituting (3), we have

$$\varphi_{mp}\, C_{lhjk} + \varphi_{lp}\, C_{mhjk} = 0$$

which, because of $\varphi_{mp} = \varphi_{pm}$, evidently implies $C_{lhjk} = 0$. This result, together with (5) leads to a contradition. Therefore, equation (18) implies $B = 0$.

We have thus proved the following

L e m m a 3. Every null geodesic collineation in a conformally 2-recurrent Riemannian manifold with symmetric 2-recurrence tensor satisfies the following condition:

(19) $$B = A_{r,s}\, g^{rs} = 0.$$

Since $B = 0$, equation (17) can be written in the form

$$g_{il} A_r C^r{}_{hjk,m} + g_{im} A_r C^r{}_{hjk,l} + g_{jl} A_r C^r{}_{hki,m} + g_{jm} A_r C^r{}_{hki,l} +$$

$$+ g_{kl} A_r C^r{}_{hij,m} + g_{km} A_r C^r{}_{hij,l} + A_i(C_{lhjk,m} + C_{mhjk,l}) +$$

$$+ A_j(C_{lhki,m} + C_{mhki,l}) + A_k(C_{lhij,m} + C_{mhij,l}) = 0.$$

Differentiating the last equation covariantly and using (3) and the equation $A_{r,l} = 0$, we have

$$\varphi_{mp} T_{ilhjk} + \varphi_{lp} T_{imhjk} = 0,$$

where

(20) $$T_{ilhjk} = g_{il}A_rC^r{}_{hjk} + g_{jl}A_rC^r{}_{hki} + g_{lk}A_rC^r{}_{hij} +$$
$$+ A_iC_{lhjk} + A_jC_{lhki} + A_kC_{lhij}.$$

Since $\varphi_{mp} = \varphi_{pm}$, this implies

$$\varphi_{mp} T_{ilhjk} = 0$$

and, since $\varphi_{mp} \neq 0$,

(21) $$T_{ilhjk} = 0.$$

The contraction of (20) with g^{il} gives $A_rC^r{}_{hjk} = 0$. The result can be formulated as follows:

L e m m a 4. If a conformally 2-recurrent Riemannian manifold with symmetric 2-recurrence tensor admits a null geodesic collineation, then the vector A_j satisfies condition

(22) $$A_rC^r{}_{hjk} = 0.$$

Now we can proceed to the main result of this paper.

T h e o r e m . If an analytic non conformally flat conformally 2-recurrent Riemannian manifold with $\varphi_{lm} = \varphi_{ml}$ admits a null geodesic collineation, then this collineation is affine.

P r o o f . Going to prove that the parallel vector field A_i vanishes, we assume on the contrary that $A_i \neq 0$ everywhere. Equations (20), (21) and (22) imply

$$A_i C_{lhjk} + A_j C_{lhki} + A_k C_{lhij} = 0.$$

Transvecting this equation with A^i, we obtain $A_r A^r C_{lhjk}=0$ and, because of (5),

(23) $$A_r A^r = 0.$$

Relation (21) leads to the formula

(24) $$\frac{R}{n-1}(A_k g_{jh} - A_j g_{kh}) - (A_k R_{jh} - A_j R_{kh}) = 0$$

whence, transvecting this with A^k and making use of (23) and (12), we have

$$R A_j A_k = 0$$

which, by our assumption, implies

(25) $$R = 0$$

It follows from (24) and (25) that

(26) $$A_k R_{jh} = A_j R_{kh}.$$

This implies, by an elementary algebraic argument,

(27) $$R_{ij} = e A_i A_j,$$

where e is an analytic function on M.

If $R_{ij} = 0$, then $C_{hijk} = R_{hijk}$ and our theorem follows immediately from [6]. In the remaining case, we have $e \neq 0$ on some open subset V.
Equation (27) immediately implies

(28) $$R_{ij,k} = c_k R_{ij},$$

where $c_k = e_{,k}/e$.
Relation (7) yields now

$$w_i A_h = w_h A_i$$

with

(29) $$w_i = a_{ir} A^r,$$

whence $w_i = SA_i$ for some function S. Therefore

(30) $$a_{ir} R^r{}_j = SR_{ij}$$

and, because of (25)

(31) $$a_{rs} R^{rs} = 0.$$

Relation (9), equation $A_{i,j} = \Theta$ and Ricci identity give

(32) $$a_{kr} R^r{}_{ijm} + a_{ir} R^r{}_{kjm} = 0.$$

Contracting the last equation with g^{km} and using (30), we obtain

(33) $$a_{rs} R^r{}_{ij}{}^s = SR_{ij}.$$

It follows easily from (1), (30), (31) and (25) that

(34) $$a_{rs} C^r{}_{ij}{}^s = \frac{1}{n-2}(nS - a)R_{ij},$$

where $a = a_{rs} g^{rs}$.

Differentiating (29) and using (8) and (23) we have $w_{i,l} = A_i A_l$. On the other hand, from $w_i = SA_i$ and $A_{i,l} = 0$ we have $w_{i,l} = S_{,l}A_i$. Comparing these equations, we obtain

(35) $$A_l = S_{,l}.$$

Differentiating (34) covariantly and using (35), (28), (8), (9), (22), (34) and relation $A_{i,j} = 0$, we find

$$\frac{E}{n-2}\varphi_{lm}R_{ij} = (A_l c_m + A_m c_l)R_{ij} + (c_l c_m + c_{l,m})\frac{E}{n-2}R_{ij},$$

which leads to

(36) $$E\varphi_{lm} = (n-2)(A_l c_m + A_m c_l) + Eb_{lm},$$

where $E = nS-a$, $b_{lm} = c_l c_m + c_{l,m}$.

Relation $R_{ij,lm} = b_{lm}R_{ij}$ implies

$$EC_{hijk,lm} = ER_{hijk,lm} - \frac{Eb_{lm}}{n-2}(g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}).$$

On the other hand equations (36) and (2) imply

$$EC_{hijk,lm} = (n-2)(A_l c_m + A_m c_l)C_{hijk} + Eb_{lm}R_{hijk} - \frac{Eb_{lm}}{n-2}(g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}).$$

Comparing these relations and using (1) and (36), we obtain

(37) $$ER_{hijk,lm} = E\varphi_{lm}R_{hijk} - (A_l c_m + A_m c_l)(g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}).$$

Differentiating (32) twice covariantly and using (8), (9), (37), (32) and (30), we have

$$(A_l c_p + A_p c_l)(d_{km}R_{ij} + d_{im}R_{kj} - d_{ij}R_{km} - d_{kj}R_{im}) =$$

$$= E(A_k R_{lijm,p} + A_k R_{pijm,l} + A_i R_{lkjm,p} + A_i R_{pkjm,l}),$$

where $d_{ij} = a_{ij} - Sg_{ij}$.

This equation can be written in the form

$$c_p(d_{km}A_l R_{ij} + d_{im}A_l R_{kj} - d_{ij}A_l R_{km} - d_{kj}A_l R_{im}) +$$

$$+ c_l(d_{km}A_p R_{ij} + d_{im}A_p R_{kj} - d_{ij}A_p R_{km} - d_{kj}A_p R_{im}) =$$

$$= E(A_k R_{lijm,p} + A_k R_{pijm,l} + A_i R_{lkjm,p} + A_i R_{pkjm,l})$$

and after substitution of (26), in the form

$$A_i\left[c_p(d_{km}R_{lj} - d_{kj}R_{lm}) + c_l(d_{km}R_{pj} - d_{kj}R_{pm}) - E(R_{pkjm,l} + R_{lkjm,p})\right] =$$

$$= -A_k\left[c_p(d_{im}R_{lj} - d_{ij}R_{lm}) + c_l(d_{im}R_{pj} - d_{ij}R_{pm}) - E(R_{pijm,l} + R_{lijm,p})\right].$$

This equation we can shortly write as follows

$$A_i P_{pkmlj} = - A_k P_{pimlj}$$

whence, by a standard calculation, we have

$$A_i P_{pkmlj} = 0.$$

Our assumption implies thus

(38) $$c_p(d_{km}R_{lj} - d_{kj}R_{lm}) + c_l(d_{km}R_{pj} - d_{kj}R_{pm}) = E(R_{pkjm,l} + R_{lkjm,p}).$$

Contracting now (28) with g^{ki} and using (25) and the well known relation $R^{r}{}_{,jr} = \frac{1}{2} R_{,j}$, we obtain

$$c_r R^r{}_j = 0. \tag{39}$$

By contraction of (38) with g^{lp} and substitution of (39), we have

$$E R^r{}_{kjm,r} = 0. \tag{40}$$

If $E = 0$, equation $E_{,j} = (n-2)A_j$ implies $A_j = 0$. If $R^r{}_{kjm,r} = 0$, we use well-known relation $R^r{}_{kjm,r} = R_{kj,m} - R_{km,j}$ and apply Theorem C. Hence, in both cases $A_j = 0$. This contradition to our assumption shows our assertion.

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